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A HEAT CONDUCTION PROBLEM AND CERTAIN PRODUCTS OF THE MULTIVARIABLE *H*-FUNCTION WITH TWO GENERAL CLASSES OF POLYNOMIALS

By

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(*Received : September 3, 1996*)

ABSTRACT

Special functions and their applications are becoming increasingly important to boundary value problems in partial differential equations of engineering and physics. So far as its technical applications is concerned, space research and nuclear reactor also give rise to several problems of the applications of special functions. The aim of this paper is to discuss the applications of certain products of the multivariable *H*-function with two general classes of polynomials in obtaining a solution of the partial differential equation

$$\frac{\partial\theta}{\partial t}=L\,\frac{\partial^2\theta}{\partial x^2}$$

related to a (simple but typical) problem of heat conduction in a rod with one end at zero degree and the other end insulated. The results so derived may be found useful for computing different values of $f(x)$ for several special functions (in one and more arguments).

1. INTRODUCTION As an example of the applications of certain products of the multivariable *H*-function with two general classes of polynomials in applied mathematics, we shall consider the problem of obtaining a solution of heat conduction problem in one dimensional rod with constant thermal properties having no source, with one end at zero degree and the other end insulated, such that no solutions of heat equation develop exponentially in time.

Now, we formulate a simple (but typical) problem of heat flow to find a formula for the temperature $\theta(x, t)$ in one dimensional rod $0 \leq x \leq F$ must satisfy the heat equation

$$\frac{\partial\theta}{\partial t} = L\,\frac{\partial^2\theta}{\partial x^2}, \quad t \geq 0 \qquad \text{...(1)}$$

and the initial conditions

$$\theta\,(0, t) = 0 \qquad \text{...(2)}$$

$$\frac{\partial\theta}{\partial x}\,(E, t) = 0 \qquad \text{...(3)}$$

$$\theta\,(x, t) \text{ is finite when } t \to \infty \qquad \text{...(4)}$$

$$\theta\,(x, 0) = f(x) \qquad \text{...(5)}$$

Here we shall consider

$$f(x) = \sin\left(\frac{\pi x}{F}\right)^{\gamma-1} S_n^m\left[y\left(\sin\frac{\pi x}{F}\right)^h\right] S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right]$$
$$H\left\{z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1}, \ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right\}, \qquad \text{...(6)}$$

where the multivariable H-function is the H-function of several complex variables introduced by Srivastava and Panda ([8]; see also [6], p.251) and for the general class of polynomials ([5]; see also [7]).

2. THE MAIN INTEGRAL

$$\int_0^F \left(\sin\frac{\pi x}{F}\right)^{\gamma-1} \sin\,(2v+1)\left(\frac{\pi x}{2F}\right) S_n^m\left[y\left(\sin\frac{\pi x}{F}\right)^h\right]$$
$$S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right] H\left(z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1}, \ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right)dx$$
$$= F\,2^{1-\gamma}\sin\,(2v+1)\left(\frac{\pi}{4}\right)\sum_{s=0}^{(n/m)}\frac{(-n)_{ms}}{s!}A_{n,\,s}\,(2^h\,y)^s$$
$$\sum_{s'=0}^{(n'/m')}\frac{(-n')_{m's'}}{s'!}A_{n',\,s'}\,[2^{h'}\,y']^{s'}\; H^{0,\,\lambda+1\,:\,(u',\,v');\ldots;(u^{(r)},v^{(r)})}_{A+1,\,C+2:\,[B',D'];\ldots;[B^{(R)},\,D^{(r)}]}$$
$$\left(\begin{array}{l} [1-\gamma-hs-h's' : 2h_1;\ldots;2h_r],..[(a):\theta':\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}],[(1-2\gamma-2hs-2h's-2v)/4 : h_1;\ldots;h_r],[(3-2\gamma-2hs-2h's'+2v)/4;h_1;\ldots;h_r]: \end{array}\right.$$
$$\left.\begin{array}{l} [(b'):\phi'];\ldots;[(b^{(r)});\phi^{(\rho)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{array}\; z_1 4^{-h_1};\ldots;z_r 4^{-h_r}\right) \qquad \text{...(7)}$$

where $\mathrm{Re}\left(\gamma + \sum_{i=1}^{r} h_i d_j^i/\delta_j^i\right) > 0,\ h_i > 0,\ h > 0,\ h' > 0,$

$$|\arg (z_i)| < T_i \frac{\pi}{2}, T_i > 0, 1 \le i \le r,$$

$$1 \le j \le u^{(i)},$$

and m and m' are arbitrary integers and the coeffients $A_{n,s}$ and $A_{n',s'}$ $(n, n', s, s' \ge 0)$ are constants, real or complex.

The integral in (7) can be established by making use of the definition of the general class of polynomials ([8]; see also [6]) and a result recently obtained by Sharma ([3], p.372).

3. SOLUTION OF THE PROBLEM

The solution of the problem will be found to be

$$\theta(x,t) = 2^{2-\gamma-hs-h's'} \sum_{v=0}^{\infty} \sin(2v+1)\left(\frac{\pi}{4}\right) e^{-(2v+1)(\pi/2F)Lt}$$

$$\sin\left(v+\frac{1}{2}\right)\left(\frac{\pi x}{F}\right) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s} y^{s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n',s'} y^{s'}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [l-\gamma-hs-h's':2h_1;\ldots;2h_r] \\ [(c):\psi';\ldots;\psi^{(r)}], \end{matrix} \right.$$

$$[(a):\theta';\ldots;\theta^{(r)}]:$$

$$[(1-2\gamma-2hs-2h's'-2v)/4:h_1;\ldots;h_r], [(3+2v-2\gamma-2hs-2h's')/4;h_1;\ldots;h_r];$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 4^{-h_1}, \ldots, z_r 4^{-h_r} \right) \quad \ldots(8)$$

where $h > 0, h' > 0, h_i > 0, \mathrm{Re}\left(\gamma + \sum_{i=1}^{r} h_i d_j^i/\delta_j^i\right) > 0,$

$$|\arg (z_i)| < T_i \frac{\pi}{2}, T_i > 0, 1 \le i \le r, 1 \le j \le u^{(i)},$$

and m and m' are arbitrary integers and the coefficients $A_{n,s}$ $(n, s \ge 0)$ and $A_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

The general solution of the problem expressed in section 1 by setting $\theta = X(x)\ T(t)$ can be easily established as ([4], p.75)

$$\theta(x,t) = \sum_{v=0}^{\infty} B_v \sin\left(\left(v+\frac{1}{2}\right)\frac{\pi x}{F}\right) e^{\{-(2v+1)\pi/2F\} Lt} \quad \ldots(9)$$

If $t = 0$, then by virtue of (5) we have

$$\left(\sin\frac{\pi x}{F}\right)^{\gamma-1} S_n^m\left[y\left(\sin\frac{\pi x}{F}\right)^h\right] S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right]$$

$$H\left(z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1}, \ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right) = \sum_{v=0}^{\infty} B_v \sin\left(v+\frac{1}{2}\right)\left(\frac{\pi x}{F}\right) \qquad \ldots(10)$$

Multiplying both the sides of (10) by $\sin\,(w+\frac{1}{2})\,\frac{\pi x}{F})$, integrating with respect to x from 0 to F and using (7) along with the orthogonality property of sines, we get

$$B_w = 2^{2-\gamma-hs-h's'} \sin\,(2w+1)\,\frac{\pi}{4}) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n',s'}\, y'^{s'}\, H^{0,\,\lambda+1\,:\,(u',\,v');\ldots;(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D'];\ldots;[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix}[1-\gamma-hs-h's' : 2h_1;\ldots;2h_r],\ldots\ [(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}],[(1-2\gamma-2hs-2h's')/4{:}h_1;\ldots;h_r];\ [(3-2\gamma-2hs-2h's'+2w)/4 : h_1;\ldots;h_r)]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1\, 4^{-h_1},\ldots, z_r\, 4^{-h_r}\right) \qquad \ldots(11)$$

Putting the values of B_w from (11) in (9), we get the related solution.

4. APPLICATIONS AND PARTICULAR CASES

(A) By applying our results given in (7) and (8) to the case of Hermite polynomials ([9], p.106, eq. (5.54) and [7], p.158) by setting

$$S_n^2(x) \to x^{n/2} H_n\left(\frac{1}{2\sqrt{x}}\right)$$

in which case $m = 2$, $A_{n,s} = (-1)^s$ and also letting $m' = 2$. $A_{n',s'} = (-1)^{s'}$. We have the following interesting consequences of the main results :

(A.1)

$$\int_0^F \sin\left(\frac{\pi x}{F}\right)^{\gamma-1} (\sin\,(2v+1)\,\frac{\pi x}{2F})\left[y\left(\sin\frac{\pi x}{F}\right)^h\right]^{n/2} H_n\left[\frac{1}{2\sqrt{y\,(\sin\frac{\pi x}{F})^h}}\right]$$

$$\left[y' (\sin \frac{\pi x}{F})^{h'} \right]^{n'/2} H_{n'}\left[\frac{1}{2\sqrt{y' (\sin \frac{\pi x}{F})^{h'}}} \right]$$

$$H\left(z_1(\sin \frac{\pi x}{F})^{h_1}, \ldots, z_r (\sin \frac{\pi x}{F})^{h_r} \right) dx$$

$$= F.2^{1-\gamma-n-n'-hs-h's'} \sin ((2v+1)\frac{\pi}{4}) \sum_{s=0}^{n/2} \frac{(-n)_{2s}}{s!} (-\frac{y}{4})^s$$

$$\sum_{s'=0}^{(n'/2)} \frac{(-n')_{2s'}}{s'!} (\frac{-y'}{4})^{s'} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} (1-\gamma-hs-h's'); 2h_1;\ldots;2h_r], [(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}],[(1-2\gamma-2hs-2h's'-2v)/4;h_1;\ldots;h_r][(3-2\gamma-2hs-2h's'+2v)/4:h_1;\ldots;h_r]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(\rho)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 4^{-h_1},\ldots, z_r 4^{-h_r} \right) \ldots(12)$$

which holds true under the same conditions needed for (7).

(A.2)

$$\theta(x,t) = 2^{2-\gamma+n+n'-hs-h's'} \sum_{v=0}^{\infty} \sin(2v+1)\frac{\pi}{4}) e^{-[(2v+1)\pi/2F] Lt}$$

$$\sin(v+\frac{1}{2})\frac{\pi x}{F}) \sum_{s=0}^{[n/2]} \frac{(-n)_{2s}}{s!} (-\frac{y}{4})^s \sum_{s'=0}^{(-n'/2)} \frac{(-n')_{2s'}}{s'!} (-\frac{y'}{4})^{s'}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)});}_{A+1,C+2:[B',D'];\ldots;[B^{(r)}.D^{(r)}];} \left(\begin{matrix} [1-\gamma-hs-h's':2h_1;\ldots;2h_r] \\ [(c):\psi;\ldots;\psi^{(r)}] \end{matrix} \right.$$

$$[(a):\theta';\ldots;\theta^{(r)}]:$$

$$[(1-2\gamma-2hs-2h's'-2v)/4:h_1;\ldots;h_r], [(3+2v-2\gamma-2hs-2h's')/4; h_1;\ldots;h_r]:$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 4^{-h_1},\ldots, z_r 4^{-h} \right) \ldots(13)$$

valid under the same conditions as obtainable from (8).

(B) For the Laguerre polynomials ([9], p.101, Eq. 15, 1.6) and [7], p.159), setting $S_n^1(x) \to L_n^{(\alpha')}(x)$ in which case

$m = 1, A_{n,s} \binom{n+\alpha'}{n} (\frac{1}{(\alpha'+1)_k})$ and also taking $m'_{\cdot} = 1$.

$A_{n',s'} \binom{n'+\alpha''}{n'} (\frac{1}{(\alpha''+1)_{k'}})$ the results in (7) and (8) reduce to the following formulae :

(B.1)

$$\int_0^F \left(\sin \frac{\pi x}{F}\right)^{\gamma-1} \sin\left((2v+1)\frac{\pi x}{2F}\right) L_n^{\alpha'}\left[y\left(\sin\frac{\pi x}{F}\right)^h\right] L_{n'}^{\alpha''}\left[y'\left(\sin\frac{\pi x}{F}\right)^{h'}\right]$$

$$H\left(z_1\left(\sin\frac{\pi x}{F}\right)^{2h_1},\ldots, z_r\left(\sin\frac{\pi x}{F}\right)^{2h_r}\right)dx$$

$$= F.2^{1-\gamma-hs-h's'} \sin\left((2v+1)\frac{\pi}{4}\right) \sum_{s=0}^{(n)} \binom{n+\alpha'}{n-s}\frac{(-y)^s}{s!} \sum_{s'=0}^{(n')} \binom{n'+\alpha''}{n'-s'}\frac{(-y')^{s'}}{s'!}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',d'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}1-\gamma-hs-h's':2h_1;\ldots;2h_r], \\ [(c):\psi';\ldots;\psi^{(r)}]\end{matrix}\right.$$

$$[(a):\theta';\ldots;\theta^{(r)}]:$$
$$[(1-2\gamma-2hs-2h's'-2v)/4):h_1;\ldots;h_r],\ [(3-2\gamma-2hs-2h's'+2v)/4:h_1,\ldots,h_r]:$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)});\delta^{(r)}];\end{matrix}\ z_1 4^{-h_1};\ldots;z_r 4^{-h_r}\right) \qquad \ldots(14)$$

valid under the same conditions as obtainable from (7).

(B.2)

$$\theta(x,t) = 2^{2-\gamma+n+n'-hs-h's'} \sum_{v=0}^{\infty} \sin(2v+1)\frac{\pi}{4})\ e^{-[(2v+1)\pi/2F]Lt}$$

$$\sin\left(v+\frac{1}{2}\right)\frac{\pi x}{F}) \sum_{s=0}^{n}\frac{(-y)^s}{s!}\binom{n+\alpha'}{n-s}\sum_{s'=0}^{n'}\frac{(-y')^{s'}}{s'!}\binom{n'+\alpha''}{n'-s'}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)});}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]:}\left(\begin{matrix}[1-\gamma-hs-h's':2h_1;\ldots;2h_r] \\ [(c):\psi';\ldots;\psi^{(r)}],\end{matrix}\right.$$

$$[(a):\theta';\ldots;\theta^{(r)}]:$$
$$[(1-2\gamma-2hs-2h's'-2v)/4:h_1;\ldots;h_r],\ [(3+2v-2\gamma-2hs-2h's')/4:h_1;\ldots;h_r]:$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]: \\ [(d').\delta'];\ldots;[(d^{(r)}).\delta^{(r)}]:\end{matrix}\ z_1 4^{-h_1},\ldots,z_1 4^{-h_r}\right) \qquad \ldots(15)$$

holds true under the same conditions as those required for (8).

(C) for the Jacobi polynomials ([9], p.68, Eq. (4.3.2) and [7], p.159), setting $S_n^1(x) \to P_n^{(\alpha',\beta')}(1-2x)$ and $S_{n'}^1(x) \to p_{n'}^{(\alpha'',\beta'')}(1-2x)$ in which case $m = 1$, $A_{n,s} = \binom{n+\alpha'}{n}\frac{(\alpha'+\beta'+n+1)_s}{(\alpha'+1)_s}$ and also let $m' = 1$.

$$A_{n',s'} = \binom{n'+\alpha''}{n'}\frac{(\alpha''+\beta''+n'+1)_{s'}}{(\alpha''+1)_{s'}}$$

From equations (7) and (8), we obtain the following results.

(C.1)

$$\int_0^F (\sin \frac{\pi x}{F})^{\gamma - 1} \sin (2v + 1) \frac{\pi x}{2F}) \, p_n^{(\alpha', \beta')}\left[1 - 2y \left(\sin \frac{\pi x}{F} \right)^h \right]$$

$$P_{n'}^{(\alpha'', \beta'')}\left[1 - 2y' \left(\sin \frac{\pi x}{F} \right)^{h'} \right] H \left(z_1 (\sin \frac{\pi x}{F})^{2h_1}, \ldots, z_r (\sin \frac{\pi x}{F})^{2h_r} \right) dx$$

$$= F.2^{1 - \gamma - hs - h's'} \sin (2v + 1) \frac{\pi}{4}) \sum_{s=0}^{n} (-y)^s \binom{n + \alpha'}{n - s} \binom{\alpha' + \beta' + n + s}{s}$$

$$\sum_{s'=0}^{(n')} (-y')^{s'} \binom{n' + \alpha''}{n' - s'} \binom{\alpha'' + \beta'' + n' + s'}{s'}$$

$$H_{A+1, C+2 : [B', D'];\ldots;[B^{(r)}, D^{(r)}]}^{0, \lambda + 1 : (u', v');\ldots;(u^{(r)}, v^{(r)})} \left(\begin{matrix} [1 - \gamma - hs - h's' : 2h_1 ;\ldots: 2h_r], \\ [(c) : \psi' ;\ldots; \psi^{(r)}], \end{matrix} \right.$$

$$[(a) : \theta' ;\ldots; \theta^{(r)}]:$$

$$[(1 - 2\gamma - 2hs - 2h's' - 2v)/4v : h_1;\ldots;h_r] \; [(3 - 2\gamma - 2hs - 2h's' + 2v)/4 : h_1 ;\ldots; h_r]:$$

$$\left. \begin{matrix} [(b') ; \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]; \\ (d') : \delta'];\ldots; [(d^{(r)}); \delta^{(r)}]; \end{matrix} \; z_1 4^{-h_1} ;\ldots; z_r 4^{-h_4} \right), \qquad \ldots(16)$$

valid under the same conditions needed for (7).

(C.2)

$$\theta (x, t) = 2^{2 - \gamma - hs - h's'} \sum_{v=0}^{\infty} \sin (2v + 1) \frac{\pi}{4}) \, e^{[-(2v + 1) \pi/2F] Lt}$$

$$\sin (v + \frac{1}{2}) \frac{\pi x}{F}) \sum_{s=0}^{n} (-y)^s \binom{n + \alpha'}{n - s} \binom{\alpha' + \beta' + n + s}{s}$$

$$\sum_{s'=0}^{n'} (-y')^{s'} \binom{n' + \alpha''}{n' - s'} \binom{\alpha'' + \beta'' + n' + s'}{s'}$$

$$H_{A+1, C+2 : [B', D'];\ldots;[B^{(r)} : D^{(r)}]}^{0, \lambda + 1 : (u', v');\ldots;(u^{(r)}, v^{(r)}]} \left(\begin{matrix} [1 - \gamma - hs - h's' : 2h_1;\ldots;2h_r], \\ [(c) : \psi' ;\ldots; \psi^{(r)}], \end{matrix} \right.$$

$$[(a) : \theta' ;\ldots; \theta^{(r)}]:$$

$$[(1 - 2\gamma - 2hs - 2h's' - 2v)/4 : h_1;\ldots;h_r], [(3 - 2\gamma - 2hs - 2h's' + 2v)/4 : h_1 ;\ldots; h_r]:$$

$$\left. \begin{matrix} [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]: \\ [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \; z_1 4^{-h_1} ;\ldots; z_r 4^{-h} \right) \qquad \ldots(17)$$

valid under the same conditions as obtainable from (8).

(D) Taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$ and replacing $D^{(i)}$ by $D^{(i)} + 1$, $\forall\ i = 1, \ldots, v$ in (7) and (8), we get

(D.1)

$$\int_0^F (\sin \frac{\pi x}{F})^{\gamma - 1} \sin (2v + 1) \frac{\pi x}{2F}) S_n^m \left[y (\sin \frac{\pi x}{F})^h \right] S_{n'}^{m'} \left[y' \left(\sin \frac{\pi x}{F} \right)^{h'} \cdot \right]$$

$$F^{A : B' ;\ldots; B^{(r)}}_{C : D' ;\ldots; D^{(r)}} \left(\begin{matrix} [1 - (a) : \theta' ;\ldots; \theta^{(r)}] : [1 - (b') : \phi'] ;\ldots; [1 - (b^{(r)}) : \phi^{(r)}]; \\ [1 - (c) : \psi' ;\ldots; \psi^{(r)}] : [1 - (d') : \delta'] ;\ldots; [1 - (d^{(r)}) : \delta^{(r)}] ; \end{matrix} \right.$$

$$\left. - z_1 (\sin \frac{\pi x}{F})^{2h_1} ,\ldots, - z_r (\sin \frac{\pi x}{F})^{2h_r} \right) dx$$

$$= F . 2^{1 - \gamma - hs - h's'} \sin ((2v + 1) \frac{\pi}{4}) \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n, s} y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n', s'} y^{s'}$$

$$\frac{\Gamma (\gamma + hs + h's')}{\Gamma [(3 + 2\gamma + 2hs + 2h's' + 2v)/4]\ \Gamma [(1 + 2\gamma + 2hs + 2h's' - 2v)/4]}$$

$$F^{A + 1 : B' ;\ldots; B^{(r)}}_{C + 2 : D' ;\ldots; D^{(r)}} \left(\begin{matrix} [\gamma + hs + h's' : 2h_1 ;\ldots; 2h_r]; \\ [1 - (c) : \psi' ;\ldots; \psi^{(r)}] \end{matrix} \right.$$

$$[1 - (a) : \theta' ;\ldots; \theta^{(r)}] :$$

$$[(3 + 2\gamma + 2hs + 2h's' + 2v)/4 : h_1 ;\ldots; h_r], [(1 + 2\gamma + 2hs + 2h's' - 2v)/4 : h_1 ;\ldots; h_r] :$$

$$\left. \begin{matrix} [1 - (b') : \phi'] ;\ldots; [1 - (b^{(r)}) : \phi^{(r)}]; \\ [1 - (d') : \partial'] ;\ldots; [1 - (d^{(r)}); \delta^{(r)}]; \end{matrix} - z_1 4^{-h_1} ,\ldots, - z_r 4^{-h_r} \right), \qquad \ldots(18)$$

valid under the same conditions as required in (7).

(D.2)

$$\theta (x, t) = 2^{2 - \gamma - hs - h's'} \sum_{v = 0}^{\infty} \sin (2v + 1) \frac{\pi}{4}) e^{- [(2v + 1) \pi/2F] Lt}$$

$$\sin ((v + \frac{1}{2}) \frac{\pi x}{F}) \sum_{s = 0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n, s} y^s \sum_{s' = 0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} A_{n', s'} y'^{s'}$$

$$\frac{\Gamma (\gamma + hs + h's')}{\Gamma [(3 + 2\gamma + 2hs + 2h's' + 2v)/4]\ \Gamma [(1 + 2\gamma + 2hs + 2h's' - 2v)/4]}$$

$$F^{A + 1 : B' ;\ldots; B^{(r)}}_{C + 2; D' ;\ldots; D^{(r)}} \left(\begin{matrix} [\gamma + hs + h's' : 2h_1 ;\ldots; 2h_r] \\ [1 - (c) : \psi' ;\ldots.; \psi^{(r)}]; \end{matrix} \right.$$

$$[1 - (a) : \theta' ;\ldots: \theta^{(r)}] :$$

$$[(3 + 2\gamma + 2hs + 2h's' + 2v)/4 : h_1 ;\ldots; h_r], [(1 + 2\gamma + 2hs + 2h's' - 2v)/4 : h_1 ;\ldots; h_r] :$$

$$\left. \begin{matrix} [1 - (b') : \phi'] ;\ldots; [1 - (b^{(r)}) : \phi^{(r)}] ; \\ [1 - (d') : \delta'] ;\ldots; [1 - (d^{(r)}) : \delta^{(r)}] ; \end{matrix} - z_1 4^{h_1} ,\ldots, - z_r 4^{-h_r} \right) \qquad \ldots(19)$$

valid under the same conditions as obtainable from (8).

(E) Letting $n' \to 0$ the results in (12) through (19) follow as particular case of the results recently obtained by Sharma [3].

(F) Taking $n \to 0$ and $n' \to 0$, the results in (7) and (8) reduce to the results recently obtained by Chaurasia ([1], p.172, eq.(2.1) and p.173, eq.(3.1)).

The importance of our results lies in its manifold generality. In view of the generality of the *H*-function of several complex variables, on specializing the various parameters and variables in the *H*-function of several complex variables, we obtain, from our results, several integrals and solutions involving a remarkably wide variety of useful functions (or product of several such functions), which are expressible in terms of *E, F, G* and *H* functions of one and several variables.

Secondly, by suitably specializing the coefficients $A_{n,s}$ and $A_{n',s'}$ and making a free use of the special cases of $S_n^m[x]$ listed by Srivastava and Singh [7], our results can be reduced to a large number of integerals and solutions of the problem (and also its particular cases) involving a product of generalized Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials. Gould-Hopper polynomials, Brafman polynomials and their various combinations. Thus the results presented in this paper would at once yield a very large number of results involving a large variety of polynomials and various special functions occuring in the problems of mathematical analysis, applied mathematics and mathematical physics.

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STABILITY OF TWO RIVLIN-ERICKSEN ELASTICO-VISCOUS SUPERPOSE FLUIDS

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ABSTRACT

The stability of the plane interface separating two Rivilin-Ericksen elastico-viscous superposed fluids of uniform densities has been studied. The stability analysis has been carried out, for mathematical simplicity, for two highly viscous fluids of equal kinematic viscosities and equal kinematic viscoelasticities. It is found that the system is stable for stable configuration and unstable for unstable configuration. The behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity parameters are examined analytically.

1. INTRODUCTION The instability of the plane interface separating two fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors, and Chandrasekhar [2] has given a detailed account of these investigations. Roberts [5] has extended the analysis to the case of two fluids of equal kinematic viscosities in presence of vertical magnetic field while Gerwin [3] has studied the case of compressible streaming fluids. The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field has been studied by Bhatia [1]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities.

Sharma and Sharma [6] have studied the stability of the plane interface separating two viscoelastic (Oldroydian) superposed fluids of uniform densities in another study Sharma [7] has studied the instability of the plane interface between two Oldrydian viscoelastic superposed conducting fluids in the presence of a uniform magnetic field. With the growing importance of non-Newtonian fluids in modern technology and industries, the further investigations on such fluids are desirable. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [4] constitutive relations. Rivlin-Ericksen elastico- viscous fluid is one such fluid. It is this class of elastico-viscous fluids we are interested in, particularly to study the stability of the plane interface separating two imcompressible superposed Rivlin-Ericksen fluids of uniform densities. This aspect forms the subject matter of the present paper wherein we have carried out the stability analysis for two fluids of equal kinematic viscosities, equal kinematic viscoelasticities and different uniform densities.

2. PERTURBATION EQUATIONS Consider a static state, in which an incompressible Rivlin- Ericksen elastico-viscous fluid of variable density is arranged in horizontal strata and the pressure p and density ρ are functions of the vertical coordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u, v, w)$, $\delta\rho$ and δp denote the perturbations in velocity (0,0,0), density ρ and pressure p, respectively. Then the linearized perturbation equations relevant to the problem are

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla \delta p + \vec{g}\delta\rho + \rho \left(\nu + \nu' \frac{\partial}{\partial t}\right) \nabla^2 \vec{v} + \left(\frac{\partial \mu}{dz} + \frac{\partial}{\partial t}\frac{\partial \mu'}{dz}\right)\left(\frac{\partial w}{\partial \bar{x}} + \frac{\partial \vec{v}}{\partial z}\right) \quad \ldots(1)$$

$$\nabla \cdot \vec{v} = 0 \quad \ldots(2)$$

$$\frac{\partial}{\partial t}(\delta\rho) = -wD\rho, \quad \ldots(3)$$

where $\nu \left(= \frac{\mu}{\rho}\right)$ and $\nu' \left(= \frac{\mu'}{\rho}\right)$ denote respectively the kinematic viscosity and the kinematic viscoelasticity of the fluid, $\vec{g} = (0, 0, -g)$ is the acceleration due to gravity $\bar{x} = (x, y, z)$ and $D = \frac{d}{dz}$. Equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion.

Analyzing the disturbances into normal modes, we assume that the perturbed quantities have the space and time dependence of the form

$$f(z) \exp(ik_x x + ik_y y + nt), \quad \ldots(4)$$

where k_x, k_y are the wave numbers along the x and y directions respectively, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number, n is the growth rate which is, in general, a complex constant and $f(x)$ is some function of z.

For perturbations of the form (4), equations (1)-(3) become

$$\rho n u = - ik_x \,\delta p + \rho(\nu + \nu' n)(D^2 - k^2)u + (D\mu + nd\mu')(ik_x w + Du) \qquad ...(5)$$

$$\rho n v = - ik_y \delta p + \rho(\nu + \nu' n)(D^2 - k^2)v + (D\mu + nd\mu')(ik_y w + Dv), \qquad ...(6)$$

$$\rho n w = - D\delta p - g\delta\rho + \rho(\nu + \nu' n)(D^2 - k^2)w + 2(D\mu + nd\mu')dw \qquad ...(7)$$

$$ik_x u + ik_y v + Dw = 0 \qquad ...(8)$$

and

$$n\delta\rho = - wD\rho \qquad ...(9)$$

Multiplying equations (5) and (6) by $-ik_x$ and $-ik_y$, respectively, adding the resultant equations and using equation (8) in it, we obtain

$$\rho n Dw = - k^2\delta p + \rho(\nu + \nu' n)(D^2 - k^2)Dw + (D\mu + nD\mu')(D^2 + k^2)w \qquad ...(10)$$

Substituting for $\delta\rho$ from (9) in equation (7) and eliminating δp between equations (7) and (10), we obtain the equation in w :

$$n[D(\rho Dw) - k^2\rho w] - [D\{\rho(v + v' n)(D^2 - k^2)Dw\}$$

$$- k^2\rho(\nu + \nu' n)(D^2 - k^2)w] + \frac{gk^2}{n}(D\rho)w$$

$$- [D\{(D\mu + nd\mu')(D^2 + k^2)w\} - 2k^2 D\,(\mu + n\mu')Dw] = 0 \qquad ...(11)$$

3. TWO SUPERPOSED RIVLIN-ERICKSEN FLUIDS SEPARATED BY A HORIZONTAL BOUNDARY

Here we consider the case when two Rivlin-Ericksen superposed fluids of uniform densities ρ_1, ρ_2 uniform viscosities μ_1, μ_2 and uniform viscoelasticities μ_1', μ_2' are separated by a horizontal boundary at z=0. The subscripts 1 and 2 distinguish the lower and the upper fluids, respectively. Then, in each region of constant ρ constant μ and constant μ', equation (11) becomes

$$(D^2 - k^2)(D^2 - q^2)w = 0 \qquad ...(12)$$

where

$$q^2 = k^2 + \frac{n}{\nu + \nu' n}.$$

Since w must vanish both when $z \to \infty$ (in the upper fluid) and $z \to -\infty$ (in the lower fluid), the general solution of equation (12) can be written as

$$w_1 = A_1 e^{+kz} + A_2 e^{+q_1 z},\ (z < 0) \qquad \text{...(13)}$$

$$w_2 = A_3 e^{-kz} + A_4 e^{-q_2 z},\ (z > 0) \qquad \text{...(14)}$$

where A_1, A_2, A_3, A_4 are constants of integration,

$$q_1 = \sqrt{k^2 + \frac{n}{\nu_1 + \nu_1' n}} \text{ and } q_2 = \sqrt{k^2 + \frac{n}{\nu_2 + \nu_2' n}}\,, \qquad \text{...(15)}$$

In writing the solutions (13) and (14), it is assumed that q_1 and q_2 are so defined that their real parts are positive.

4. BOUNDARY CONDITIONS

The solutions (13) and (14) must satisfy certain boundary conditions. The boundary conditions to be satisfied at the interface $z = 0$ are :

$$w, \qquad \text{...(16)}$$

$$Dw, \qquad \text{...(17)}$$

and

$$(\mu + \mu' n)(d^2 + k^2)w, \qquad \text{...(18)}$$

must be continuous.

Integrating equation (11) across the interface $z = 0$, we obtain another condition

$$[\rho_2 Dw_2 - \rho_1 Dw_1]_{z=0} - [\tfrac{1}{n}(\mu_2 + \mu_2' n)(D^2 - k^2)Dw_2$$

$$- \tfrac{1}{n}(\mu_1 + \mu_1' n)(D^2 - k^2)Dw_1]_{z=0}$$

$$= -\frac{gk^2}{n^2}[\rho_2 - \rho_1]w_0 - \frac{2k^2}{n}(\mu_2 + n\mu_2' - \mu_1 - n\mu_1')(Dw)_0 \qquad \text{...(19)}$$

where w_0, $(Dw)_0$ are the common values of w_1, w_2 and Dw_1, Dw_2 at $z = 0$.

5. DISPERSION RELATION AND DISCUSSION

Applying the boundary conditions (16)-(19) to the solutions (13) and (14), we obtain

$$A_1 + A_2 = A_3 + A_4\ (= w_0) \qquad \text{...(20)}$$

$$kA_1 + q_1 A_2 = -kA_3 - q_2 A_4\ (= Dw_0). \qquad \text{...(21)}$$

$$(\mu_1 + \mu_1' n)\,[2k^2 A_1 + (q_1^2 + k^2)A_2] = (\mu_2 + \mu_2' n)$$

$$[2k^2 A_3 + (q_2^2 + k^2)A_4]\ (= \mu + \mu' n)\,(d^2 + k^2)w_0) \qquad \text{...(22)}$$

$$[-k\rho_2A_3-\rho_2q_2A_4-k\rho_1A_1-\rho_1A_1-\rho_1q_1A_2]-\frac{1}{n}(\mu_2-\mu_2'n)(-q_2)$$

$$(q_2^2-k^2)A_4-\frac{1}{n}(\mu_1+\mu_1'n)q_1(q_1^2-k^2)A_2]+\frac{gk^2}{2n^2}(\rho_2-\rho_1)$$

$$(A_1+A_2+A_3+A_4)+\frac{k^2}{n}(\mu_2+n\mu_2'-\mu_1-n\mu_1')$$

$$(kA_1+q_1A_2-kA_3-q_2A_4)=0 \qquad \text{...(23)}$$

Eliminating the constant A_1, A_2, A_3, A_4 from equations (20)-(23), we obtain

$$\begin{vmatrix}
1 & 1 & -1 & -1 \\
k & q_1 & k & q_2 \\
2k^2(\mu_1+\mu_1'n) & \begin{array}{l}[(\mu_1+\mu_1'n)\\(q_1^2+k^2)]\end{array} & -2k^2(\mu_2+\mu_2'n) & \begin{array}{l}[-(\mu_2+\mu_2'n)\\(q_2^2+k^2)]\end{array} \\
\begin{array}{l}[-\alpha_1+\frac{R}{2}\\+\frac{k^2}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\\-\nu_1\alpha_1-n\nu_1'\alpha)]\end{array} & \begin{array}{l}[\frac{R}{2}+\frac{k}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-v_1\alpha_1-\\-n\nu_1'\alpha_1)q_1]\end{array} & \begin{array}{l}[-\alpha_2+\frac{R}{2}\\-\frac{k^2}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\\-\nu_1\alpha_1-n\nu_1'\alpha_1)]\end{array} & \begin{array}{l}[\frac{R}{2}-\frac{k}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\nu_1\alpha_1\\-n\nu_1'\alpha_1)q_2]\end{array}
\end{vmatrix}=0 \qquad \text{...(24)}$$

where $\alpha_1=\dfrac{\rho_1}{\rho_1+\rho_2}$, $\alpha_2=\dfrac{\rho_2}{\rho_1+\rho_2}$, $R=\dfrac{gk}{n^2}(\alpha_2-\alpha_1)$.

The determinant can be reduced by substracting the first column from the second, the third column from the fourth and adding the first column to the third. By this procedure, we obtain

$$\begin{vmatrix}
q_1-k & 2k & q_2-k \\
\begin{array}{l}[(\nu_1\alpha_1+n\nu_1'\alpha_1)\\(q_1^2-k^2)]\end{array} & \begin{array}{r}[-2k^2(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)]\end{array} & \begin{array}{r}[-(\nu_2\alpha_2+n\nu_2'\alpha_2)\\(q_2^2-k^2)]\end{array} \\
\begin{array}{l}[\frac{k}{n}(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)\\(q_1-k)+\alpha_1]\end{array} & R-1 & \begin{array}{r}[-\frac{k}{n}(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)\\(q_2-k)+\alpha_2]\end{array}
\end{vmatrix}=0 \qquad \text{...(25)}$$

Evaluating the determinant (25), we obtain the following characteristic equation

$$\begin{aligned}
&(q_1 - k)[-2k^2 (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \{-\tfrac{k}{n} (\nu_2\alpha_2 \\
&+ n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_2 - k) + \alpha_2\} + (R - 1) (\nu_2\alpha_2 \\
&+ n\nu_2'\alpha_2) (q_2^2 - k^2)] - 2k \, [(\nu_1\alpha_1 + n\nu_1'\alpha_1)(q_1^2 - k^2) \\
&\{-\tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_2 - k) + \alpha_2\} + \\
&+ (\nu_2\alpha_2 + n\nu_2'\alpha_2) (q_2^2 - k^2) (\tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \\
&(q_1 - k) + \alpha_1\}] + (q_2 - k) \, [(\nu_1\alpha_1 + n\nu_1'\alpha_1)(q_1^2 - k^2)(R - 1) \\
&+ 2k^2 (\nu_1\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \{ \tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 \\
&- \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_1 - k) + \alpha_1\}] = 0. \qquad \text{...(26)}
\end{aligned}$$

The dispersion relation (26) is quite complicated as the values of q_1 and q_2 involve square roots. We therefore make the assumption that the two fluids are of high viscosity and high viscoelasticity. Under this assumption, we have

$$q = k \left[1 + \frac{n}{k^2 (\nu + \nu' n)} \right]^{1/2} = k + \frac{n}{2k (\nu + \nu' n)}, \qquad \text{...(27)}$$

so that

$$q_1 - k = \frac{n}{2k (\nu_1 + \nu_1' n)} \text{ and } q_2 - k = \frac{n}{2k (\nu_2 + \nu_2' n)} \qquad \text{...(28)}$$

Substituting the values of $q_1 - k$ and $q_2 - k$ from (28) in equation (26) and putting $\nu_1 = \nu_2 = \nu$, $\nu_1' = \nu_2' = \nu'$ (the case of equal kinematic viscosities and equal kinematic viscoelasticities, for mathematical simplicity), we obtain the following dispersion relation

$$(1 + 2k^2\nu')n^2 + (2k^2\nu)n - gk(\alpha_2 - \alpha_1) = 0. \qquad \text{...(29)}$$

For the potentially stable arrangement $\alpha_1 > \alpha_2$ equation (29) does not involve any change of sign and so does not allow any positive root. The system is therefore stable. This result is also true when both the fluids are viscous (Chandrasekhar [2]) or Oldroydian viscoelastic (Sharma [7]).

For the unstable configuration $\alpha_2 > \alpha_1$, there is at least one change of sign in equation (29) and so equation (29) has one positive root. The occurence of positive root implies that the system is unstable. The potentially unstable arrangement, therefore, remains unstable for the Rivlin-Ericksen elastico-viscous fluid.

We now examine the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity analytically. Since $\alpha_2 > \alpha_1$ equation (29) has one positive root, let n_0 denote the positive root. Then

$$(1 + 2k^2\nu')n_0^2 + (2k^2\nu)n_0 - gk(\alpha_2 - \alpha_1) = 0. \qquad ...(30)$$

To study the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity we examine the natures of $\frac{dn_0}{d\nu}$ and $\frac{dn_0}{d\nu'}$ analytically.

If follows from equation (30) that

$$\frac{dn_0}{d\nu} = -\frac{k^2 n_0}{(1 + 2k^2\nu')n_0 + \nu k^2}, \qquad ...(31)$$

and

$$\frac{dn_0}{d\nu'} = -\frac{k^2 n_0^2}{(1 + 2k^2\nu')n_0 + \nu k^2}. \qquad ...(32)$$

From (31) and (32), we find that the growth rate decreases with the increase in kinematic viscosity as well as with the increase in kinematic viscoelasticity.

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A GENERAL FRACTIONAL INTEGRAL OPERATOR, A GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS AND THE MULTIVARIABLE *H*-FUNCTION

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ABSTRACT

The main object of this paper is to derive a fractional integral operator (involving a generalized polynomial set) of the product of a general class of multivariable polynomials and the *H*-function of several complex variables. On account of general nature of the operator, the multivariable polynomials and the *H*-function of several complex variables, our formula can be shown to yield a number of interesting (known or new) results. Some new special cases of the formula are mentioned briefly.

1. INTRODUCTION

(a) General Fractional Integral Operator

The familier fractional integral operator (FIO) is defined and represented as follows :

$$ {}_cR_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} f(t)\, dt,\ Re(\mu) > 0 \qquad ...(1.1)$$

The *FIO* (1.1) defines the classical Riemann-Liouville *FIO* of order μ when $c = 0$ and when $c \to \infty$ it may be identified with the definition of the familier Weyl *FIO* of order μ. A number of generalizations of *FIO* are proposed recently. Agrawal, Pareek and Saigo [1,p. 55, Eq. (1.2)] have recently introduced and studied a generalization of *FIO* (1.1) which is defined by representing in the following manner :

$$ {}_cR_{x;\, r,\, s,\, q;\, m,\, k,\, l}^{\rho;\, \alpha,\, \beta,\, \tau;\, A,\, B,\, \mu}\{f(x)\}$$

$$= \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} S_n^{\alpha,\beta,\tau}[z(1-\tfrac{t}{x})^{\rho}; r, s, q, A, B, m, k, l]\, f(t)\, dt \qquad \text{...(1.2)}$$

where $\mathrm{Re}(\mu) > 0$ and $S_n^{\alpha,\beta,\tau}[z]$ stands for the generalized polynomial set due to Raizada [6, p.64, Eq. (2.1.8)].

The explicit form of this generalized polynomial set ([6,p.71, Eq. (2.3.4)] : see also [1, p.56, Eq. (1.5)]) is

$$S_n^{\alpha,\beta,\tau}[x; r, s, q, A, B, m, k, l] = B^{qn} x^{l(m+n)} (l - \tau x^r)^{sn}$$

$$l^{m+n} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{j=0}^{m+n} \sum_{i=0}^{j} \frac{(-1)^j (-j)_i (\alpha)_j (-v)_{u)} (-\alpha-qn)_i}{i!\, j!\, u!\, v!\, (1-\alpha-j)_i}$$

$$\left(-\frac{\beta}{\tau} - sn\right)_v \left(\frac{i+k+ru}{l}\right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r}\right)^v \left(\frac{Ax}{B}\right)^j \qquad \text{...(1.3)}$$

In this paper we shall study the following special case of *FIO* (1.2) defined and represented in the following manner :

$${}_c R_{x:r,q;m,k,l}^{\rho;\alpha,\beta,0;1,0,\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_x^c (x-t)^{\mu-1}$$

$$S_n^{\alpha,\beta,0}[z(1-t/x)^{\rho}; r, q, 1, 0, m, k, l]\, f(t)\, dt \qquad \text{...(1.4)}$$

where $Re(\mu) > 0$ and

$$S_n^{\alpha,\beta,0}[x; r, q, 1, 0, m, k, l] = x^{qn+l(m+n)}\, l^{m+n}$$

$$\sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{(-v)_u}{v!\, u!} \left(\frac{\alpha+qn+k+ru}{l}\right)_{m+n} (\beta x^r)^v \qquad \text{...(1.5)}$$

It may be pointed out here that *FIO* (1.4) is sufficiently general in nature as it involves a generalized polnomial set which unifies and extends a large number of classical polynomials introduced and studied by various research workers such as Chatterjea [2], Gould and Hopper [3], Krall and Frink [4] and Singh and Shrivastava [8], etc.

(b) A-General class of multivariable Polynomials

Srivastava and Garg [10,p.685] introduced and studied a general class of multivariable polynomial in the following manner

$$S_n^{m_1,\ldots,m_s}[x_1, \ldots, x_s]$$

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$$= \sum_{k_1, \ldots, k_s = 0}^{m_1k_1 + \ldots + m_sk_s \leq n} (-n)_{m_1k_1 + \ldots + m_sk_s} A(n; k_1, \ldots, k_s) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_s^{k_s}}{k_s!} \quad \ldots(1.6)$$

where $m_1, \ldots, m_s$ are arbitrary positive integers and the coefficients $A(n; k_1, \ldots, k_s)$ $(n, k_i \geq 0, i = 1, \ldots, s)$ are arbitrary constant real or complex.

(c) The Multivariable *H*-function

The multivariable H-function occurring in this paper has been defined by Srivastava and Panda [13]. We shall use the following contracted notation (Srivastava et al. [12, p.251, Eq.(C.1)] :

$$H[z_1, \ldots, z_p]$$

$$= H^{O, N; M_1, N_1; \ldots; M_p, N_p}_{P, Q; P_1, Q_1; \ldots; P_p, Q_p} \left[\begin{matrix} z_1 \\ \vdots \\ z_p \end{matrix} \middle| \begin{matrix} (a_j; A_j', \ldots, A_j^{(p)})_{1, P}; \\ (b_j; B_j', \ldots, B_j^{(p)})_{1, Q}; \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j', C_j')_{1, P_1}; \ldots; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1}; \ldots; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix} \right] \quad \ldots(1.7)$$

to denote the H-function of p-complex variables here all the capital letters are assumed to be positive real numbers for the standardization purposes; the definition of multivariable H-function given by (1.7) will, however, be meaningful even if some of these quantities are zero. The details of this function can be found in paper and book referred to above.

2. MAIN RESULT

$${}_cR^{\rho; \alpha, \beta, 0; 1, 0, \mu}_{x: r, q; m, k, l} \{[(x-a)^v + \xi]^\sigma (x-a)^\lambda$$

$$S_{n'}^{m_1, \ldots, m_s} [y_1(x-a)^{\lambda_1} \{(x-a)^v + \xi\}^{-\tau_1}, \ldots, y_s(x-a)^{\lambda_s} \{(x-a)^v + \xi\}^{-\tau_s}]$$

$$H[z_1 (x-a)^{\rho_1} \{(x-a)^v + \xi\}^{-\sigma_1}, \ldots, z_p(x-a)^{\rho_p} \{(x-a)^v + \xi\}^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1, \ldots, k_s = 0}^{m_1k_1 + \ldots + m_sk_s \leq n'} \sum_{w=0}^{\infty} \sum_{\delta=0}^{\infty} \frac{(-1)^{w+\delta}}{w!\, \delta!}$$

$$\frac{B(m, n, u, v)\, D(n', k_i, y_i)\xi}{(\mu \text{ [illegible] } \delta)}$$

$$(x-a)^{\lambda + vu \text{ [illegible] } \delta + \sum_{i=1}^{s} \lambda_i k_i} (x-c)^{\mu + A + \delta}\, x^{-A}$$

$$H^{0,\,N+2\,:\,M_1,\,N_1;\ldots;M_p,\,N_p}_{P+2,\,Q+2\,:\,P_1,\,Q_1;\ldots;P_p,\,Q_p}\left[\begin{array}{c|} z_1(x-a)^{\rho_1}\xi^{-\sigma_1} \\ \vdots \\ z_p\,(x-a)^{\rho_p}\,\xi^{-\sigma_p} \end{array}\right.$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i;\ \rho_1,\ldots,\rho_p),$$

$$(b_j\,;B_j',\ldots,B_j^{(p)})_{1,\,Q'}$$

$$(1-w+\sigma-\sum_{i=1}^{s}\tau_i k_i\,;\sigma_1,\ldots,\sigma_p),\ (a_j\,;A_j',\ldots,A_j^{(p)})_{1,\,p}:$$

$$(\delta-\lambda-vw-\sum_{i=1}^{s}\lambda_i k_i\,;\rho_1,\ldots,\rho_p),\ (1+\sigma-\sum_{i=1}^{s}\tau_i k_i\,;\sigma_1,\ldots,\sigma_p):$$

$$\left.\begin{array}{l}(c_j',C_j')_{1,\,P_1};\ldots;(c_j^{(p)},C_j^{(p)})_{1,\,P_p}\\ (d_j',D_j')_{1,\,Q_1};\ldots;(d_j^{(p)},D_j^{(p)})_{1,\,Q_p}\end{array}\right] \qquad \ldots(2.1)$$

where for convienience

$$B(m,n,u,v)=\frac{(-v)_u}{v!\,u!}\left(\frac{\alpha+qn+k+ru}{l}\right)_{m+n} l^{m+n}\,\beta^v\,z^{A/\rho} \qquad \ldots(2.2)$$

$$D(n',k_i,y_i)=(-n')_{m_1k_1+\ldots+m_sk_s}\,A(n'\,;k_1,\ldots,k_s)\,\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!} \qquad \ldots(2.3)$$

$$A=qn\rho+l\rho(m+n)+rv\rho \qquad \ldots(2.4)$$

and (in addition to the appropriate convergence and existence conditions [12,p.252]), we assume that

$$\min\{v,\rho,\lambda,\sigma,\lambda_i,\tau_i,\rho_j,\sigma_j\}>0\ (i=1,\ldots,s,\ j=1,\ldots,p),$$

$$|\arg(x-a)^{\rho}/\xi|<\pi,\ c\geq a\geq 0,$$

$$\mathrm{Re}\left[\lambda+\sum_{i=1}^{s}\lambda_i k_i+\sum_{j=1}^{p}\rho_j\zeta_i\right]>-1\ (\zeta_j=\min_{1\leq k\leq m_j}[Re(d_k^{(j)}/D_k^{(j)})],$$

$\mathrm{Re}\,[\mu+qn\rho+l\rho(m+n)+rv\rho]>0\quad(v=1,\ldots,m+n)$ and the multiple series occuring on right hand side of (2.1) is absolutely convergent.

Derivation of formula (2.1):

Using the defintion of *FIO* (1.4) in (2.1), we find that

$$\text{L.H.S. of (2.1)}=\frac{1}{\Gamma(\mu)}\int_c^x(x-t)^{\mu-1}\,\{(t-a)^{\lambda}\,[(t-a)^{v}+\xi]^{\sigma}$$

$$S_n^{\alpha,\beta,0}[z(1-t/x); r, q, 1, 0, m, k, l]$$

$$S_{n'}^{m_1,\dots,m_s}[y_1(t-a)^{\lambda_1}\{(t-a)^{v}+\xi\}^{-\tau_1},\dots,y_s(t-a)^{\lambda s}\{(t-a)^{v}+\xi\}^{-\tau_s}]$$

$$H[z_1(t-a)^{\rho_1}\{(t-a)^{v}+\xi\}^{-\sigma_1},\dots,z_p(t-a)^{\rho_p}(t-a)^{v}+\xi\}^{-\sigma_p}]\}dt$$

...(2.5)

Now replacing in (2.5) the generalized polynomial set $S_n^{\alpha,\beta,0}[z(1-t/x)^{\rho}: r, q, 1, 0, m, k, l]$ the general class of multivariable polynomials,

$$S_{n'}^{m_1,\dots,m_s}[y_1(t-a)^{\lambda_1}\{(t-a)^{v}+\xi\}^{-\tau_1},\dots y_s(t-a)^{\lambda_s}\{(t-a)^{v}+\xi\}^{-\tau_s}]$$

by their series given by (1.5) and (1.6) respectively and multivariable H-function by its Mellin-Barnes contour integral given by (1.7) and interchanging the order of integeration and summation, we get

L.H.S. of (2.1)

$$=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{k_1,\dots,k_s=0}^{m_1k_1+\dots+m_sk_s\le n'} x^{-A}$$

$$B(m,n,u,v)\,D(n',k_i,y_i)\frac{1}{(2\pi i)^p}\int_{L_1}\dots,\int_{L_p}\phi(\eta_1,\dots,\eta_p)$$

$$\prod_{j=1}^{p}\{\theta_j(\eta_j)(z_j)^{\eta_j}\}\ \{\int_c^x(x-t)^{\mu+A-1}(t-a)^{\lambda+\sum_{i=1}^{s}\lambda_ik_i+\sum_{j=1}^{p}\rho_j\eta_j}$$

$$[(t-a)^{v}+\xi]^{\sigma-\sum_{i=1}^{s}\tau_ik_i-\sum_{j=1}^{p}\sigma_j\eta_j}\,dt\}\,d\eta_1,\dots,d\eta_p \qquad ...(2.6)$$

Applying the binomial expansion

$$(x+\xi)^{\lambda}=\xi^{\lambda}\sum_{w=0}^{\infty}\frac{(-1)^{w}(-\lambda)_{w}}{w!}\left(\frac{x}{\xi}\right)^{w},\left(\left|\frac{x}{\xi}\right|<1\right) \qquad ...(2.7)$$

Changing the order of integration and summation and using the well-known formula (cf,[7,p.22]) :

$$\int_c^x(x-t)^{\mu-1}(t-a)^{\lambda}dt$$

$$=(x-c)^{\mu}(x-a)^{\lambda}\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}\Gamma(\lambda+1)}{(\mu+\lambda)\,\delta!\,\Gamma(\lambda-\delta+1)}\left(\frac{x-c}{x-a}\right)^{\delta} \qquad ...(2.8)$$

where Re $(\mu)>0$, Re$(\lambda)>-1$ and $c\ge a\ge 0$, we obtain

$$\text{L.H.S. of } (2.1) = \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s \le n'} \sum_{w=0}^{\infty} \sum_{\delta=0}^{\infty}$$

$$B(m,n,u,v)\, D(n',k_i,y_i)\, \frac{(-1)^{w+\delta}}{w!\,\delta!}\, \xi^{\sigma - w - \sum_{i=1}^{s} \tau_i k_i}$$

$$x^{-A} (\mu + A + \delta)^{-1} (x-c)^{\mu+\delta+A} (x-a)^{\lambda + vw - \delta + \sum_{i=1}^{s} \lambda_i k_i}$$

$$\left(\frac{1}{2\pi i}\right)^{p} \int_{L_1} \ldots \int_{L_p} \frac{\Gamma\left(w - \sigma + \sum_{i=1}^{s} \tau_i k_i + \sum_{j=1}^{p} \sigma_j \eta_j\right)}{\Gamma\left(-\sigma + \sum_{i=1}^{s} \tau_i k_i + \sum_{j=1}^{p} \sigma_j \eta_j\right)}$$

$$\frac{\Gamma\left(1 + \lambda + vw + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \eta_j\right)}{\Gamma\left(1 - \delta + \lambda + vw + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \eta_j\right)} \phi(\eta_1,\ldots,\eta_p)$$

$$\left[\prod_{j=1}^{p} \theta_j(\eta_j) z_j^{\eta_j}\right] \xi^{-\sum_{j=1}^{p} \sigma_j \eta_j} (x-a)^{\sum_{j=1}^{p} \rho_j \eta_j} ; d\eta_1,\ldots,d\eta_p \qquad \ldots(2.9)$$

Interpreting the multiple Mellin-Barnes contour integral as an *H*-function of p-variables, we led finally to the fractional integral formula (2.1) under the (sufficient) conditions of validity stated already with (2.1).

3. SPECIAL CASES

The importance of our generalized fractional integral formula lies in its manifold generality. First of all our ***FIO*** (1.4) involves the generalized polynomial set which can be reduced to a large number of polynomials (see e.g.[6]). Secondly, the general class of multi-variable polynomials involved in (2.1) can be reduced to a large spectrum of polynomials of one and several variables. Thirdly, the formula (2.1) can be suitably applied to derive the corresponding results involving a remarkably wide variety of potentially useful functions (or product of several such functions) which are expressible in terms of the ***E, G*** and ***H*** functions of one, two or more variables. Our main result thus provides unification and extension of various (known or new) results on *FIO*. For the sake of illustration, we mention the following interesting special cases of (2.1):

(i) Taking $a = c$ in (2.1) and using the well-known Gauss summation theorem to sum the δ-series, we get after a little simplification

$$
{}_{c}R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\ \{(x-c)^{\lambda}\,[(x-c)^{\upsilon}+\xi]^{\sigma}
$$

$$
S^{m_1,\ldots,m_s}_{n'}\,[y_1(x-c)^{\lambda_1}\,\{(x-c)^{\upsilon}+\xi\}^{-\tau_1},\ldots,y_s(x-c)^{\lambda_s}\,\{(x-c)^{\upsilon}+\xi\}^{-\tau_s}]
$$

$$
H[z_1(x-c)^{\rho_1}\,\{(x-c)^{\upsilon}+\xi\}^{-\sigma_1},\ldots,z_p\,(x-c)^{\rho_p}\,\{(x-c)^{\upsilon}+\xi\}^{-\sigma_p}]\}
$$

$$
=\frac{1}{\Gamma(\mu)}\sum_{\upsilon=0}^{m+n}\ \sum_{u=0}^{\upsilon}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'}\ \sum_{w=0}^{\infty}\frac{(-1)^{w}}{w!}\,B(m,n,u,\upsilon)
$$

$$
D(n',k_i,y_i)\,\Sigma\,(\mu+A)x^{-A}\,\xi^{\sigma-w-\sum\limits_{i=1}^{s}\tau_ik_i}\,(x-c)^{\mu+\lambda+A+\upsilon w+\sum\limits_{i=1}^{s}\lambda_ik_i}
$$

$$
H^{0,\,N+2\,:\,M_1,\,N_1\,;\ldots;\,M_p,\,N_p}_{P+2,\,Q+2\,:\,P_1,\,Q_1\,;\ldots;\,P_p,\,Q_p}\left[\begin{array}{l} z_1\,(x-c)^{\rho_1}\,\xi^{-\sigma_1}\\ \vdots \\ z_p\,(x-c)^{\rho_p}\,\xi^{-\sigma_p}\end{array}\right|
$$

$$
(-\lambda-\upsilon w-\sum_{i=1}^{s}\lambda_ik_i\,;\,\rho_1,\ldots,\rho_p),
$$

$$
(b_j\,;\,B'_j,\ldots,B^{(p)}_j)_{1,\,Q}
$$

$$
(1-w+\sigma-\sum_{i=1}^{s}\tau_ik_i\,;\,\sigma_1,\ldots,\sigma_p),\,(a_j\,;\,A'_j,\ldots,A^{(p)}_j)_{1,\,p}\,:
$$

$$
(-\mu-A-\lambda-\upsilon w-\sum_{i=1}^{s}\lambda_ik_i\,;\,\rho_1,\ldots,\rho_p),\,(1+\sigma-\sum_{i=1}^{s}\tau_ik_i\,;\,\sigma_1,\ldots,\sigma_p)
$$

$$
\left.\begin{array}{l}(c'_j,C'_j)_{1,\,P_1}\,;\ldots;\,(c^{(p)}_j,C^{(p)}_j)_{1,\,P_p}\\ (d'_j,D'_j)_{1,\,Q_1}\,;\ldots;\,(d^{(p)}_j,D^{(p)}_j)_{1,\,Q_p}\end{array}\right] \qquad \ldots(3.1)
$$

(*ii*) **Further taking $a = 0$ in (2.1), we get**

$$
{}_{c}R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}
$$

$$
\{x^{\lambda}\,(x^{\upsilon}+\xi)^{\sigma}\,S^{m_1,\ldots,m_s}_{n'}\,[y_1x^{\lambda_1}\,(x^{\upsilon}+\xi)^{-\tau_1},\ldots,y_s\,x^{\lambda_s}\,(x^{\upsilon}+\xi)^{-\tau_s}]
$$

$$
H[z_1\,x^{\rho_1}\,(x^{\upsilon}+\xi)^{-\sigma_1},\ldots,z_p\,x^{\rho_p}\,(x^{\upsilon}+\xi)^{-\sigma_p}]\}
$$

$$
=\frac{1}{\Gamma(\mu)}\sum_{\upsilon=0}^{m+n}\ \sum_{u=0}^{\upsilon}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'}\ \sum_{w=0}^{\infty}\ \sum_{\delta=0}^{\infty}\frac{(-1)^{w+\delta}}{w!\,\delta!\,(\mu+A+\delta)}
$$

$$B(m, n, u, v)\, D(n', k_i, y_i)\, \xi^{\sigma - \sum_{i=1}^{s} \tau_i k_i - w} (x - c)^{\mu + A + \delta})\, x^{\lambda + vw - \delta - A + \sum_{i=1}^{s} \tau_i k_i}$$

$$H^{0, N+2 : M_1, N_1 ;\dots; M_p, N_p}_{P+2, Q+2 : P_1, Q_1 ;\dots; P_p, Q_p} \left[\begin{array}{l|} z_1\, x^{\rho_1}\, \xi^{-\sigma_1} \\ : \\ z_p\, x^{\rho_p}\, \xi^{-\sigma_p} \end{array} \right.$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, \dots, \rho_p),$$

$$(b_j ; B_j', \dots, B_j^{(p)})_{1, Q},$$

$$(1 - w + \sigma - \sum_{i=1}^{s} \tau_i k_i ;\ \sigma_1, \dots, \sigma_p), (a_j ; A_j', \dots, A_j^{(p)})_{1, p} :$$

$$(\delta - \lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, \dots, \rho_p), (1 + \sigma - \sum_{i=1}^{s} \tau_i k_i ;\ \sigma_1, \dots, \sigma_p);$$

$$\left. \begin{array}{l} (c_j', C_j')_{1, P_1} ;\dots; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1} ;\dots; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{array} \right] \qquad \dots(3.2)$$

(*iii*) Again taking $c = 0$ in (3.1), we find that

$$R^{\rho ; \alpha, \beta, 0 ; 1, 0, \mu}_{x ; r, q ; m, k, l} \{x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$S^{m_1, \dots, m_s}_{n'} [y_1\, x^{\lambda_1} (x^{v} + \xi)^{-\tau_1}, \dots, y_s x^{\lambda s} (x^{v} + \xi)^{-\tau_s}]$$

$$H[z_1\, x^{\rho_1} (x^{v} + \xi)^{-\sigma_1}, \dots, z_p\, x^{\rho_p} (x^{v} + \xi)^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1, \dots, k_s = 0}^{m_1 k_1 + \dots + m_s k_s \le n'} \sum_{w=0}^{\infty} \frac{(-1)^{w}}{w!}$$

$$\xi^{\sigma - \omega - \sum_{1=1}^{\sigma} \tau_1 \kappa_1}\, x^{\mu + \lambda + v + w + \sum_{i=1}^{s} \lambda_i k_i}\, \Gamma(\mu + A)\, B(m, n, u, v)\, D(n', k_i, y_i)$$

$$H^{0, N+2 : M_1, N_1 ;\dots; M_p, N_p}_{P+2, Q+2 : P_1, Q_1 ;\dots; P_p, Q_p} \left[\begin{array}{l|} z_1\, x^{\rho_1}\, \xi^{-\sigma_1} \\ : \\ z_p\, x^{\rho_p}\, \xi^{-\sigma_p} \end{array} \right.$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ;\ \rho_1, \dots, \rho_p)$$

$$(b_j' ; B_j', \dots, B_j^{(p)})_{1, Q},$$

$$(1 - w + \sigma - \sum_{i=1}^{s} \tau_i k_i ; \sigma_1 ,\ldots, \sigma_p), (a_j ; A'_j ,\ldots, A_j^{(p)})_{1, P} :$$

$$(- \mu - A - \lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1 ,\ldots, \rho_p), (1 + \sigma - \sum_{i=1}^{s} \tau_i k_i ; \sigma_1,\ldots,\sigma_p):$$

$$\left.\begin{matrix} (c'_j, C'_j)_{1, P_1} ;\ldots; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d'_j, D'_j)_{1, Q_1} ;\ldots; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix}\right] \qquad \ldots(3.3)$$

(iv) If we take $n' = 0$ and $\lambda_i, T_i \to 0$ $(i = l, \ldots, s)$ in (3.3), we get

$$R_{x; r, q; m, k, l}^{\rho; \alpha, \beta, 0; 1, 0, \mu} \{ x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$H[z_1 x^{\rho_1} (x^{v} + \xi)^{-\sigma_1} ,\ldots, z_p x^{\rho_p} (x^{v} + \xi)^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{w=0}^{\infty} \frac{(-1)^{w}}{w!} B(m, n, u, v)\, \xi^{\sigma - \omega} x^{\mu + \lambda + vw} \Gamma(\mu + A)$$

$$H_{P+2, Q+2, : P_1, Q_1 ;\ldots; P_p, Q_p}^{0, N+2 : M_1, N_1 ;\ldots; M_p, N_p} \left[\begin{matrix} z_1 x^{\rho_1} \xi^{-\sigma_1} \\ : \\ z^{p} x^{\rho_p} \xi^{-\sigma_p} \end{matrix} \right|$$

$$(- \lambda - vw; \rho_1,\ldots, \rho_p), (1 - w + \sigma; \sigma_1,\ldots,\sigma_p), (a_j; A'_j,\ldots A_j^{(p)})_{1, P} :$$

$$(b_j, B'_j,\ldots, B_j^{(p)})_{1, Q}, (- \mu - A - \lambda - vw; \rho_1,\ldots,\rho_p), (1 + \sigma; \sigma_1, \ldots, \sigma_p) :$$

$$\left.\begin{matrix} (c'_j, C'_j)_{1, P_1} ;\ldots; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d'_j, D'_j)_{1, Q_1} ;\ldots; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{matrix}\right] \qquad \ldots(3.4)$$

Further, if we put $m = n = \alpha = k = 0$ in (3.4), we arrive at a recent result due to Srivastava et al. [9,p.563, Eq.(2.1)], which contains another known result due to Srivastava and Goyal [11,p.644, Eq.(2.1)] as a special case.

The conditions of validity of the fractional integral formula (3.1) through (3.4) are obtainable fairly easily from those of their parent formula (2.1).

(v) Now, appealing to the known relationship (c.f. [13, p.272, Eq.(4.7)], see also [12,p.253, Eq.(C.9)]), it is not difficult to derive the following fractional integral formula as a special case of (3.4):

$$R_{x; r, q; m, k, l}^{\rho; \alpha, \beta, 0; 1, 0, \mu} \{x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$F^{P:P_1,\dots,P_p}_{Q:Q_1,\dots,Q_p}\left[\begin{matrix}(a_j\,;A_j',\dots,A_j^{(p)})_{1,P}:\\(a_j\,;B_j',\dots,B_j^{(p)})_{1,Q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p};\\(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p};\end{matrix}\; z_1\,x^{\rho_1}\,(x^{v}+\xi)^{-\sigma_1},\dots,z_p x^{\rho_p}\,(x^{v}+\xi)^{-\sigma_p}\right]\Big\}$$

$$=\xi^{\sigma}\,x^{\mu+\lambda}\,\frac{\Gamma(\lambda+1)}{\Gamma(\mu)}\sum_{v=0}^{m+n}\;\sum_{u=0}^{v}\;B(m,n,u,v)\;\frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F^{P+2:P_1;\dots;P_p;0}_{Q+2:Q_1;\dots;Q_p;0}\left[\begin{matrix}(1+\lambda\,;\rho_1,\dots,\rho_p,v),(-\sigma\,;\sigma_1,\dots,\sigma_p,1),\\(1+\lambda+\mu+A;\rho_1,\dots,\rho_p,v),(-\sigma\,;\sigma_1,\dots,\sigma_p,0),\end{matrix}\right.$$

$$\begin{matrix}(a_j\,;A_j',\dots,A_j^{(p)})_{1,P}:(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p}:\\(b_j\,;B_j',\dots,B_j^{(p)})_{1,Q}:(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p}:\end{matrix}$$

$$\left. z_1\,x^{\rho_1}\,\xi^{-\sigma_1},\dots,z_p x^{\rho_p}\,\xi^{-\sigma_p},-\frac{x^{v}}{\xi}\right] \qquad \dots(3.5)$$

(*vi*) Further, putting $\sigma=0,\ v=\rho_1=\dots=\rho_p=1$ and $\sigma_i\to 0$ $(i=1,\dots,p)$ in (3.5), we get

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\left[x^{\lambda}F^{P:P_1;\dots;P_p}_{Q:Q_1;\dots;Q_p}\left|\begin{matrix}(a_j\,;A_j',\dots,A_j^{(p)})_{1,P}:\\(b_j;B_j',\dots,B_j^{(p)})_{1,Q}:\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p};\\(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p};\end{matrix}\;z_1\,x,\dots,z_p x\right]$$

$$=\frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)}\sum_{v=0}^{m+n}\;\sum_{u=0}^{v}\;B(m,n,u,v)\,\frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F^{P+1:P_1;\dots;P_p}_{Q+1:Q_1;\dots;Q_p}\left[\begin{matrix}(1+\lambda\,;1,\dots,1):\\(1+\lambda+\mu+A;1,\dots,1):\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p};\\(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p};\end{matrix}\;z_1x,\dots,z_px\right] \qquad \dots(3.6)$$

(*vii*) From (3.6), we obtain easily the following formulas involving Lauricella hypergeometric functions of p variables :

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\,\{x^{\lambda}\,[a_1,\dots,a_p,b_1,\dots,b_p\,;1+\lambda;z_1x,\dots,z_px]\}$$

$$= \frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} B(m,n,u,v)\, \frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F_B^{(p)}\,[a_1,\ldots,a_p, b_1,\ldots,b_p\,;\, 1+\lambda+\mu+A; z_1x,\ldots,z_px]\} \qquad \ldots(3.7)$$

$$R_{x;\,r,\,q,\,m,\,k,\,l}^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}\,\{x^{\lambda}\prod_{i=1}^{p}(1-z_ix)^{-\mu_i}\}$$

$$= \frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} B(m,n,u,v)\, \frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F_D^{(p)}\,[1+\lambda, b_1,\ldots,b_p\,;\, 1+\lambda+\mu+A; z_1x,\ldots,z_px] \qquad \ldots(3.8)$$

If in (3.8), we put $\rho = m = n = v = \alpha = k = 0$ and replace λ by $\lambda - 1$, μ by $\mu - \lambda$, we arrive at a result due to Srivastava and Goyal [11,p.649, Eq. (3.6)] which for $p = 2$ reduces to another known result due to Lavoie et al. [5,p.260].

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A SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION OF HEAT CONDUCTION IN A ROD UNDER THE ROBIN CONDITION

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ABSTRACT

The role of special functions and their applications is becoming increasingly of much significance in technical applications. Space research and nuclear reactor also give rise to several problems of application of special functions.

This paper has been designed to discuss the application of certain products involving the multivariable *H*-function and two general class of polynomials in establishing a solution of the partial differential equation

$$\frac{\partial\theta}{\partial t} = \mu \, \frac{\partial^2\theta}{\partial x^2}$$

related to a problem of heat conduction in a rod under the Robin condition. The result gives a number of known particular cases on specializing the parameters and may prove to be useful in several interesting situations appearing in the literature on mathemaitcal analysis, applied mathematics and mathematical physics.

1. INTRODUCTION. As an example of the application of the multivariable *H*-function in applied mathematics, the problem of obtaining a solution of a problem of flow of heat in a uniform rod under third kind of boundary conditions, i.e., the problem of heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends inside the medium) will be consider in this problem. If the thermal coefficient is constant and their is no source of thermal energy, then a function $u(x, t)$ representing the temperature in a rod $(0 \le x \le L)$ must satisfy the heat equation

$$\frac{\partial\theta}{\partial t} = \mu \frac{\partial^2\theta}{\partial x^2} \qquad \text{...(1.1)}$$

and the initial conditions

$$\theta(x, 0) = f(x) \qquad \text{...(1.2)}$$

$$\frac{\partial\theta}{\partial x}(0, t) - h\theta(0, t) = 0 \qquad \text{...(1.3)}$$

$$\frac{\partial\theta}{\partial x}(L, t) + h\theta(L, t) = 0;\ h > 0 \qquad \text{...(1.4)}$$

Here we shall consider

$$f(x) = \left(\sin\frac{\pi x}{L}\right)^{\sigma - 1} S_n^m\left[y\left(\sin\frac{\pi x}{L}\right)^{2\rho}\right] S_{n'}^{m'}\left[y'\left(\sin\frac{\pi x}{L}\right)^{2\rho'}\right]$$

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix} z_1\left(\sin\frac{\pi x}{L}\right)^{2k_1} \\ \vdots \\ z_r\left(\sin\frac{\pi x}{L}\right)^{2k_r}\end{matrix}\right) \qquad \text{...(1.5)}$$

(For the multivariable H-function and a general class of polynomials, see [7] and [6], respectively).

The expression

$$e^{-\mu E_q^2 t}(m \cos E_q x + n \sin E_q x) \qquad \text{...(1.6)}$$

satisfies the equation (1.1), Equation (1.6) also satisfies (1.3) and (1.4) on the condition that

$$E_q n - hm = 0 \qquad \text{...(1.7)}$$

and $E_q(n \cos E_q L - m \sin E_q L) + h(m \cos E_q L + n \sin E_q L) = 0$ (1.8)

From (1.6) and (1.7), we get

$$A/B = Eq/h \qquad \text{...(1.9)}$$

and $\tan E_q L = 2E_q h/(E_q^2 - h^2)$ ...(1.10)

where E_q is the qth positive root of (1.14). Then the solution of our problem takes the following elegant from

$$\theta(x, t) = \sum_{q=1}^{\infty} R_q \left(\cos E_q x + \frac{h}{E_q} \sin E_q x\right) e^{-\mu E_q^2 t} \qquad \text{...(1.11)}$$

The following results will be required :

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \ \sin \frac{E_p \pi x}{L}\, dx$$

$$= L.2^{1-\sigma} \sin \frac{E_p \pi}{2} \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma \pm E_p + 1}{2})}, R(\sigma) > 0 \quad ...(1.12)$$

$$\int_0^l (\sin \frac{\pi x}{L})^{\sigma - 1} \cos \frac{E_p \pi x}{L}\, dx$$

$$= L.2^{1-\sigma} \cos \frac{E_p \pi}{2} \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma \pm E_p + 1}{2})}, R(\sigma) > 0 \quad ...(1.13)$$

$$\int_0^L (\cos E_q x + \frac{h}{E_q} \sin E_q x)\,(\cos E_p x + h_l\, E_p \sin E_p x)\, dx$$

$$= \begin{cases} 2(E_q^2)^{-1}\,[(E_q^2 + h^2)L + 2h], & p = q \\ 0, & p \neq q \end{cases} \quad ...(1.14)$$

where E_q is positive root of the non-algebraic equation

$$\tan EL = \frac{2hE}{(E^2 - h^2)} \quad ...(1.15)$$

We shall require the following definition (cf. Srivastava [6], p.1, eqn.(1.1)]

$$S_n^m\,[x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!}\, F_{n,s}\, x^s, n = 0, 1, 2, \ldots \quad ...(1.16)$$

where m is arbitrary positive integer and the coefficients $F_{n,s}\,(n, s \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $F_{n,s}$ the polynomials $S_n^m\,[x]$ can be reduced to the well-known classical orthogonal polynomials such as Jacobi, Hermite, Legendre, Tchebycheff and Laguerre polynomials, etc.

2. THE MAIN INTEGRAL FORMULAE

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} S_n^m \left[\, y\,(\sin \frac{\pi x}{L})^{2\rho} \,\right]$$

$$S_{n'}^{m'}\left[y' \left(\sin \frac{x\pi}{L}\right)^{2\rho'} \right] H^{0,\, \lambda\, :\, (u',\, v')\, ;\ldots;\, (u^{(r)},\, v^{(r)})}_{A,\, C\, :\, [B',\, D']\, ;\ldots;\, [B^{(r)},\, D^{(r)}]} \begin{pmatrix} z_1 \left(\sin \frac{\pi x}{L}\right)^{2k_1} \\ \vdots \\ z_r \left(\sin \frac{\pi x}{L}\right)^{2k_r} \end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'}\, y'^{s'} \sin \frac{E_p \pi}{2}$$

$$H^{0,\, \lambda+1\, :\, (u',\, v')\, ;\ldots;\, (u^{(r)},\, v^{(r)})}_{A+1,\, C+2\, :\, [B',\, D']\, ;\ldots;\, [B^{(r)},\, D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r],\ [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] \,;\ldots;\, [(b^{(r)}) : \phi^{(r)}] \,;\, z_1\, 4^{-k_1} \\ \vdots \\ [(d') : \delta'] \,;\ldots;\, [(d^{(r)}) : \delta^{(r)}] \,;\, z_r\, 4^{-k_r} \end{matrix} \right) \qquad \ldots(2.1)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\text{Re}\,(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)} / \delta_j^{(i)}) > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|\arg (z_i)| < T_i \pi/2$, $T_i > 0$, m, m' are arbitrary positive integers and the coefficients $F_{n,s}\,(n, s \ge 0)$ and $F_{n',s'}\,(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\int_0^L \left(\sin \frac{\pi x}{L}\right)^{\sigma-1} \cos \frac{E_p\, \pi x}{L}\, S_n^m \left[y \left(\sin \frac{\pi x}{L}\right)^{2\rho} \right]$$

$$S_{n'}^{m'}\left[y' \left(\sin \frac{\pi x}{L}\right)^{2\rho'} \right] H^{0,\, \lambda\, :\, (u',\, v')\, ;\ldots;\, (u^{(r)},\, v^{(r)})}_{A,\, C\, :\, [B',\, D']\, ;\ldots;\, [B^{(r)},\, D^{(r)}]} \begin{pmatrix} z_1 \left(\sin \frac{\pi x}{l}\right)^{2k_1} \\ \vdots \\ z_r \left(\sin \frac{\pi x}{l}\right)^{2k_r} \end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'}\, y'^{s'} \cos \frac{E_p \pi}{2}$$

$$H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}
\left(\begin{array}{l}
[1-\sigma-2\rho s-2\rho' s' : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}] : \\
[(c):\psi',\ldots,\psi^{(r)}],\left[\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1,\ldots,k_r\right] : \\
\qquad [(b'):\phi'] \,;\ldots;\, [(b^{(r)}):\phi^{(r)}] \,;\ z_1\, 4^{-k_1} \\
\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\qquad [(d'):\delta'] \,;\ldots;\, [(d^{(r)}):\delta^{(r)}] \,;\ z_r\, 4^{-k_r}
\end{array}\right) \quad \ldots(2.2)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\mathrm{Re}\,(\sigma + \sum_{i=1}^{r} k_i\, d_j^{(r)}/d_j^{(i)}) > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|\arg(z_i)| < T_i\,\pi/2$, $T_i > 0$, m, m' are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex. The integral formulae in (2.1) and (2.2) can be derived by making use of the definition of the multivariable *H*-function (Srivastava and Panda [7] and Srivastava [8], p.251) and of the results in (1.12), (1.13) and (1.16) respectively.

3. SOLUTION OF THE PROBLEM The solution of the problem to be established

$$\theta(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/m]} \sum_{s'=0}^{[n'/m']} \frac{(-n)_{ms}}{s!}$$

$$F_{n,s}\, y^{s} \frac{(-n')_{m's'}}{s'!}\, F_{n',s'}\, y^{s'} \ \frac{E_q^2}{[(E_q^2+h^2)\,L+2h]}\, e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q}\sin E_q x)\,(\cos\frac{E_q\pi}{2} + \frac{h}{E_q}\sin\frac{E_q\pi}{2})$$

$$H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}
\left(\begin{array}{l}
[1-\sigma-2\rho s-2\rho' s' : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}] : \\
[(c):\psi',\ldots,\psi^{(r)}],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1,\ldots,k_r] : \\
\qquad [(b'):\phi'] \,;\ldots;\, [(b^{(r)}):\phi^{(r)}] \,;\ z_1\, 4^{-k_1} \\
\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\qquad [(d'):\delta'] \,;\ldots;\, [(d^{(r)}):\delta^{(r)}] \,;\ z_r\, 4^{-k_r}
\end{array}\right) \quad \ldots(3.1)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\mathrm{Re}\,(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)}/\delta_j^{(i)} > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|arg\,(z_i)| < T_i\,\pi/2$, $T_i > 0$, m, m' are arbitrary positive integers and the coefficients $F_{n,s}\,(n, s \ge 0)$ and $F_{n',s'}\,(n', s' > 0)$ are arbitrary constants, real or complex. Derivation of (3.1).

If $t = 0$ then from (1.5) and (1.11), we have

$$(\sin \frac{\pi x}{L})^{\sigma - 1}\, S_n^m \left[y\,(\sin \frac{\pi x}{L})^{2\rho} \right] S_{n'}^{m'} \left[y'\,(\sin \frac{\pi x}{L})^{2\rho'} \right]$$

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]} \begin{pmatrix} z_1\,(\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,(\sin \frac{\pi x}{L})^{2k_r} \end{pmatrix}$$

$$= \sum_{q=1}^{\infty} R_q\,(\cos E_q x + \frac{h}{E_q} \sin E_q x) \qquad \ldots(3.2)$$

Multiply both sides of (3.2) by

$$(\cos E_p x + \frac{h}{E_p} \sin E_p x)$$

then integrate with respect to x from 0 to L, and use (1.4), (2.1) and (2.2) to obtain

$$R_p = 2LE_p^2\,[(E_p^2 + h^2)L + 2h]^{-1}\,2^{1 - \sigma - 2\rho s - 2\rho' s'}$$

$$\sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!}\,F_{n,s}\,y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}\,F_{n',s'}\,y'^{s'}$$

$$(\cos \frac{E_p \pi}{2} + \frac{h}{E_p} \sin \frac{E_p \pi}{2})\, H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k_1, \ldots, 2k_r],\ [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)},\ [\frac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi']\,;\ldots;\,[(b^{(r)}) : \phi^{(r)}]\,;\, z_1\,4^{-k_1} \\ \vdots \\ [(d') : \delta']\,;\ldots;\,[(d^{(r)}) : \delta^{(r)}]\,;\, z_r\,4^{-k_r} \end{matrix} \right) \qquad \ldots(3.3)$$

Substitute the values of R_p from (3.3) in (3.2), the desired solution (3.1) is established.

4. SPECIAL CASES AND APPLICATIONS

(*i*) For $r = 1$, the results in (2.1), (2.2) and (3.1) reduce to the following results involving Fox's H-function [5]

$$\int_0^L (\sin\frac{\pi x}{L})^{\sigma-1} \sin\frac{E_p \pi x}{L}\, S_n^m\left[y\,(\sin\frac{\pi x}{L})^{2\rho} \right]$$

$$.\, S_{n'}^{m'}\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'} \right] H_{B,D}^{u,v}\left[z\,(\sin\frac{\pi x}{L})^{2k} \left| \begin{matrix} [(b):\phi] \\ [(d):\delta] \end{matrix} \right. \right] dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \sin\frac{E_p\pi}{2}\, H_{B+1,D+2}^{u,v+1}\left[z\,4^{-k} \left| \begin{matrix} [1-\sigma-2\rho s-2\rho's' : 2k],\ [(b):\phi] \\ [(d):\delta],\ [\frac{1-\sigma-2\rho s-2\rho's' \pm E_p}{2} : k] \end{matrix} \right. \right] \quad ...(4.1)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, Re $(\sigma + kd_j/\delta_j > 0$, $1 \le j \le u$, m and m' are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$(T = \sum_{1}^{v} \phi_i - \sum_{v+1}^{B} \phi_i + \sum_{1}^{u} \delta_i - \sum_{u+1}^{D} \delta_i)$$

$$\int_0^L (\sin\frac{\pi x}{L})^{\sigma-1} \cos\frac{E_p \pi x}{L}\, S_n^m\left[y \sin\frac{\pi x}{L})^{2\rho} \right]$$

$$.\, S_{n'}^{m'}\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'} \right] H_{B,D}^{u,v}\left[z\,(\sin\frac{\pi x}{L})^{2k} \left| \begin{matrix} [(b):\phi] \\ [(d):\delta] \end{matrix} \right. \right] dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!}\, F_{n,s}\, y^s\, F_{n,s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \cos\frac{E_p\pi}{2}\, H_{B+1,D+2}^{u,v+1}\left[z4^{-k} \left| \begin{matrix} [1-\sigma-2\rho s-2\rho's' : 2k],\ [(b):\phi] \\ [(d):\delta],\ [\frac{1-\sigma-2\rho s-2\rho's' \pm E_p}{2} : k] \end{matrix} \right. \right] \quad ...(4.2)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, Re $(\sigma + k\, d_j/\delta_j > 0$, $1 < j \le u$,

$|\arg(z)| < T, \pi/2$, $T > 0$, m and m' are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\theta(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho's'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'}\, y'^{s'} \quad \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q} \sin E_q x)(\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})$$

$$H^{u,\,v+1}_{B+1,\,D+2}\left[z4^{-k} \,\middle|\, \begin{matrix} [1-\sigma-2\rho s-2\rho's' : 2k],\ [(b):\phi] \\ [(d):\delta]\ [\dfrac{1-\sigma-2\rho s-2\rho's' \pm E_q}{2} : k] \end{matrix} \right] \quad \ldots(4.3)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, $\mathrm{Re}\,(\sigma + kd_j/\delta_j) > 0$, $1 \le j \le u$, $|\arg(z)| < T\pi/2$, $T > 0$, m and m' are arbitrary positive integer and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants real or complex.

(*ii*) Letting $n \to 0$, $n' \to 0$ and $y = y' = 1$ the results in (2.1), (2.2), (3.1), (4.1), (4.2) and (4.3) follow as particular cases of results recently obtained by Chaurasia ([1], eqns. (16) and (17), p.79, eqn. (18), p.80, eqn. (21), p.81 and eqns. (22) and (23), p.82).

(*iii*) Taking $n' \to 0$ $y = y' = 1$ the results in (2.1), (2.2), (3.1), (4.1), (4.2) and (4.3) follow as particular cases of the results recently obtained by Chaurasia and Girdhari Lal in [2].

(*iv*) By applying our results derived in (2.1), (2.2) and (2.3) to the case of Hermite polynomials ([10], p.106, eqn. (5.5.4) and [9], p.158) by setting

$$S_n^2\,[x] \to x^{n/2} H_n\,(\frac{1}{2\sqrt{x}})$$

in which case $m = 2$, $F_{n,s} = (-1)^s$ and also letting $m' = 2$, $F_{n',s'} = (-1)^{s'}$, we have the following interesting consequences of the main result :

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma-1} \sin \frac{E_p \pi x}{L} \left[y\,(\sin \frac{\pi x}{L})^{2\rho} \right]^{n/2}$$

$$.\, H_n \left[\frac{1}{2\sqrt{y\,(\sin \frac{\pi x}{L})^{2\rho}}} \right] \left[y'\,(\sin \frac{\pi x}{L})^{2\rho'} \right]^{n'/2}.$$

$$. H_{n'}\left[\frac{1}{2\sqrt{y'}\,(\sin\frac{\pi x}{L})^{2\rho'}}\right] H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1\,(\sin\frac{\pi x}{L})^{2k_1} \\ : \\ z_r\,(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'}\sum_{s=0}^{[n/2]}\frac{(-n)_{2s}}{s!}\,(-y)^s\sum_{s'=0}^{[n'/2]}\frac{(-n')_{2s'}}{s'!}$$

$$(-y')^{s'}\sin\frac{E_p\pi}{2}\,H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}] : \\ [(c):\psi',\ldots,\psi^{(r)}],\ [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1,\ldots,k_r] : \end{matrix}\right.$$

$$\left.\begin{matrix} (b'):\phi']\,;\ldots;\,[(b^{(r)}):\phi^{(\rho)}]\,;\, z_1\,4^{-k_1} \\ : \\ [(d'):\delta\,;\ldots;\,[(d^{(r)}):\delta^{(r)}];\, z_r\,4^{-k_r}\end{matrix}\right) \qquad \ldots(4.4)$$

which holds true under the same conditions as those required for (2.1)

$$\int_0^L (\sin\frac{\pi x}{L})^{\sigma-1}\cos\frac{E_p\pi x}{L}\left[y\,(\sin\frac{\pi x}{L})^{2\rho}\right]^{n/2}$$

$$.H_n\left[\frac{1}{2\sqrt{y}\,(\sin\frac{\pi x}{L})^{2\rho}}\right]\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'}\right]^{n'/2} H_{n'}\left[\frac{1}{2\sqrt{y'}\,(\sin\frac{\pi x}{L})^{2\rho'}}\right]$$

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1\,(\sin\frac{\pi x}{L})^{2k_1} \\ : \\ z_r\,(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'}\sum_{s=0}^{[n/2]}\frac{(-n)_{2s}}{s!}\,(-y)^s\sum_{s'=0}^{[n'/2]}\frac{(-n')_{2s'}}{s'!}$$

$$(-y')^{s'}\cos\frac{E_p\pi}{2}\,H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)}.\,D^{(r)}]}$$

$$\left(\begin{array}{l}[1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}], [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r] : \end{array}\right.$$

$$\left.\begin{array}{ll}(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(\rho)}] ; & z_1\, 4^{-k_1} \\ \cdot & \vdots \\ [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}]; & z_r\, 4^{-k_r}\end{array}\right) \qquad \ldots(4.5)$$

valid under some conditions as needed for (2.2)

$$\theta(x, t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2']} \frac{(-n)_{2s}}{s!} (-y)^s$$

$$\frac{(-n')_{2s'}}{s'!} -y')^{s'} \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t} (\cos E_q x + \frac{h}{E_q} \sin E_q x)$$

$$(\cos \frac{E_q t}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})\, H^{0,\, \lambda+1 : (u', v') ;\ldots; (u^{(r)}, v^{(r)})}_{A+1,\, C+2 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]}$$

$$\left(\begin{array}{l}[1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}, [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1, \ldots, k_r] : \end{array}\right.$$

$$\left.\begin{array}{ll}[(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}] ; & z_1 4^{-k_1} \\ & \vdots \\ [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}] ; & z_r\, 4^{-k_r}\end{array}\right) \qquad \ldots(4.6)$$

which holds true under the same conditions as those required for (3.1)

(*v*) Taking $\lambda = A_i$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$, $D^{(i)} = D^{(i)} + 1$, $\forall\, i = 1, 2, \ldots, r$ in (2.1), (2.2) and (3.1), we get

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma-1} \sin \frac{E_p \pi x}{L}\; S_n^m\, [(y \frac{\pi x}{L})^{2\rho}]$$

$$.S_{n'}^{m'}\left[y'(\sin \frac{\pi x}{L})^{2\rho'}\right] F^{A : B' ;\ldots; B^{(r)}}_{C : D' ;\ldots; D^{(r)}} \left(\begin{array}{l}[1-(a) : \theta', \ldots, \theta^{(r)}] : \\ [1-(c) : \psi', \ldots, \psi^{(r)}] : \end{array}\right.$$

$$\left.\begin{array}{ll}[1-(b') : \phi'] ;\ldots; [1-(b^{(r)}) : \phi^{(r)}]; & -z_1 (\sin \dfrac{\pi x}{L})^{2k_1} \\ [1-(d') : \delta'] ;\ldots; [1-(d^{(r)}) : \delta^{(r)}]; & \vdots \\ & -z_r (\sin \dfrac{\pi x}{l})^{2k_r}\end{array}\right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} \; F_{n,s}\, y^{s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \; \sin\frac{E_p\pi}{2} \; \frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\dfrac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}\right)}$$

$$.F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{array}{l}[\sigma+2\rho s+2\rho' s' : 2k_1,\ldots,2k_r],\\ [1-(c):\psi',\ldots,\psi^{(r)}],\end{array}\right.$$

$$[1-(a):\theta',\ldots,\theta^{(r)}] : [1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}];$$

$$[\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2} : k_1,\ldots,k_r]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]$$

$$\left. -z_1 4^{-k_1},\ldots,-z_r\, 4^{-k_r}\right) \qquad \ldots(4.7)$$

valid under the same conditions as obtainable from (2.1).

$$\int_0^L (\sin\frac{nx}{L})^{\sigma-1} \cos\frac{E_p\pi x}{L}\, S_n^m\left[y\,(\sin\frac{\pi x}{L})^{2\rho}\right]$$

$$S_{n'}^{m'}\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'}\right] F^{A:b';\ldots;B^{(r)}}_{C:d';\ldots;D^{(r)}}\left(\begin{array}{l}[1-(a):\theta',\ldots,\theta^{(r)}:\\ [1-(c):\psi',\ldots,\psi^{(r)}]:\end{array}\right.$$

$$\left.\begin{array}{ll}[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; & -z_1(\sin\frac{\pi x}{L})^{2k_1}\\ & \vdots \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; & -z_r(\sin\frac{\pi x}{L})^{2k_r}\end{array}\right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} \; F_{n,s}\, y^{s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \cos\frac{E_p\pi}{2} \; \frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\dfrac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}\right)}$$

$$.F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{array}{l}[\sigma+2\rho s+2\rho' s': 2k_1,\ldots,2k_r],\ [1-(a):\theta',\ldots,\theta^{(r)}]:\\ [1-(c):\psi',\ldots,\psi^{(r)}],[\dfrac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}:k_1,\ldots,k_r]:\end{array}\right.$$

$$\left.\begin{array}{ll}[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; & -z_1\, 4^{-k_1}\\ & \vdots \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; & -z_r 4^{-k_r}\end{array}\right) \qquad \ldots(4.8)$$

valid under the same conditions as need for (2.2)

$$\theta(x,t)=\frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}}\sum_{q=1}^{\infty}\sum_{s=0}^{[n/m]}\sum_{s'=0}^{[n'/m']}\frac{(-n)_{ms}}{s!}$$

$$.F_{n,s}\,y^{s}\,\frac{(-n')_{m',s'}}{s'!}\,F_{n',s'}\,y'^{s'}\,\frac{E_q^2}{[(E_q^2+h^2)L+2h]}\,e^{-\mu E_q^2 t}$$

$$(\cos E_q x+\frac{h}{E_q}\sin E_q x)\,(\cos\frac{E_q\pi}{2}+\frac{h}{E_q}\sin\frac{E_q\pi}{2})$$

$$\frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\dfrac{\sigma+2\rho s+2\rho' s'\pm E_q+1}{2}\right)}\,\frac{\prod_{j=1}^{A}\Gamma(1-a_j)\prod_{j=1}^{B'}\Gamma(1-b_j')\ldots\prod_{j=1}^{B^{(r)}}\Gamma(1-b_j^{(r)})}{\prod_{j=1}^{C}\Gamma(1-c_j)\prod_{j=1}^{D'}\Gamma(1-d_j')\ldots\prod_{j=1}^{D^{(r)}}\Gamma(1-d_j^{(r)})}$$

$$.\,F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{matrix}[\sigma+2\rho s+2\rho' s':2k_1,\ldots,2k_r],\ [1-(a):\theta',\ldots,\theta^{(r)}]: \\ [1-(c):\psi',\ldots,\psi^{(r)}],\left[\dfrac{\sigma+2\rho s+2\rho' s'\pm E_q+1}{2}:k_1,\ldots,k_r\right]\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\ldots;[1-b^{(r)}:\phi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-d^{(r)}:\delta^{(r)}];\end{matrix}\begin{matrix}-z_1\,4^{-k_1} \\ \vdots \\ -z_r\,4^{-k_r}\end{matrix}\right) \qquad \ldots(4.9)$$

valid under the same conditions as obtainable from (3.1).

(*vi*) For the Laguerre polynomials ([10]; p.101, eqn. 1.5, 1.6) and ([9], p.158) setting $S_n^1(x)\rightarrow L_n^{(\alpha')}(x)$,

in which case $m=1$ and $F_{n,s}=\binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_s}$

and also taking $m'=1$ and $F_{n',s'}=\binom{n'+\alpha''}{n'}\frac{1}{(\alpha''+1)_{s'}}$

the results in (2.1), (2.2) and (3.1) reduce to the following formulae :

$$.\int_0^L(\sin\frac{\pi x}{L})^{\sigma-1}L_n^{\alpha'}\left[y(\sin\frac{\pi x}{L})^{2\rho}\right]L_{n'}^{\alpha''}\left[y'(\sin\frac{\pi x}{L})^{2\rho'}\right]$$

$$.\sin\frac{E_p\pi x}{L}\;H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\begin{pmatrix}z_1(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix}dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^{s}}{s!} \sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'}$$

$$\cdot \frac{(-y')^{s'}}{s'!} \sin \frac{E_p \pi}{2}\; H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D'];\ldots;[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r],\ [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(\rho)}],\ \left[\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r \right] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'];\ldots;[(b^{(r)}) : \phi^{(r)}];\ z_1\, 4^{-k_1} \\ \vdots \\ [(d') : \delta'];\ldots;[(d^{(r)}) : \delta^{(r)}];\ z_r 4^{-k_r} \end{matrix} \right) \qquad \ldots(4.10)$$

valid under the same conditions as needed for (2.1)

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma-1} \cos \frac{E_p \pi x}{L}\, L_n^{\alpha'} \left[y\, (\sin \frac{\pi x}{L})^{2\rho} \right] L_{n'}^{\alpha''} \left[y'\, (\sin \frac{\pi x}{L})^{2\rho'} \right]$$

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]} \left(\begin{matrix} z_1\, (\sin \dfrac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r (\sin \dfrac{\pi x}{L})^{2k_r} \end{matrix} \right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y')^{s}}{s!} \sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!} \cos \frac{E_p \pi}{2}$$

$$H^{0,\,\lambda+1\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2:\,[B',\,D'];\,\ldots\,;\,(B^{(r)},\,D^{(r)})]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r],\ [(a) : \theta', \ldots, \theta^{(r)}]: \\ [(c) : \psi', \ldots, \psi^{(r)}],\ [\, \dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, \ldots, k_r]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi']\,;\ldots;\, [(b^{(r)}) : \phi^{(r)}]\,;\ z_1\, 4^{-k_1} \\ \vdots \\ [(d') : \delta']\,;\ldots;\, [(d^{(r)}) : \delta^{(r)}];\ z_r\, 4^{-k_r} \end{matrix} \right) \qquad \ldots(4.11)$$

valid under the same conditions as obtainable from (2.2)

$$\theta\,(x,\,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \cdot \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^{s}}{s!}$$

$$\sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!} \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q} \sin E_q x) (\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})$$

$$H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}], [\frac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1, \ldots, k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ;\ldots; [b^{(r)} : \phi^{(r)}] ; z_1 4^{-k_1} \\ \vdots \\ [(d') : \delta'] ;\ldots; [d^{(r)} : \delta^{(r)}] ; z_r 4^{-k_r} \end{matrix} \right) \qquad \ldots(4.12)$$

valid under the same conditions as obtainable from (3.1).

The importance of our resutls lies in its manifold generality. In view of the generality of the multivariable H-function, on specializing the various parameters and variables, from our results, several integrals and solutions involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables.

Secondly, on suitably specializing the coefficients $F_{n,\,s}$ and also the coefficients $F_{n',\,s'}$ and making a free use of the special cases of $S_n^m\,[x]$ listed by Srivastava and Singh [9], our results can be reduced to a large number of integerals and the solution of the problem involving the products of generalized Hermite polynomials, Jacobi polynomials, Brafman polynomials, Gould-Hopper polynomials and their various combinations. Thus, the result derived in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occuring in the problems of mathematical analysis, applied mathematics and mathematical physics.

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EXISTENCE THEOREMS FOR COMMON SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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ABSTRACT

The aim of the present paper is to establish the stability of the common solution of two differential equations in Banach spaces via fixed point technique.

1. INTRODUCTION. At present there is an abundance of literature available on the stability of differential equations with the different notions and with the different approaches. See for example. Hirsch and Smale [2], Lakshmikantham and Leela [3] and Lasalle [4]. The stability of differential equations via the fixed point technique is well-known in the literature. Recently the present author proved a common fixed point theorem for a pair of mappings in Banach spaces and which has been applied to prove the common solution of two differential equations in Banach spaces [1]. In this paper the simultaneous stability of two differential equations in Banach spaces is established. Before proving the main resutls of this paper, we first prove a fixed point theorem which will be used in the sequel. We need the following preliminaries.

Let E denote a Banach space. A subset K of E is said to be a cone in E if (i) $k_r + k \leq K$ (ii) $\lambda k \subseteq K$, whenever $\lambda \geq 0$, and (iii) $K \cap -K = \{0\}$. The relation $x \leq y$ if and only if $y - x \in K$, defines a partial ordering in E.

Definition 1.1 : Two mappings A and B on a Banach space into itself with a order relation $\leq$ induced by the closed cone K in X, are said to be **isotone increasing** if $A(x) \leq B(A(x))$ and $B(x) \leq A(B(x))$ for all $x \in X$. If the reverse inequalities are satisfied, then A and B are called weakly **isotone decreasing** on X. Finally, A and B are called **weakly isotone** on X if they are either **weakly isotone** increasing or decreasing on X.

Theorem 1.1 : Let S be a non-empty, closed, convex and bounded subset of a Banach space X with an order relation $\leq$ **induced by the** closed cone K in X and let A, $B : S \to S$ be two completely continuous and weakly isotone mappings. Then A and B have a common fixed point.

Proof : Let S be arbitrary and consider the sequence $\{x_n\}$ in S defined by

$$x_0 = x,\ x_{2n+1} = Ax_{2n},\ x_{2n+2} = BX_{2n+1} \qquad \ldots(1.1)$$

for $n = 0, 1, 2, \ldots$.

Suppose that A and B are weakly isotone increasing on S. Then by nature of the sequence $\{x_n\}$, we get

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \qquad \ldots(1.2)$$

Since A and B are completely continuous $A(S)$ and $B(S)$ are relatively compact and hence $\overline{A(S)}$ and $\overline{B(S)}$ are compact. As $\{x_n\} \subseteq \{x_0\} \cup \overline{A(S)} \cup \overline{B(S)}$ and $\{x_n\}$ is monotone increasing, it has a unique limit point x^* in S such that $\lim\limits_{n \to \infty} x_n = x^*$, and every subsequence of $\{x_n\}$ converges to the same limit point. By continuity of A and B, we obtain

$$x^* = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} A(x_{2n}) = A\,(\lim_{n \to \infty} x_{2n}) = Ax^*$$

$$x^* = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} B(x_{2n+1}) = B\,(\lim_{n \to \infty} x_{2n+1}) = Bx^*$$

Thus x^* is a common fixed point of A and B. Similarly, if A and B are weakly isotone decreasing on S, then it can be shown that the sequence $\{x_n\}$ in S defined by (1.1) is monotone decreasing and converges to the unique limit point which is again a common fixed point of A and B. This completes the proof.

Corollary 1.1 : Let S be a non-empty, closed, convex and bounded subset of a Banach space X with an order relation $\leq$ induced by the closed cone K in X and let $A, B : S \to S$ be two weakly isotone mappings. If A is continuous and B is a completely continuous mapping on S, then A and B have a common fixed point.

Proof : Let $x \in S$ be arbitrary and consider a sequence $\{x_n\}$ in S defined by (1.1).

Now suppose that A and B are weakly isotone increasing on S. Then by nature of the sequence, we get

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \qquad \ldots(1.3)$$

Since A is continuous and B is completely continuous, AoB is completely continuous on S. Therefore it follows that $AB(S)$ is relatively compact and hence $\overline{AB(S)}$ and $\overline{B(S)}$ are compact in X. As $\{x_n\} \subseteq \{x_0, x_1\} \cup \overline{B(S)} \cup \overline{AB(S)}$, in view of (1.3) it follows that the sequence $\{x_n\}$ converges to a unique limit point. The rest of the proof is similar to Theorem 1.1. The proof is complete.

Corollary 1.2 : Let S be a non-empty, closed, convex and bounded subset of a Banach space X with an order relation $\le$ induced by the closed cone K on X and let $A, B : S \to S$ be two weakly isotone mappings. Further, if A is a contraction and B is completely continuous, then A and B have a unique common fixed point.

Proof : Since every contraction mapping is continuous, A is continuous on S. By corollary 1.2, the mapping A and B have a common fixed point. But by the Banach contraction mapping principle, it follows that A has a unique fixed point. So A and B have a unique common fixed point. This completes the proof.

In the following section, we prove the simultaneous stability of two differential equations, particularly initial value problems in Banach spaces, by an applications of the results proved in this section.

2. STABILITY OF DIFFERENTIAL EQUATIONS

Let R denote the real line and $J = [t_0, t_0 + a] \subset R$, for some t_0, $a \in R$, $a > 0$ a closed and bounded interval. Let E denote a Banach space with norm $\| . \|_E$ and an order relation $\le$ induced by a closed cone K in it. Now consider the nonlinear differential equations (in short DEs) with the same initial condition (for convenience),

$$\left.\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0 \in E \end{aligned}\right\} \qquad \text{...(2.1)}$$

and

$$\left.\begin{aligned} x' &= g(t, x) \\ x(t_0) &= x_0 \in E \end{aligned}\right\} \qquad \text{...(2.2)}$$

for $t \in J$, where $f, g \in C(J \times E, E)$.

We seek the simultaneous stability of the common solution of the DEs (1.1) and (1.2) in the space $X = C(J, E)$ of all continuous E-valued functions on J. Define a norm $\| . \|$ in X by

$$\| X \| = \sup_{t \in J} \| x(t) \|_E \qquad \text{...(2.3)}$$

Clearly X is a Banach space with this supremum norm. We introduce an order relation $\le$ in X as follows. Let $\overline{K}$ denotes a cone in X defined by

$$\overline{K} = \{x \in X \mid x(t) \in K, \text{ for all } t \in J\} \qquad \text{...(2.4)}$$

Let $x, y \in X$, then by $x \le y$ we mean $y - x \in \overline{K}$, or equivalently $y(t) - x(t) \in K$ for all $t \in J$, i.e. $x(t) \le y(t)$ for all $t \in J$. Obviously $\overline{K}$ is a closed cone in X.

Definition 2.1 : The differential equations (2.1) and (2.2) are said to have a stable common solution P if for $\varepsilon > 0$, there exists an $n > 0$ such that $\|x_0\| \le n$ implies $\| p(t) \|_E \le \varepsilon$ for all $t \in J$.

We consider the following set of assumptions :

(A_1) For given $\varepsilon > 0$ there exists a $\delta > 0$, $a\delta > 1$, such that

$$\| f(t,x) - f(t,y) \|_E \leq \delta \| x - y \|_E$$

and
$$\| g(t,x) - g(t,y) \|_E \leq \delta \| x - y \|_E$$

for all $(t, x), (t, y) \in J \times E$ whenever $\| x \|_E \leq \varepsilon$, $\| y \|_E \leq \varepsilon$.

(A_2) $f(t, \theta) = \theta = g(t, \theta)$ for all $t \in J$, where θ is the zero vector of E.

(A_3)
$$f(t, x(t)) \leq g\left(t, \int_0^t f(s, x(s))\, ds + x_0 \right)$$

and
$$g(t, x(t)) \leq f\left(t, \int_{t_0}^t f(s, x(s))\, ds + x_0 \right)$$

for $(t, x) \in J \times X$ and for a fixed element $x_0 \in E$ gien in (2.1) and (2.2).

(A_4) $f(t, x)$ and $g(t, x)$ are continuous in t, uniformly for $x \in E$ with $\| x \|_E < \infty$.

(A_5) The function $f(t, x)$ satisfies the Lipschitz condition in x, uniformly for $t \in J$, i.e. there exists a constant $L > 0$ such that

$$\| f(t,x) - f(t,y) \|_E \leq L \| x - y \|_E$$

for all $(t, x), (t, y) \in J \times E$.

Theorem 2.1 : Assume $(A_1) - (A_4)$. Then the *DEs* (2.1) and (2.2) have a stable common solution on J, whenever $\| x_0 \|_E \leq b\, \varepsilon$ for some $b \in (0, 1)$.

Proof : Let $\epsilon > 0$ be given and take $\delta = \dfrac{1-b}{a}$ for some $b \in (0, 1)$, which is possible in view of $a\delta < 1$. Then by (A_1) and (A_2), we have

$$\left.\begin{aligned} &\| f(t,x) \|_E \leq \frac{1-b}{a}\, \varepsilon \\ \text{and} \quad &\| g(t,x) \|_E \leq \frac{1-b}{a}\, \varepsilon \end{aligned}\right\} \qquad \text{...(2.5)}$$

for all $(t, x) \in J \times E$.

Define a subset $\overline{S}(\varepsilon)$ of the Banach space X by

$$\overline{S}(\varepsilon) = \{x \in X / \| x \| \leq \varepsilon\} \qquad \text{...(2.6)}$$

Clearly a subset $\overline{S}$ (ε) is a closed, convex and bounded subset of the Banach space X. Now the *DEs* (2.1) and (2.2) are equivalent to the integral equations,

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds\,, \quad t \in J \qquad \text{...(2.7)}$$

and

$$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))\, ds\,, \quad t \in J \qquad \ldots(2.8)$$

respectively.

Define two operators A and B on $\overline{S}(\varepsilon)$ by right hand sides of the equations (2.7) and (2.8), respectively. Then the problem of a common solution of the *DEs* (2.1) and (2.2) is reduced to finding the common fixed points of the operators A and B. We show that the operators A and B map $\overline{S}(\varepsilon)$ into itself. Let $x \in \overline{S}(\varepsilon)$ be arbitrary then by $(A_1) - (A_2)$, we obtain

$$\| Ax(t) \|_E \le \| x_0 \|_E + \int_{t_0}^{t} \| f(s, x(s)) \|_E \; ds$$

$$\le b\varepsilon + \frac{1-b}{a}\, \varepsilon . a = \varepsilon$$

for all $t \in J$ and hence A maps $\overline{S}(\varepsilon)$ into itelf. Similarly B also maps $\overline{S}(\varepsilon)$ into itself. Also the operators A and B are completely continuous on $\overline{S}(\varepsilon)$ in view of the hypotheses $(A_1) - (A_2)$ and (A_4). Next we show that the operators A and B are weakly isotone increasing on $\overline{S}(\varepsilon)$, then by (A_3)

$$Ax(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \le x_0 + \int_{t_0}^{t} g\left(s, \int_{t_0}^{s} f(\tau, x(\tau))\, d\tau + x_0 \right) ds$$

$$= x_0 + \int_{t_0}^{t} g(s, Ax(s))\, ds$$

$$= B(Ax(t))$$

for all $t \in J$, and so $Ax \le B(Ax)$. Similarly $Bx \le A(Bx)$ for all $x \in \overline{S}(\varepsilon)$. This shows that A and B are weakly isotone increasing on $\overline{S}(\varepsilon)$. Now an application of Theorem 1.1 yields that the operators A and B have a common fixed point in $\overline{S}(\varepsilon)$. Consequently the *DEs* (2.1) and (2.2) have a common solution in $\overline{S}(\varepsilon)$. Thus there exists a common solution p of the *DEs* (2.1) and (2.2) such that $\| p(t) \|_E \le \varepsilon$ for all $t \in J$, whenever $\| x_0 \|_E \le b\varepsilon$ for some $b \in (0, 1)$. This, in view of defintion 2.1, implies that the *DEs* (2.1) and (2.2) have a stable common solution on J. This completes the proof.

Theorem 2.2 : Assume $(A_1) - (A_5)$. Further if $La < 1$ then the *DEs* (2.1) and (2.2) have a unique stable solution on J whenver $\| x_0 \|_E \le b\varepsilon$ for some $b \in (0, 1)$.

Proof : By Theorem 2.1, the *DEs* (2.1) and (2.2) have a stable solution on J. To prove the uniqueness, let u and v be two distinct stable common solutions of the *DEs* (2.1) and (2.2) on J. Then u and v satisfy the *DEs* (2.1) and (2.2) on J. Therefore, we have

$$u(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\, ds, \ t \in J$$

and
$$v(t) = x_0 + \int_{t_0}^{t} f(s, v(s))\, ds \ , t \in J$$

Then

$$\| \ u(t) - v(t) \ \|_E \ \leq \int_{t_0}^{t} \ \| \ f(s, u(s)) - f(s, v(s)) \ \|_E \ \ ds$$
$$\leq La \ \| \ u(t) - v(t) \ \|_E$$

or
$$\| \ u - v \ \| \ \leq La \ \| \ u - v \ \|,$$

which is not possible since $La < 1$. Hence $u(t) = v(t)$ for all $t \in J$. This completes the proof.

Remark 2.1 : The assumption (A_3) in Theorems 2.1 and 2.2 can be replaced by the following assumptions $(A_3)'$.

$(A_3)'$ (*i*) $f(t, x)$ and $g(t, x)$ are non-decreasing in $x \in E$ uniformly for $t \in J$;

(*ii*) $f(t, x) \leq g(t, f(t, x))$ and $g(t, x) \leq f(t, g(t, x))$ for all $(t, x) \in J \times E$

$$(iii) \quad f(t, x(t)) \leq x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

and
$$g(t, x(t) \leq x_0 + \int_{t_0}^{t} g(s, x(s))\, ds$$

for all $(t, x) \in J \times X$ and for a fixed element $x_0 \in E$ given in (2.1) and (2.2).

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STABILITY OF STRATIFIED RIVLIN-ERICKSEN FLUID IN POROUS MEDIUM IN PRESENCE OF SUSPENDED PARTICLES AND VARIABLE MAGNETIC FIELD

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ABSTRACT

The stability of stratified Rivlin-Ericksen viscoelastic fluid in stratified porous medium is considered, separately, in presence of suspended particles (fine dust) and variable horizontal magnetic field. A variable horizontal magnetic field stabilizes certain wave-number range whereas the system was unstable for all wave numbers in the absence of magnetic field, for unstable stratifications. The growth rates are found to decrease with the increase in kinematic viscosity, kinematic viscoelasticity and magnetic field whereas the growth rates increase with the increase in medium permeability. The suspended particles (fine dust) do not affect the stability or instability, as was the situation in their absence.

1. INTRODUCTION The stability of superposed fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been treated in detail by Chandrasekhar [1]. The medium has been assumed to be non-porous and the fluid to be Newtonian. The flow through a porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. A macroscopic equation which describes incompressible flow of a Newtonian fluid of viscosity μ through a macrosocopically homogeneous and isotropic porous medium of permeability k_1 is the usual Darcy's equation wherein the usual viscous term in the equations of fluid motion is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, where $\vec{q}$ is the filter velocity of the fluid.

The effect of suspended particles on the stability of superposed fluids importance in geophysics and chemical engineering. Further motivation for the stability of fluids in presence of suspended particles is the fact that knowledge concerning fluid-particle mixtures is not commensurate with thier industrial and scientific importance. Scanlon and Segel [2] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The suspended particles were thus found to destabilize the layer. Palaniswamy and Purushotham [3] have considered the stability of shear flow of stratified fluids with fine dust and have found that the fine dust (suspended particles) increases the region of instability.

Generally the magnetic field has a stabilizing effect on the instability, but there are a few exceptions also. For example, Kent [4] has studied the effect of a horizontal magnetic field which varies in the vertical direction on the stability of parallel flows and has shown that the system is unstable under certain conditions, while in the absence of magnetic field the system is known to be stable.

Sharma [5] has studied the stability of a layer of electrically conducting Oldroyed fluid (i.e. fluid described by the Oldroyd [6] constitutive relation) heated from below in the presence of a uniform magnetic field and has found that the magnetic field has a stabilizing effect. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relation or Oldroyd's constitutive relation. Rivlin-Ericksen fluid is one such class of viscoelastic fluids. Garg, Srivastava and Singh [7] have studied the drag on a sphere oscillating in conducting dusty Rivlin-Ericksen elastico-viscous liquid. It is this class of elastico-viscous fluid we are interested in.

Owing to the importance of suspended particles and variable magnetic field in chemical engineering and geophysics, the present paper considers the stability of stratified Rivlin- Ericksen fluid in stratified porous medium in the presence of suspended particles and variable horizontal magnetic field separately.

2. EFFECT OF VARIABLE MAGNETIC FIELD

2.1 PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible, infinitely conducting, Rivlin-Ericksen fluid of variable density, kinematic viscosity and kinematic viscoelasticity arranged in horizontal strata in a porous medium of variable porosity and permeability and acted on by gravity force $\vec{g}(0, 0, -g)$ and variable horizontal magnetic field $\vec{H}(H_0(z), 0, 0)$. Consider an infinite horizontal layer of thickness d bounded by the planes $z = 0$ and $z = d$.

Let $\delta\rho$, δp, $\vec{v}(u, v, w)$ and $\vec{h}(h_x, h_y, h_z)$ denote respectively, the perturbations in density ρ, pressure p, velocity $(0,0,0)$ and magnetic field $\vec{H}(H_0(z), 0, 0)$. Then the linearized perturbation equations governing the motion of Rivlin-Ericksen fluid and Maxwell's equation are

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\,\delta\rho + \frac{\mu_e}{4\pi}[(\nabla \times \vec{h}) \times \vec{H} + (\nabla \times \vec{H}) \times \vec{h}] - \frac{\rho}{k_1}\left(\nu + \nu'\frac{\partial}{\partial t}\right)\vec{v}, \quad \text{...(1)}$$

$$\nabla \,.\, \vec{v} = 0, \quad \text{...(2)}$$

$$\varepsilon\frac{\partial}{\partial t}(\delta\rho) = -\omega\frac{\partial\rho}{dz}, \quad \text{...(3)}$$

$$\varepsilon\frac{\partial \vec{h}}{\partial t} = (\vec{H}\,.\,\nabla)\vec{v} - (\vec{v}\,.\,\nabla)\vec{H} \quad \text{...(4)}$$

$$\nabla \,.\, \vec{h} = 0. \quad \text{...(5)}$$

Equations (1) and (2) are linearized perturbed equations of motion and continuity whereas equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion. Equations (4) and (5) are linearlized perturbed, Maxwell's equations.

Analysing the disturbance into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp(ik_x x + ik_y y + nt), \quad \text{...(6)}$$

where k_x, k_y are horizontal wave numbers, $k = (k_x^2 + k_y^2)^{1/2}$ and n is in general, a complex constant.

Equations (1)-(5), using expression (6), give

$$\frac{\rho}{\varepsilon}\left[\left(1 + \frac{\varepsilon\nu'}{k_1}\right)n + \frac{\varepsilon\nu}{k_1}\right]u = -ik_x\delta p + \frac{\mu_e}{4\pi}h_z\,dH_0 \quad \text{...(7)}$$

$$\frac{\rho}{\varepsilon}\left[\left(1 + \frac{\varepsilon\nu'}{k_1}\right)n + \frac{\varepsilon\nu}{k_1}\right]v = -ik_y\delta p + \frac{\mu_e}{4\pi}(ik_x h_y - ik_y h_x) \quad \text{...(8)}$$

$$\frac{\rho}{\varepsilon}\left[\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right]w=-D\delta p-\frac{g(D\rho)}{\varepsilon n}w+\frac{\mu_e}{4\pi}\left(ik_xh_z-Dh_xh_x\frac{DH_0}{H_0}\right) \quad ...(9)$$

$$ik_xu+ik_yv+Dw=0 \quad ...(10)$$

$$ik_xh_x+ik_yh_y+Dh_z=0 \quad ...(11)$$

$$\varepsilon nh_x=ik_xH_0u-wdH_0 \quad ...(12)$$

$$\varepsilon nh_y=ik_xH_0v \quad ...(13)$$

$$\varepsilon nh_z=ik_xH_0w. \quad ...(14)$$

Eliminating u, v, h_x, h_y, h_z and δp from equations (7)-(9) and using (10)- (14), after a little algebra, we get

$$D\left[\frac{\rho}{\varepsilon}\left\{\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right\}Dw\right]-k^2\frac{\rho}{\varepsilon}\left[\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right]w$$
$$+\frac{gk^2}{\varepsilon n}(D\rho)w+\frac{\mu_ek_x^2}{4\pi\varepsilon n}\left\{H_0^2(D^2-k^2)w+D(H_0^2)Dw-\frac{H_0^2\,(Dw)(D\varepsilon)}{\varepsilon}\right\}=0 \quad ...(15)$$

which, on simplification, can be written as

$$\left[\frac{\rho}{\varepsilon}\left\{\left(1+\frac{\varepsilon\nu'}{k_1}\right)n+\frac{\varepsilon\nu}{k_1}\right\}\right](D^2-k^2)w+\left\{\frac{n}{\varepsilon}(D\rho)\left(1+\frac{\varepsilon\nu'}{k_1}\right)+\frac{n\rho}{\varepsilon^2}(D\varepsilon)\left(1+\frac{\varepsilon\nu'}{k_1}\right)\right.$$
$$\left.+\frac{n\rho}{k_1}D\nu'-\frac{n\rho\nu'}{k_1^2}dk_1+\frac{\nu}{k_1}D\rho+\frac{\rho}{k_1}D\nu-\frac{\rho\nu}{k_1^2}Dk_1\right\}Dw+\frac{gk^2(D\rho)w}{\varepsilon n}$$
$$+\frac{\mu_ek_x^2}{4\pi\varepsilon n}\left\{H_0^2(D^2-k^2)w+D(H_0^2)Dw-\frac{H_0^2}{\varepsilon}(Dw)(D\varepsilon)\right\}=0 \quad ...(16)$$

Equation (16) is the general equation governing the stability of a stratified Rivlin-Ericksen fluid in a stratified porous medium in the presence of a variable horizontal magnetic field.

2.2 THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

Assume the stratifications in density, kinematic viscosity, kinematic viscoelasticity, medium porosity, medium permeability and magnetic field of the forms

$$\rho=\rho_0e^{\beta z},\ \mu=\mu_0e^{\beta z},\ \mu'=\mu_0'e^{\beta z},\ \varepsilon=\varepsilon_0e^{\beta z},\ k_1=k_{10}e^{\beta z},\ H_0^2=H_1^2e^{\beta z} \quad ...(17)$$

where $\rho_0, \mu_0, \mu_0', \varepsilon_0, k_{10}, H_1$ and β are constants. Equations (17) imply that the kinematic viscosity $\nu\left(=\frac{\mu}{\rho}=\frac{\mu_0}{\rho_0}\right)$, the kinematic viscoelasticity $\nu'\left(=\frac{\mu'}{\rho}=\frac{\mu_0'}{\rho_0}\right)$, and the Alfvén velocity $V_A\left(=\sqrt{\frac{\mu_e H_0^2}{4\pi\rho}}=\sqrt{\frac{\mu_e H_1^2}{4\pi\rho_0}}\right)$ are constant everywhere.

Using stratifications of the form (17), equation (16) transforms to

$$\left[n+\frac{\varepsilon_0}{\kappa_{10}}(\nu_0+\nu_0'n)+\frac{k_x^2V_A^2}{n}\right](D^2-k^2)w+\frac{gk^2\beta}{n}w=0 \qquad ...(18)$$

The general solution of equation (18) is

$$w=Ae^{m_1z}+Be^{m_2z} \qquad ...(19)$$

where A, B are arbitrary constants; m_1, m_2 are given by

$$\left.\begin{aligned} m_{1,2} &= \pm k\left(1-\frac{g\beta}{nL}\right)^{1/2} \\ \text{and} \quad L &= \left[n+\frac{\varepsilon_0}{\kappa_{10}}(v_0+nv_0')+\frac{k_x^2V_A^2}{n}\right] \end{aligned}\right\} \qquad ...(20)$$

Here we consider the fluid to be confined between two rigid planes $z=0$ and $z=d$. The boundary conditions for the case of two rigid surfaces are

$$w=0 \text{ at } z=0 \text{ and } z=d. \qquad ...(21)$$

The vanishing of w at $z=0$ is satisified by the choice

$$w=A(e^{m_1z}-e^{m_2z})$$

while the vanishing of w at $z=d$ requires

$$e^{(m_1-m_2)d}=1$$

i.e.

$$(m_1-m_2)d=2is\pi$$

where s is an integer.

Inserting the values of m_1, m_2 from (20) in above equation, we obtain

$$1-\frac{g\beta}{nL}=-\frac{s^2\pi^2}{k^2d^2},$$

which on simplification gives

$$\left(1+\frac{\varepsilon_0\nu_0'}{k_{10}}\right)n^2+\frac{\varepsilon_0\nu_0}{k_{10}}n-\frac{g\beta k^2d^2}{k^2d^2+s^2\pi^2}=0. \qquad \text{...(22)}$$

If $\beta < 0$ (stable stratification), equation (22) does not admit any positive root of n and so the system is always stable for disturbance of all wave numbers.

If $\beta > 0$ (unstable stratification), the system is stable or unstable according as

$$k_x^2\, V_A^2 >< gk^2\beta/L \qquad \text{...(23)}$$

The system is clearly unstable in the absence of a magnetic field. However, the system can be completely stabilized by large enough magnetic field as can be seen from equation (23), if

$$V_A^2 > gk^2\beta/Lk_x^2 .$$

Thus if $\beta > 0$ and $k_x^2V_A^2 < gk^2B/L$ equation (22) has atleast one positive root. Let n_0 denotes the positive root of equation (22). Then

$$\left(1+\frac{\varepsilon\nu'}{k_1}\right)n_0^2+\frac{\nu\varepsilon}{k_1}n_0+\left(k_x^2\,V_A^2-\frac{gk^2\beta}{L}\right)=0 \qquad \text{...(24)}$$

as $\quad \nu_0=\dfrac{\mu_0}{\rho_0}=\dfrac{\mu}{\rho}=\nu,\ \dfrac{\varepsilon_0}{\kappa_{10}}=\dfrac{\varepsilon}{\kappa_1},\ \nu_0'=\dfrac{\mu_0'}{\rho_0}=\dfrac{\mu'}{\rho}=\nu'$

and $\quad V_A^2=\dfrac{\mu_e H_1^2}{4\pi\rho_0}=\dfrac{\mu_e H_0^2}{4\pi\rho}.$

To find the role of kinematic viscosity, kinematic viscoelasticity, medium permeablity and magnetic field on the growth rate of unstable modes, we examine the natures of

$$\frac{dn_0}{d\upsilon},\ \frac{dn_0}{d\upsilon'},\ \frac{dn_0}{dk_1}\ \text{and}\ \frac{dn_0}{dV_A}.$$

Equation (24) yields

$$\frac{dn_0}{d\nu} = -\frac{\varepsilon n_0}{2n_0\,(k_1 + \varepsilon v') + \varepsilon\nu}\,, \qquad \text{...(25)}$$

$$\frac{dn_0}{d\nu'} = -\frac{\varepsilon n_0^2}{2n_0\,(k_1 + \varepsilon\nu')\,\varepsilon\nu}\,, \qquad \text{...(26)}$$

$$\frac{dn_0}{dk_1} = \frac{\varepsilon n_0\,(\nu + n\nu')}{k_1\,[2n_0\,(k_1 + \varepsilon\nu') + \varepsilon\nu]}\,, \qquad \text{...(27)}$$

$$\frac{dn_0}{dV_A} = -\frac{2k_x^2 V_A}{2n_0\,(k_1 + \varepsilon\nu') + \varepsilon\nu}\,. \qquad \text{...(28)}$$

It is clear from equation (25)-(28) that the growth rates decrease with the increase in kinematic viscosity, kinematic viscoelasticity and magnetic field whereas the growth rates increase with the increase in medium permeability.

3. EFFECT OF SUSPENDED PARTICLES

3.1 PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible Rivlin-Ericksen fluid of variable density, kinematic viscosity and kinematic viscoelasticity arranged in horizontal strata embedded by suspended particles (fine dust) in porous medium of variable porosity and permeability. Consider an infinite horizontal fluid-particle layer of thickness d bounded by the planes $z = 0$ and $z = d$. The relevant equations of motion for the fluid are given by

$$\frac{\rho}{\varepsilon}\left[\frac{\partial\vec{V}}{\partial t} + \frac{1}{\varepsilon}(\vec{V}\,.\,\nabla)\vec{V}\right] = -\nabla p + \rho\vec{g} - \frac{\rho}{k_1}\left(\nu + \nu'\frac{\partial}{\partial t}\right)\vec{V} + \frac{kN}{\varepsilon}(\vec{u} - \vec{v}) \qquad \text{...(29)}$$

$$\nabla\,.\,\vec{v} = 0, \qquad \text{...(30)}$$

$$\varepsilon\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\rho = 0. \qquad \text{...(31)}$$

Here $\vec{u}(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and number density of the particles respectively, $k = 6\pi\rho\nu\eta$, where η is particle radius, is the Stokes drag coefficient, $\vec{u} = (l, r, s)$ and $\bar{x} = (x, y, z)$.

If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial\vec{u}}{\partial t} + \frac{1}{\varepsilon}(\vec{u}\cdot\nabla)\,\vec{u}\right] = kN\,(\vec{v} - \vec{u}) \qquad \text{...(32)}$$

$$\varepsilon \frac{\partial N}{\partial t} + \nabla \, . \, (N\vec{u}) = 0. \qquad ...(33)$$

The presence of particles adds an extra force term proportional to the velocity difference between particles and fluid and appears in the equation of motion (30). Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles (32). The buoyancy force on the particles is neglected. Inerparticle reactions are not considered either since we assume that the distances between particles are quite large compared with their diameters. These assumptions have been used in writing the equations of motion for the particles (32).

Let $\delta\rho$, δp and $\vec{v}(u, v, w)$ denote, respectively the perturbations in fluid density ρ, pressure p and velocity (0,0,0). Then the linearized perturbation equations of the fluid-particle layer become

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{v}}{\partial t} = -\nabla \delta p + \vec{g}(\delta p) - \frac{\rho}{k_1}\left(\nu + \nu' \frac{\partial}{\partial t}\right)\vec{v} + \frac{kN}{\varepsilon}(\vec{u} - \vec{v}) \qquad ...(34)$$

$$\left(\frac{m}{k}\frac{\partial}{\partial t} + 1\right)\vec{u} = \vec{v} \qquad ...(35)$$

together with equations (2) and (3).

Eliminating $\vec{u}$ from equation (34) by making use of (35) and using expression (6), equations (34), (2) and (3) yield

$$\frac{\rho}{\varepsilon}\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{K} + 1}\right]u = -ik_x\delta p \qquad ...(36)$$

$$\frac{\rho}{\varepsilon}\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{k} + 1}\right]v = -ik_y\delta p \qquad ...(37)$$

$$\frac{\rho}{\varepsilon}\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{k} + 1}\right]w = -D(\delta p) - g(\delta\rho) \qquad ...(38)$$

$$ik_x u + ik_y v + Dw = 0 \qquad ...(39)$$

$$\varepsilon n\delta\rho = -wD\rho. \qquad ...(40)$$

Eliminating *u, v* and δp from equations (36)-(38) and using (39) and (40), we obtain

$$\left[n + \frac{\varepsilon}{\kappa_1}(\nu + n\nu') + \frac{mNn/\rho}{\frac{mn}{k}+1} \right](D^2 - k^2)w + \left[\frac{1}{k_1} D\{\varepsilon(\nu + n\nu')\} - \frac{\varepsilon\,(\nu + n\nu')Dk_1}{k_1^2} + \frac{mn}{(mn/k+1)} \left\{ N\left(-\frac{1}{\rho^2} \right)D\rho + \frac{1}{\rho}DN \right\} \right]Dw + \frac{gk^2(d\rho)w}{\rho n} = 0 \quad ...(41)$$

3.2 THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

Assume the stratifications in fluid density, fluid viscosity, fluid viscoelasticity, suspended particles number density, medium porosity and medium permeability of the forms

$$\rho = \rho_0 e^{\beta z},\ \mu = \mu_0 e^{\beta z},\ \mu_0' = \mu_0' e^{\beta z},\ N = N_0 e^{\beta z},\ \varepsilon = \varepsilon_0 e^{\beta z},\ k_1 = k_{10} e^{\beta z} \quad ...(42)$$

where ρ_0, μ_0, μ_0', N_0, ε_0, k_{10} and β are constants and so the kinematic viscosity $\nu = \frac{\mu}{\rho} = \frac{\mu_0}{\rho_0} (= \nu_0)$ and the kinematic viscoelasticity $\nu' = \frac{\mu'}{\rho} = \frac{\mu_0'}{\rho_0} (= \nu_0')$ are contrants everywhere.

Using stratifications of the form (42), equation (41) transfoms to

$$\left[n + \frac{\varepsilon_0}{\kappa_{10}}(\nu_0 + \nu_0' n) + \frac{mN_0 n/\rho_0}{\left(\frac{mn}{k}+1\right)} \right](D^2 - k^2)w + \frac{gk^2\beta}{n}w = 0 \quad ...(43)$$

The general solution of equation (43) is

$$w = Ae^{q_1 z} + Be^{q_2 z} \quad ...(44)$$

where *A, B* are constants; q_1, q_2 are given by

$$q_{1,2} = \pm k \left[1 - \frac{q\beta}{nL_2} \right]^{1/2} \quad ...(45)$$

and

$$L_2 = \left[n + \frac{\varepsilon_0}{\kappa_{10}}(\nu_0 + \nu_0' n) + \frac{mN_0 n/\rho_0}{\left(\frac{mn}{k}+1\right)} \right] \quad ...(46)$$

Here also we consider the fluid to be confined between two rigid planes at $z = 0$ and $z = d$. The boundary conditions are

$$w = 0 \text{ at } z = 0 \text{ and } z = d \qquad \ldots(47$$

The vanishing of w at $z = 0$ is satisfied by the choice

$$\omega = A\,(e^{q_1 z} - e^{q_2 z}),$$

while the vanishing of w at $z = d$ requires

$$e^{(q_1 - q_2)d} = 1$$

i.e. $\qquad (q_1 - q_2)d = 2is\pi$

where s is an integer.

Inserting the values of q_1, q_2 form (45) in above equation, we obtain

$$1 - \frac{g\beta}{nL_2} = -\frac{s^2\pi^2}{d^2},$$

which on simplification yields

$$\left(1 + \frac{\varepsilon_0 \nu_0'}{k_{10}}\right)n^3 + \left[\frac{k}{m}\left(1 + \frac{\varepsilon_0}{k_{10}}\nu_0'\right) + \frac{\varepsilon_0 \nu_0}{k_{10}} + \frac{kN_0}{\rho_0}\right]n^2$$
$$+\left[\frac{\nu_0 \varepsilon_0}{k_{10}}\frac{k}{m} - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right]n - \frac{q\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\frac{k}{m} = 0. \qquad \ldots(48$$

If $\beta < 0$ (stable stratification), equation (48) does not allow any positive root of n and so the system is always stable for disturbances of all wave numbers.

If $\beta > 0$ (unstable stratification), equation (48) has one positive root and so the system is unstable for disturbances of all wave numbers. Let n_0 denote the positive root of equation (48). Then

$$\left[1 + \frac{\varepsilon v'}{k_1}\right]n_0^3 + \left[\frac{k}{m} + \frac{\varepsilon v}{k_1} + \frac{k}{m} + \frac{\varepsilon v'}{k_1} + \frac{kN}{\rho}\right]n_0^2$$
$$+\left[\frac{v\varepsilon}{k_1}\frac{k}{m} - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right]n_0 - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\frac{k}{m} = 0 \qquad \ldots(49)$$

as $\quad \nu_0 = \dfrac{\mu_0}{\rho_0} = \dfrac{\mu}{\rho} = \nu,\ \nu_0' = \dfrac{\mu_0'}{\rho_0} = \dfrac{\mu'}{\rho} = \nu',\ \dfrac{N_0}{\rho_0} = \dfrac{N}{\rho}$

and $\quad \dfrac{\varepsilon_0}{k_{10}} = \dfrac{\varepsilon}{k_1}.$

To find the roles of fluid viscosity, fluid viscoelasticity, suspended particles number density and medium permeability on the growth rates of unstable modes, we examine the natures of $\dfrac{dn_0}{d\nu}, \dfrac{dn_0}{d\nu'}$, $\dfrac{dn_0}{dN}$ and $\dfrac{dn_0}{dk_1}$ analytically. Equation (49) yields

$$\frac{dn_0}{d\nu} = -\frac{n_0\varepsilon\left(n_0 + \dfrac{k}{m}\right)}{k_1\left[3n_0^2\left(1 + \dfrac{\varepsilon\nu'}{k}\right) + 2n_0\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k\varepsilon}{mk_1}\nu' + \dfrac{kN}{\rho}\right) + \dfrac{\nu\varepsilon}{k_1}\dfrac{k}{m} - \dfrac{g\beta k^2 d^2}{k^2d^2 + s^2\pi^2}\right]} \qquad \text{...(50)}$$

$$\frac{dn_0}{d\nu'} = -\frac{n_0^2\varepsilon\left(n_0 + \dfrac{k}{m}\right)}{k_1\left[3n_0^2\left(1 + \dfrac{\varepsilon\nu'}{k_1}\right) + 2n_0\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k\varepsilon}{mk_1}\nu' + \dfrac{kN}{\rho}\right) + \dfrac{\nu\varepsilon}{k_1}\dfrac{k}{m} - \dfrac{g\beta k^2 d^2}{k^2d^2 + s^2\pi^2}\right]} \qquad \text{...(51)}$$

$$\frac{dn_0}{dN} = \frac{n_0\, k/\rho}{3n_0^2\left(1 + \dfrac{\varepsilon\nu'}{k_1}\right) + 2\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k\varepsilon}{mk_1}\nu' + \dfrac{kN}{\rho}\right)} \qquad \text{...(52)}$$

and

$$\frac{dn_0}{dk_1} = \frac{n_0\varepsilon\left[n_0^2\nu' + n_0\left(\nu + \dfrac{k}{m}\nu'\right) + \nu\dfrac{k}{m}\right]}{k_1^2\left[3n_0^2\left(1 + \dfrac{\varepsilon\nu'}{k_1}\right) + 2n_0\left(\dfrac{k}{m} + \dfrac{\varepsilon\nu}{k_1} + \dfrac{k}{m}\dfrac{\varepsilon\nu'}{k_1} + \dfrac{kN}{\rho}\right) + \dfrac{\nu\varepsilon}{k_1}\dfrac{k}{m} - \dfrac{g\beta k^2 d^2}{k^2d^2 + s^2\pi^2}\right]} \qquad \text{...(53)}$$

It is clear from equations (50) and (51) that the growth rates decrease or increase with the increase in kinematic viscosity and kinematic viscoelasticity according to

$$3n_0^2\left(1 + \frac{\varepsilon\nu'}{k_1}\right) + 2n_0\left(\frac{k}{m} + \frac{\varepsilon\nu}{k_1} + \frac{k\varepsilon\nu'}{mk_1} + \frac{kN}{\rho}\right) + \frac{\nu\varepsilon}{k_1}\frac{k}{m} > < \frac{g\beta^2 d^2}{k^2d^2 + s^2\pi^2} \qquad \text{...(54)}$$

Equation (52) implies that the growth rates increase with the increase in the suspended particles number density. However, the growth rates increase or decrease with the increase in medium permeability according to (54) as is evident from equation (53).

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INSTABILITY OF TWO ROTATING VISCOELASTIC SUPERPOSED FLUIDS WITH SUSPENDED PARTICLES IN POROUS MEDIUM

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ABSTRACT

The instability of the plane interface between two Oldroydian viscoelastic superposed fluids permeated with suspended particles and uniform rotation in porous medium is considered. The cases of two uniform viscoelastic fluids separated by a horizontal boundary and exponentially varying density, viscosity and suspended particles number density are considered. The system is found to be stable for potentially stable configuration/stable stratification and unstable for potentially unstable configuration/unstable stratification. The viscoelasticity, suspended particles, rotation and porous medium effects do not affect the stability or instability of the system.

1. INTRODUCTION. A detailed account of the instability of the plane interface between two Newtonian viscous fluids under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar[1], Bhatia and Steiner[2] have studied the problem of thermal instability of a Maxiwellian viscoelastic fluid in the presence of rotation and have found that the rotation has a destablizing effect in contrast to the stabilizing effect on Newtonian fluid. Bhatia and Steiner[3] have also considered the problem of thermal instability of a Maxwellian fluid in hydromagnetics and have found that the magnetic field has stabilizing effect on viscoelastic fluid just as in the case of Newtonian fluid. Eltayeb[4] has studied the convective instability in a rapidly rotating Oldroydian viscoelastic fluid. An experimental demonstration by Toms and Strawbridge[5] reveals that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the theoretical model of oldroyd fluid.

In geophysical situations, more often than not, the fluid is not pure, but may instead be permeated with suspended (or dust) particles. Scanlon and Segel[6] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The effect of suspended particles was thus found to destablize the layer. The medium has been considered to be non- porous in all the above studies.

The flow through porous medium has been of considerable interest in recent years particularly among petroleum engineers and geophysical fluid dynamicists. A macroscopic equation which describes incompressible flow of Newtonian fluid of viscosity μ through a homogeneous and isotropic porous medium of permeability k_1 is the Darcy's equation. As a result of this macroscopic law, the usual viscous term in the equations of fluid motion is replaced by the resistance term $-(\mu/k_1)\,\vec{q}$, where $\vec{q}$ is the filter velocity of the fluid. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma[7]. The effects of suspended particles and medium permeability were found to destabilize the layer.

The present paper attempts to study the stability of the plane interface separating two incompressible superposed rotating, Oldroydian viscoelastic fluids, permeated with suspended particles, in a porous medium. The constitutive relations of the type of Oldroyd fluid were proposed and studied by Oldryod[8]. The knowledge regarding fluid-particle mixtures is not commensurate with thier scientific and industrial importance. The analysis would be relevant to the stability of some polymer solutions like a dilute solution of methyl methacrylate in n-butyl acetate and to the stability of some Maxwellian viscoelastic fluids and the problem finds its usefulness in several geophysical situations and in chemical technology. These aspects form the motivation for the present study.

2. PERTURBATION EQUATIONS. Let $T_{ij}, z_{ij}, e_{ij}, \delta_{ij}, \mu, \lambda, \lambda_0 (< \lambda)$, p, v_i, x_i and d/dt denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the Kronecker delta, the viscosity, the stress relaxation time, the strain retardation time, the isotropic pressure, the velocity vector, the position vector and the mobile operator. Then the Oldroydian viscoelastic fluid is described by the constitutive relations

$$T_{ij} = -p\delta_{ij} + z_{ij},$$

$$\left(i + \lambda \frac{d}{dt}\right) z_{ij} = 2\mu \left(1 + \lambda_0 \frac{d}{dt}\right) e_{ij}, \qquad \text{...(2.1)}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Relations of the type (2.1) were posed and studied by Oldroyd[8]. Oldroyd[8] also showed that many rheology equations of state of general validity, reduce to (2.1) when linearized. When $\lambda_0 = 0$ the fluid is termed as Maxwellian viscoelastic whereas $\lambda = \lambda_0 = 0$ yields the fluid to be Newtonian viscous fluid.

Consider a static state in which an incompressible Oldroydian viscoelastic fluid permeated with suspended partiles is arranged in horizontal strata in porous medium and the pressure p and density ρ are functions of the vertical coordinate z only. A uniform rotation

$\vec{\Omega}(0, 0, \Omega)$ pervades the whole system. The character of the equilibrium of this initial static state is determined, as usual by supposing that the system is slightly disturbed and then by following its further evolution.

Let ρ, μ, p, $\boldsymbol{u}\,(u, v, w)$ and $\vec{\Omega}\,(0, 0, \Omega)$ denote respectively, the density, the viscosity, the pressure, the velocity of pure fluid and the uniform rotation; $\boldsymbol{v}\,(\bar{x}, t)$ and $N\,(\bar{x}, t)$ denote the velocity and number density of the particles respectively. $k = \sigma\pi\rho\upsilon\eta$ where η is the particle radius is the Stokes' drag coefficient, $\boldsymbol{v} = (l, r, s)$, $\bar{x} = (x, y, z)$ and $\vec{\lambda'} = (0, 0, 1)$. Let $\in$, k_1 and g stand for medium porosity, medium permeability and acceleration due to gravity respectivley. Then the equations of motion and continuity for the rotating Oldroydian viscoelastic fluid permeated with suspended particles in a porous medium are

$$\frac{\rho}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial u}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{u}.\nabla)\,\boldsymbol{u}\right]=\left(1+\lambda\frac{\partial}{\partial t}\right)[-\nabla(p-\frac{\rho}{2}|\vec{\Omega}\times\boldsymbol{x}|^2)$$

$$-\rho g\vec{\lambda'}+\frac{kN}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})+\frac{2\rho}{\varepsilon}(\boldsymbol{u}\times\vec{\Omega})]-(1+\lambda_0\frac{\partial}{\partial t})\frac{\mu}{k_1}\boldsymbol{u} \qquad \ldots(2.2)$$

$$\nabla.\boldsymbol{u}=0 \qquad \ldots(2.3)$$

Since the density of a fluid particle moving with the fluid remains unchanged, we have

$$\varepsilon\frac{\partial\rho}{\partial t}+(\boldsymbol{u}\,.\,\nabla)\rho=0. \qquad \ldots(2.4)$$

In the equations of motion (2.2), by assuming a uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term proportional to the velocity differences between the particles and the fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion of the particles. The distances between particles are assumed quite large compared with their diameter so that interparticle reactions are ignored. The effects of pressure, gravity and Darcian force on the suspended particles are negligibly small and therefore ignored. If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions, are

$$mN\left[\frac{\partial v}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\,.\,\nabla)\boldsymbol{v}\right]=kN\,(\boldsymbol{u}-\boldsymbol{v}), \qquad \ldots(2.5)$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla\,.\,(N\boldsymbol{v})=0. \qquad \ldots(2.6)$$

Let $\delta\rho$, δp, $\boldsymbol{u}(u, v, w)$ and $\boldsymbol{v}\,(l, r, s)$ denote respectively, the perturbations in fluid density ρ, fluid pressure p, fluid velocity (0, 0,

0) and particle velocity (0, 0, 0). Then the linearized perturbation equations of the fluid-particle layer are

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\rho}{\varepsilon}\frac{\partial \boldsymbol{u}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla\delta p+\boldsymbol{g}\delta\rho+\frac{kN}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})+\frac{2\rho}{\varepsilon}(\boldsymbol{u}+\overrightarrow{\Omega})\right]-\left(1+\lambda_0\frac{\partial}{\partial t}\right)\frac{\mu}{k_1}\boldsymbol{u} \qquad ...(2.7)$$

$$\nabla \,.\, \boldsymbol{u}=0, \qquad ...(2.8)$$

$$\varepsilon\frac{\partial}{\partial t}\delta\rho=-w\frac{d\rho}{dz}, \qquad ...(2.9)$$

$$\left(\frac{m}{k}\frac{\partial}{\partial t}+1\right)\boldsymbol{v}=\boldsymbol{u}. \qquad ...(2.10)$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp\,(ik_x x+ik_y y+nt), \qquad ...(2.11)$$

where k_x, k_y are wave numbers along x and y-directions, $k^2=k_x^2+k_y^2$ and n is in general, a complex constants.

For perturbations of the form (2.11), writing component equations and eliminating u, v, $\delta\rho$ and δp Eqs. (2.7)-(2.10) yield

$$\left[(1+\lambda n)(zn+1)\,\text{[illegible]}\,(1+\lambda_0 n)(zn+1)\frac{\varepsilon v}{k_1}\right][D(\rho Dw)-k^2\rho w]$$
$$+n(1+\lambda n)\,[D(mNDw)-mNk^2w)+$$
$$+\left[gk^2\frac{(1+\lambda n)\,(zn+1)}{n}\right](D\rho)w+4(1+\lambda n)^2\,(zn+1)^2\,\Omega^2$$
$$\left[D\left\{\frac{\rho Dw}{(1+\lambda n)(zn+1)n+(1+\lambda n)\dfrac{mnN}{\rho}+(1+\lambda_0 n)(zn+1)\dfrac{v\varepsilon}{k_1}}\right\}\right]\Bigg]$$
$$=0 \qquad ...(2.12)$$

where $v\ (=\mu/\rho)$ stands for kinematic viscosity.

3. TWO UNIFORM VISCOELASTIC FLUIDS SEPARATED BY A HORIZONTAL BOUNDARY

Here we consider the case when two superposed uniform Oldroydian viscoelastic fluids of densities ρ_1 and ρ_2 uniform viscosities μ_1 and μ_2 uniform suspended particles densities N_1 and N_2 are separated by a horizontal boundary at $z=0$. The subscripts 1 and 2 distinguish the lower and the upper fluids respectively. Then, in each region of constant ρ, contant μ and constant mN, Eq. (2.12) reduces to

$$(D^2 - \chi^2)w = 0, \qquad \text{...(3.1)}$$

where

$$\chi = \frac{k}{\left[1 + \dfrac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{\left[n(1+\lambda n)(zn+1)n(1+\lambda n)\dfrac{mN}{\rho} + (1+\lambda_0 n)(zn+1)\dfrac{\upsilon\varepsilon}{k_1}\right]^2}\right]^{1/2}}$$

and in case of highly viscous fluid

$$\chi = \frac{k}{\left[1 + \dfrac{1}{2}\dfrac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{\left[n(1+\lambda n)(zn+1) + n(1+\lambda n)\dfrac{mN}{\rho} + (1+\lambda_0 n)(zn+1)\dfrac{\upsilon\varepsilon}{k_1}\right]^2}\right]}$$

...(3.2)

The general solution of Eq. (3.1) is

$$w = Ae^{+\chi z} + be^{-\chi z} \qquad \text{...(3.3)}$$

where A and B are arbitrary constants. The boundary conditions to be statisfied in the present problem are

(*i*) The velocity w should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(*ii*) $w(z)$ is continuous at $z = 0$.

(*iii*) The pressure should be continuous across the interface.

Applying the boundary conditions (*i*) and (*ii*), we have

$$w_1 = Ae^{+\chi z} \ (z < 0), \qquad \text{...(3.4)}$$

$$w_2 = Ae^{-\chi z} \ (z > 0), \qquad \text{...(3.5)}$$

where the same constant A has been chosen to ensure the continuity of w at $z = 0$.

Here we assume the kinematic viscosities of both fluids to be equal i.e. $\upsilon_1 = \upsilon_2 = \nu$ (Chandrasekhar[1], p.443) and $\dfrac{mN}{\rho} = \dfrac{mN_1}{\rho_1} = \dfrac{mN_2}{\rho_2} (= M)$ as these simplifying assumptions do not obscure any of the essential features of the problem. Integrating Eq. (2.12) across the interface, $z = 0$ we obtain the boundary condition

$$\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\upsilon\varepsilon}{k_1}\right]\Delta_0 (\rho D w)$$

$$+ n(1+\lambda n)\,\Delta_0(mNDw) + (1+n\lambda)(zn+1)\frac{gk^2}{n}\Delta_0(\rho)w_0$$

$$+ \frac{4(1+\lambda n)^2(zn+1)^2\,\Omega^2}{\left[(1+\lambda n)(zn+1)n + (1+\lambda n)\frac{mnN}{\rho} + (1+\lambda_0 n)(zn+1)\frac{\varepsilon\upsilon}{k_1}\right]}\Delta_0(\rho Dw)$$

$$= 0 \qquad \text{...(3.6)}$$

where w_0 is the common value of w at $z = 0$.

Applying the boundary condition (3.6) to the solutions (3.4) and (3.5), we obtain

$$1 + \frac{mn\,(1+\lambda n)\,(N_1+N_2)}{(\rho_1+\rho_2)\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon\upsilon}{k_1}\right]}$$

$$- \frac{(1+\lambda_n)(zn+1)gk^2(\alpha_2-\alpha_1)}{n\chi\left[(1+\lambda)(zn+1)n + (1+\lambda_0)(zn+1)\frac{\varepsilon\upsilon}{k_1}\right]}$$

$$+ \frac{4(1+\lambda n)^2(zn+1)^2\Omega^2(\alpha_2+\alpha_1)}{\left[(1+\lambda n)(zn+1)n + (1+\lambda n)Mn + (1+\lambda_0 n)(zn+1)\frac{\varepsilon\upsilon}{k_1}\right]\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon\upsilon}{k_1}\right]}$$

$$= 0, \qquad \text{...(3.7)}$$

where $$\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1+\rho_2}.$$

Equation (3.7), after substituting the value of χ from (3.2) and simplification, yields

$$A_{13}n^{13} + A_{12}n^{12} + A_{11}n^{11} + \ldots + A_2n^2 + A_1n + A_0 = 0, \qquad \text{...(3.8)}$$

where $$A_{13} = \lambda^4 z^4,$$

$$A_0 = -gk(\alpha_2-\alpha_1)\frac{\varepsilon\upsilon}{k_1}\left[\frac{\varepsilon^2\upsilon^2}{k_1^2} + 2\Omega^2\right] \qquad \text{...(3.9)}$$

and the coefficients $A_1 - A_{12}$ being quite lengthy and not needed in the discussion of stability, have not been written here.

For the potentially stable arrangement $(\alpha_2 < \alpha_1)$ all the coefficients of Eq. (3.8) are positive. So, all the roots of Eq. (3.8) are either real and negative or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots.

The system is therefore stable in each case.

For the potentially unstable arrangement $(\alpha_2 > \alpha_1)$, the constant term is negative and so there is atleast one change of sign in Eq. (3.8). Therefore, Eq. (3.8) allows atleast one positive root of n meaning thereby instability of the system.

4. SPECIAL CASE.

In the absence of rotation, Eq. (3.7) after some algebraic simplifications, gives

$$A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0, \qquad \text{...(4.1)}$$

where

$$A_4 = (\rho_2 + \rho_1)\lambda z,$$

$$A_3 = \left[(\rho_2 + \rho_1)\left(z + \lambda + \lambda_0 z \frac{\varepsilon v}{k_1}\right) + m\lambda(N_1 + N_2) \right]$$

$$A_2 = \left[(\rho_2 + \rho_1)\left(1 + \frac{z v \varepsilon}{k_1} + \lambda_0 \frac{\varepsilon v}{k_1}\right) + m(N_1 + N_2) - (\rho_1 + \rho_2)\lambda z g k(\alpha_2 - \alpha_1) \right]$$

$$A_1 = \left[(\rho_2 + \rho_1) \left\{ \frac{\varepsilon v}{k_1} - (\alpha_2 - \alpha_1) g k (z + \lambda) \right\} \right]$$

and

$$A_0 = -(\rho_2 + \rho_1) g k (\alpha_2 - \alpha_1). \qquad \text{...(4.2)}$$

For the potentially stable arrangement $(\alpha_2 < \alpha_1)$ all the coefficients of Eq. (4.1) are positive. So, all the roots of Eq. (4.1) are either real and negative or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots. The system is, therefore, stable in each case.

For the potentially unstable arrangement $(\alpha_2 > \alpha_1)$ the constant term is negative and so there is atleast one change of sign in Eq. (4.1). Equation (4.1) allows atleast one positive root of n thereby meaning instability of the system.

5. THE CASE OF EXPONENTIALLY VARYING DENSITY, VISCOSITY AND SUSPENDED PARTICLES NUMBER DENSITY

Let us assume the stratifications in density, viscosity and suspended particles number density of the form

$$\rho = \rho_0 e^{\beta z}, \ \mu = \mu_0 e^{\beta z}, \ n = N_0 e^{\beta z} \qquad \text{...(5.1)}$$

where ρ_0, μ_0, N_0 and β are constants. Equations (5.1) imply that the coefficient of kinematic viscosity ν is constant everywhere.

Using the stratification of the form (5.1), Eq. (2.12) transforms to

$$\left[(1 + \lambda n)(zn + 1)n + (1 + \lambda_0 n)(zn + 1)\frac{\varepsilon v_0}{k_1} + M_0 n(1 + \lambda_n) \right.$$

$$+\frac{4\Omega^2(1+\lambda n)^2\,(zn+1)^2}{(1+\lambda n)(zn+1)n+(1+\lambda n)M_0 n+(1+\lambda_0 n)(zn+1)\,\dfrac{\varepsilon v_0}{k_1}}\Bigg]D^2w$$

$$+\Bigg[(1+\lambda n)(zn+1)n+(1+\lambda_0 n)(zn+1)\,\frac{\varepsilon v_0}{k_1}+M_0 n(1+\lambda n)$$

$$+\frac{4\Omega^2(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n+(1+\lambda n)M_0 n+(1+\lambda_0 n)(zn+1)\,\dfrac{\varepsilon v_0}{k_1}}\Bigg]\beta Dw$$

$$-\Bigg[(1+\lambda n)(zn+1)n+(1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}+M_0 n(1+\lambda n)$$

$$-\frac{g\beta(1+\lambda n)(zn+1)}{n}\Bigg]k^2 w=0 \qquad \text{...(5.2)}$$

The general solution of Eq. (5.2) is

$$w=A_1e^{q_1 z}+A_2e^{q_2 z}, \qquad \text{...(5.3)}$$

where A_1, A_2 are two arbitrary constant and q_1, q_2 are the roots of the equation

$$\Bigg[(1+\lambda n)(zn+1)n+(1+\lambda_0 n)(zn+1)\frac{\varepsilon \nu_0}{k_1}+M_0 n(1+\lambda n)$$

$$+\frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n+(1+\lambda n)M_0 n+(1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg]q^2$$

$$+\Bigg[(1+\lambda n)(zn+1)n+(1+\lambda_0 n)(zn+1)\frac{\varepsilon n_0}{k_1}+M_0 n(1+\lambda n)$$

$$+\frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n+(1+\lambda n)M_0 n+(1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg]\beta q$$

$$-\Bigg[(1+\lambda n)(zn+1)n+(1+\lambda n)(zn+1)\frac{\varepsilon v_0}{k_1}+M_0 n(1+\lambda n)$$

$$-\frac{g\beta(1+\lambda n)(zn+1)}{n}\Bigg]k^2=0. \qquad \text{...(5.4)}$$

If the fluid is suspposed to be confined between two rigid planes at $z = 0$ and $z = d$, then the vanishing of w at $z = 0$ is statisfied by the choice

$$w = A(e^{q_1 z} - e^{q_2 z}), \qquad \text{...(5.5)}$$

while the vanishing of w at $z = d$ requires

$$\exp\,(q_1 - q_2)d = 1, \qquad \text{...(5.6)}$$

which implies that

$$(q_1 - q_2)d = 2im\pi \qquad \text{...(5.7)}$$

where m is an integer.

Equation (5.4) gives

$$\begin{aligned}
q_{1,\,2} = \Bigg[& -\frac{\beta}{2}\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n) \\
& + \frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg\} \\
& \pm \frac{1}{2}\Bigg[\beta^2\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n) \\
& + \frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg\}^2 \\
& + 4k^2\Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n) \\
& + \frac{4\Omega^2\,(1+\lambda n)^2(zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\dfrac{\varepsilon v_0}{k_1}}\Bigg\} \\
& \Bigg\{(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n\,(1+\lambda n) \\
& - \frac{g\beta(1+\lambda n)(zn+1)}{n}\Bigg\}\Bigg]^{1/2}
\end{aligned}$$

$$\left[(1+\lambda n)(zn+1)n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1} + M_0 n(1+\lambda n) \right.$$

$$\left. + \frac{4\Omega^2 (1+\lambda n)^2 (zn+1)^2}{(1+\lambda n)(zn+1)n + (1+\lambda n)M_0 n + (1+\lambda_0 n)(zn+1)\frac{\varepsilon v_0}{k_1}}\right]^{-1} \quad ...(5.8)$$

Inserting the values of q_1, q_2 from (5.8) in (5.7) and simplifying, we obtain

$$A_7 n^7 + A_6 n^6 + A_5 n^5 + \ldots + A_2 n^2 + A_1 n^1 + A_0 = 0, \quad ...(5.9)$$

where

$$A_7 = (\beta^2 d^2 + 4m^2\pi^2 + 4k^2 d^2)\lambda^2 z^2,$$

$$A_6 = (\beta^2 d^2 + 4m^2\pi^2 + 4k^2 d^2)\{2\lambda z(\lambda + z) + 2\lambda\lambda_0 z^2 \frac{\varepsilon v_0}{k_1} + 2M_0 \lambda^2 z\},$$

$$A_0 = 4k^2 d^2\,(-g\beta\,\varepsilon v_0/k_1), \quad ...(5.10)$$

and the coefficients $A_1 - A_5$ being quite lengthy and not needed in the discussion of stability have not been written here.

If $\beta < 0$ (stable stratification), Eq. (5.9) does not admit of any positive root of n and the system is always stable for disturbances of all wave-numbers.

If $\beta > 0$ (unstable stratification), the constant term in eq. (5.9) is negative. Equation (5.9), therefore, has atleast one positive real root and so the system is unstable for all wave numbers.

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CONTOUR INTEGRALS ASSOCIATED WITH TWO GENERAL CLASS OF POLYNOMIALS FOX'S *H*-FUNCTION AND THE MULTI-VARIABLE *H*-FUNCTION

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ABSTRACT

Here, we present two contour integrals based on a lemma, for the multivariable H-function [8] involving the product of Fox's H-function [1] and two general class of polynomials [6] with essentially arbitrary coefficients. By assigning suitable suitable values to the coefficients, the main result of two theorems can be reduced to integrals involving for instance the classical polynomials of Jacobi, Laguerre and Hermite. The multivariable H-function occuring in each of our main results can be reduced to the generalised Lauricella's hypergeometric functions of several complex variables.

1. INTRODUCTION AND NOTATIONS Srivastava and Panda [8] have introduced the multivariable H-function

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta'\,;\ldots;\,\theta^{(r)}]: \\ [(c):\psi'\,;\ldots;\,\psi^{(r)}]:\end{matrix}\;\begin{matrix}[(b'):\phi']\,;\ldots;\,[(b^{(r)}):\phi^{(r)}]\,; \\ [(d'):\delta']\,;\ldots;\,[(d^{(r)}):\delta^{(r)}]\,;\end{matrix}\;z_1,\ldots,z_r\right) \qquad \ldots(1.1)$$

The defining integral of the above function, its various special cases and other details can be found in the paper referred to above. For the sake of brevity

$$\Lambda_i = \sum_{j=1}^{\lambda}\theta_j^{(i)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} < 0 \qquad \ldots(1.2)$$

$$\Omega_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} - \sum_{j=1+v^{(i)}}^{B^{(i)}} - \sum_{j=1}^{C}\psi_j^{(i)}$$

$$+\sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=1+u^{(i)}}^{D^{(i)}}\delta_j^{(i)} > 0,\ \forall\, i\ \varepsilon\ 1,\ldots,r. \qquad \ldots(1.3)$$

2. LEMMA If $h > 0$, $c > -Re(y)$, $Re(\rho) > 0$, m, m' are arbitrary positive integers and $A_{n,\alpha}$, $A_{n',\alpha'}$ are arbitrary constants, real or complex then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} S_n^m\left(l(x+y)^{-1}\right) S_{n'}^{m'}\left(l'(x+y)^{-1}\right)$$

$$.\,H_{P,Q}^{M,N}\left[\,y'(x+y)^{-\sigma}\;\middle|\;\begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]dt$$

$$= \sum_{g=1}^{M}\;\sum_{G=0}^{\infty}\;\sum_{\alpha}^{[n/m]}\;\sum_{\alpha'}^{[n'/m']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}(y')\,\frac{\eta_G\,(-n)_{m\alpha}A_{n,\alpha}}{\alpha!}\,l^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!}A_{n',\alpha'}\,(l')^{\alpha'}\,e^{-hy}\,\frac{h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}}{\Gamma(\rho+\alpha+\alpha'+\sigma\eta_G)}, \qquad \ldots(2.1)$$

given that

$$\sigma > 0,\, E_j > 0,\, j = 1, \ldots, \rho;\, F_j > 0,\, j = 1, \ldots, Q;$$

and $|\arg(y')| < \dfrac{T\pi}{2}$, $T > 0$

$$\left(T = \sum_{j=1}^{n} E_j - \sum_{j=n+1}^{P} E_j + \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{Q} F_j\right).$$

Proof of the Lemma The assertion (1) of the Lemma follows at once by applying the definition of Fox's H-function and multivariable H-function in conjunction with the well-known Hankel's contour integral for the Gamma function which is given in the following form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\sigma}\,dt = \frac{h^{\rho-1}\,e^{-hy}}{\Gamma(\rho)} \qquad \ldots(2.2)$$

$h > 0$, where for convergence $c > -Re(y)$ and $Re(\rho) > 0$.

3. THE MAIN CONTOUR INTEGRAL

The main results of this paper are the contour integrals in the following theorem :

Theorem-1 With Λ_i and Ω_i defined by (1.2) and (1.3) respectively, let $\Lambda_i \leq 0$ and $|\arg(z_i)| < \Omega_i\,\pi/2,\ \forall\, i = 1, \ldots, r$ ...(3.1)

where the both equalities hold for suitably restricted values of the complex variables $z_1, \ldots, z_r$. Now, the function $H_{P,Q}^{M,N}$ be defined by [4] and [5], general class of polynomials be defined by [6] and le $h > 0$ $c > -Re(y)$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\, S_n^m\,(l(x+y)^{-1})\, S_{n'}^{m'}\,(l'(x+y)^{-1})$$

$$H_{P,Q}^{M,N}\left[\, y'\,(x+y)^{-\sigma} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right] .\, H(z_1(x+y)^{-\rho_1},\ldots, z_r(x+y)^{-\rho_r})\, dt$$

$$= \sum_{g=1}^{m}\ \sum_{G=0}^{\infty}\ \sum_{\alpha}^{[n/m]}\ \sum_{\alpha'}^{[n'/m']}\ \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,(y')^{\eta_G}\,\frac{(-n)_{m\alpha}\,A_{n,\alpha}}{\alpha!}\,(l)^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!}\,A_{n',\alpha'}\,(l')^{\alpha'}\,h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}\,e^{-hy}$$

$$H^{\,0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:\\ [(c):\psi';\ldots;\psi^{(r)}],\end{matrix}\right.$$

$$[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];$$

$$[1-\rho-\alpha-\alpha'-{}^{\sigma}\eta_G;\,\rho_1,\ldots,\rho_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];$$

$$\left. z_1h^{\rho_1},\ldots,z_rh^{\rho_r}\right) \quad \ldots(3.2)$$

provided that $\rho_j > 0$, $j = 1, \ldots, r$, $\sigma > 0$, $|\arg y'| < \frac{T\pi}{2}$ and

$Re\,(\rho) > -c \sum_{j=1}^{r} \rho_j$.

Theorem-2 Under the hypothesis preceding the assertion (3) of the theorem-1

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\, H_{P,Q}^{M,N}\left[\, y'\,(x+y)^{-\sigma} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$$

$$S_n^m\,(l(x+y)^{-1})\, S_{n'}^{m'}\,(l'(x+y)^{-1})\, H(z_1(x+y)^{\rho_1},\ldots, z_r(x+y)^{\rho_r})\, dt$$

$$= \sum_{g=1}^{m}\ \sum_{G=0}^{\infty}\ \sum_{\alpha}^{[n/m]}\ \sum_{\alpha'}^{[n'/m']}\ \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,(y')^{\eta_G}\,\frac{(-n)_{m\alpha}\,A_{n,\alpha}}{\alpha!}\,(l)^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!}\,A_{n',\alpha'}\,(l')^{\alpha'}\,h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}\,e^{-hy}$$

$$H^{\,0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:\\ [(c):\psi';\ldots;\psi^{(r)}]:\end{matrix}\right.$$

$$[\rho+\alpha+\alpha'+\sigma\eta_G;\,\rho_1,\ldots,\rho_r]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];$$

$$[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];$$

$$\left. z_1h^{-\rho_1},\ldots,z_rh^{-\rho_r}\right) \quad \ldots(3.3)$$

provided that $\rho_j > 0$, $j = 1, \ldots, r$, $\sigma > 0$, $|\arg y'| < \frac{T\pi}{2}$ and

$Re\,(\rho) > -c \sum_{j=1}^{r} \rho_j$.

Proofs of Theorem 1 and 2. The contour integrals formula of Theorem 1 can be proved by replacing the multivariable H-function in the integrand by its multiple contour integral [7]. Now we change the order of integration, evaluate the innermost integral by appealing to the assertion (2) of the lemma and then interpret the resulting multiple contour integral as an H-function of r-complex variables.

4. APPLICATIONS

The contour integral formulas of Theorem 1 and 2 of the preceding section possesses manifold generality. The multivariable H-function may be transformed into G-functions, E-functions, Lauricella's functions, Hypergeometric functions, Legendre functions, Bessel functions, Hermite functions, Laguerre, functions and other functions in one, two or more arguments.

On other hand we can get several integral formulas for such special polynomials $\left(S_n^m\,[y] \right)_{n=0}^{\infty}$ by giving the values to n and n'.

Particular cases of Lemma (*I*) By applying our Lemma to the case of Hermite polynomials ([10], p.106, Eqn. (5.5.4) and [9]) by setting $S_n^2(x) \to x^{n/2}\, H_n\,(\frac{1}{2\sqrt{x}})$ in which case $m = 2$, $A_{n,\,s} = (-1)^s$ and also letting $m' = 2$, $A_{n',\,s'} = (-1)^{s'}$, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}\,(x+y)^{-\rho}\,.\,H_{P,Q}^{M,N}\left[\, y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\left[(l(x+y)^{-1})^{n/2} H_n\!\left(\frac{1}{\sqrt{l(x+y)^{-1}}} \right) \right] \left[(l'(x+y)^{-1})^{n'/2}\, H_{n'}\!\left(\frac{1}{\sqrt{l'\,(x+y)^{-1}}} \right) \right] dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2]} \frac{(-1)^G\, \phi(\eta_G)}{G!\, F_g}\, (y')^{\eta_G}\, \frac{(-n)_{2s}\, 2^{n}}{s!} \left(\frac{-l}{4} \right)^{s}$$

$$\frac{(-n')_{2s'}\, 2^{n'}}{s'!} \left(\frac{-l'}{4} \right)^{s'} \frac{e^{-hy}\, h^{\rho + \sigma\eta_G + s + s' - 1}}{\Gamma(\rho + {}^{\sigma}\eta_G + s + s')} \qquad \ldots(4.1)$$

which holds under the same conditions as those required for (1).

(*II*) For the Laguerre polynomials ([10], p.101, Eqn.(5.1.6) and [9]) setting $S_n(x) \rightarrow L_n^{\alpha}(x)$ in which case $m = 1$ and $A_{n,s}\binom{n+\alpha}{n}\left(\frac{1}{(\alpha+1)_k}\right)$ and also taking $m' = 1$ and $A_{n',s'} = \binom{n'+\alpha'}{n'}\left(\frac{1}{(\alpha'+1)_{k'}}\right)$ we obtain the Lemma in the following form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} \,.\, H_{P,Q}^{M,N}\left[y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\left[\sum_{s=0}^{n}\binom{n+\alpha}{n-s}\frac{(-l(x+y)^{-1})^s}{s!}\right]\left[\sum_{s'=0}^{n'}\binom{n'+\alpha'}{n'-s'}\frac{(-l'(x+y)^{-1})^s}{s'!}\right]dt$$

$$= \sum_{g=1}^{m}\sum_{G=0}^{\infty}\sum_{s=0}^{n}\sum_{s'=0}^{n'} \frac{(-1)^s\,\phi(\eta_G)}{G!\,F_g}(y')^{\eta_G}\frac{e^{-hy}h^{\rho+\sigma\eta_G+s+s'-1}}{\Gamma(\rho+\sigma\eta_G+s+s')}$$

$$\binom{n+\alpha}{n-s}\frac{(-x)^s}{s!}\binom{n'+\alpha'}{n'-s'}\frac{(-x')^{s'}}{s'!} \qquad \text{...(4.2)}$$

which holds under the same conditions as those required for (1).

(*III*) For Lauricella's functions taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$, $D^{(i)} =$ $\forall\, i = 1, \ldots, r$, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} \,.\, H_{P,Q}^{M,N}\left[y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, f_Q) \end{matrix} \right]$$

$$S_n^m\,(l(x+y)^{-1})\,S_{n'}^{m'}\,(l'\,(x+y)^{-1})$$

$$F\begin{matrix} A : B' ;\ldots; B^{(r)} \\ C : D' ;\ldots; D^{(r)} \end{matrix}\left(\begin{matrix} [1-(a):\theta' ;\ldots; \theta^{(r)}] : \\ [1-(c):\psi' ;\ldots; \psi^{(r)}] : \end{matrix}\right.$$

$$\left.\begin{matrix} [1-(b'):\phi'] ;\ldots; [1-(b^{(r)}):\phi^{(r)}];\, z_1\,(x+y)^{-\rho_1} \\ \vdots \\ [1-(d'):\delta'] ;\ldots; [1-(d^{(r)}):\delta^{(r)}];\, z_2\,(x+y)^{-\rho_r} \end{matrix}\right) dt$$

$$= \sum_{g=1}^{m}\sum_{G=0}^{\infty}\sum_{\alpha=0}^{[n/m]}\sum_{\alpha'=0}^{[n'/m']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}(y')^{\eta_G}\,h^{\sigma\eta_G+\alpha+\alpha'+\rho-1}e^{-hy}$$

$$\frac{(-n)_{m\alpha}}{\alpha!}A_{n,\alpha}\,(l)^{\alpha}\,\frac{(-n')_{m'\alpha'}A_{n',\alpha'}}{\alpha'\,!}\,(l')^{\alpha'}$$

$$. F^{A : B' ;\dots; B^{(r)}}_{C+1 : D' ;\dots; D^{(r)}} \left(\begin{matrix} [1-(a) : \theta' ;\dots; \theta^{(r)}] : \\ [1-(c) : \psi' ;\dots; \psi^{(r)}], \end{matrix} \right.$$

$$[1-(b') : \phi'] ;\dots; [1-(b^{(r)}) : \phi^{(r)}];$$

$$[1-\rho+\sigma\eta_G - \alpha - \alpha' : \rho_1 ,\dots \rho_r] : [1-(d') : \delta'] ;\dots; [1-(d^{(r)}) : \delta^{(r)}] ;$$

$$\left. z_1 h^{\rho_1} ;\dots; z_r h^{\rho_r} \right) \quad \dots(4.3)$$

valid under the same conditions as given in the equation (4).

(*IV*) For Hermite polynomial in the theorem (1), we get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{hx} (x+y)^{-\rho} . H^{M,N}_{P,Q} \left[y'(x+y)^{-\sigma} \middle| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\left[(l(x+y)^{-1})^{n/2} H_n \left(\frac{1}{\sqrt{l(x+y)^{-1}}} \right) \right] \left[(l'(x+y)^{-1})^{n'/2} H_{n'} \left(\frac{1}{\sqrt{l'(x+y)^{-1}}} \right) \right] dt$$

$$. H \left(z_1 (x+y)^{-\rho_1} ,\dots z_r (x+y)^{-\rho_r}\right] dt$$

$$= \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2']} \frac{(-1)_G \, \phi(\eta_G)}{G! \, F_g} (y')^{\eta_G} h^{\sigma\eta_G + s + s' + \rho - 1} e^{-hy}$$

$$\frac{(-n)_{2s} \, 2^n}{s!} \left(\frac{-l}{4} \right)^s \frac{(-n')_{2s'} \, 2^{n'}}{s'!} \left(\frac{-l'}{4} \right)^{s'}$$

$$H^{0, \lambda : (u', v') ;\dots; (u^{(r)}, v^{(r)})}_{A, C+1 : [B', D'] ;\dots; [B^{(r)}, D^{(r)}]} \left(\begin{matrix} [(a) : \theta' ;\dots; \theta^{(r)}] : \\ [(c) : \psi' :\dots; \psi^{(r)}] : \end{matrix} \right.$$

$$[(b') : \phi'] ;\dots; [(b^{(r)}) : \phi^{(r)}] ;$$

$$[1-\rho-\sigma\eta_G - s - s' ; \rho_1 ,\dots, \rho_r] : [(d') : \delta'] ;\dots; [(-d^{(r)}) : \delta^{(r)}] ;$$

$$\left. \begin{matrix} z_1 h^{\rho_1} \\ : \\ z_2 h^{\rho_2} \end{matrix} \right) \quad \dots(4.4)$$

which holds under the same conditions as given in equation (4).

(*V*) Also by applying our theorem 1 to the case of Laguerre polynomials, we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{hx} (x+y)^{-\rho} . H^{M,N}_{P,Q} \left[y'(x+y)^{-\sigma} \middle| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$L_n^{(\alpha)}(l(x+y)^{-1})\, L_{n'}^{(\alpha')}(l'(x+y)^{-1}).H(z_1(x+y)^{-\rho_1},\ldots, z_r(x+y)^{-\rho_r}]dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{s=0}^{[n]} \sum_{s'=0}^{[n']} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} (y')^{\eta_G} h^{\sigma\eta_G + s + s' - 1} e^{-hy}\, dt$$

$$\frac{(-l)^s}{s!} \binom{n+\alpha}{n-s} \frac{(-l')^{s'}}{s'!} \binom{n'+\alpha'}{n'-s'}$$

$$H^{\,0,\lambda : (u',v');\ldots;(u^{(r)},v^{(r)})}_{\,A,C+1 : [B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [(a):\theta';\ldots;\theta^{(r)}] : \\ [(c):\psi';\ldots;\psi^{(r)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [1-\rho+\sigma\eta_G - s - s'; \rho_1,\ldots,\rho_1] : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \begin{matrix} z_1 h^{\rho_1} \\ \vdots \\ z_2 h^{\rho_r} \end{matrix} \right) \quad \ldots(4.5)$$

valid for the same conditions as given in equation (4).

(*VI*) Letting $n' \to 0$, the above five results of particular cases reduce to the known results obtained by Chaurasia and Tyagi [3].

(*VII*) Letting $n' \to 0$, $n \to 0$ the above five results of particular cases reduce to the known results obtained by V.B.L. Chaurasia [2].

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USE OF THE MULTIVARIABLE *H*-FUNCTION AND GENERAL POLYNOMIALS IN LINEAR FLOW OF HEAT IN AN ANISOTROPIC FINITE SOLID MOVING IN A CONDUCTING MEDIUM

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ABSTRACT

The use of special functions and their applications is being increasing important in technical applications. Space research, nuclear reactor, wood technology, soil mechanics and the mechanics of solids of fibrous structure give rise to several problems of the application of special functions.

The aim of this paper is to discuss the application of certain product of Fox's H-function, general polynomials, the generalized prolate spheroidal wave function and the multivariable *H*-function in obtaining formal solution of the partial differential equation related to a problem of liner flow of heat in an anisotropic finite solid moving in a conducting medium. Thus, the present study unifies and extends a number of results scattered in the literature.

1. INTRODUCTION We shall consider a problem of linear heat flow in an anisotropic finite solid moving in a conducting medium, given by following partial differential equation (see [6])

$$\frac{\partial}{\partial y}\left[(1-y^2)\frac{\partial U}{\partial y}\right] - \frac{pcv}{\lambda'}\frac{\partial U}{\partial y} + \frac{Q(y)}{\lambda'} = \frac{pc}{\lambda'}\frac{\partial U}{\partial t} \qquad \text{...(1.1)}$$

with the law of conductivity $K = \lambda'(1-y^2)$, $Q(y)$ is the intensity of a continuous source of heat situated inside this solid.

Let the initial temperature of the rod be given by

$$U(y, 0) = F(y) \qquad \text{...(1.2)}$$

assuming

$$v = (\rho - \sigma)/q,$$

$$Q(y) = -(\rho + \sigma)\,\lambda' y\,\frac{\partial U}{\partial y} - (s', y)^2\,\lambda' U,$$

$$q = pc/\lambda'$$

a solution of (1.1) takes the form

$$U(y, t) = \sum_{l=0}^{\infty} B_l \exp(-T_l t)\,\phi_l^{\rho,\sigma}(s', y) \qquad \text{...(1.3)}$$

where $B_l = (l + p)(l + p + \rho + \sigma + 1)/q$.

Consider

$$F(y) = (1-y)^{\alpha-\rho}(1+y)^{\beta-\sigma} H_{P,Q}^{M,N}\left[z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[(1-y)^{\alpha_1 h_1}(1+y)^{\alpha_1 k_1},\ldots,(1-y)^{\alpha_s h_s}(1+y)^{\alpha_s k_s}\right]$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}$$

$$\left(z_1(1-y)^{h'}(1+y)^{k'},\ldots,z_r(1-y)^{h^{(r)}}(1+y)^{k^{(r)}} \right) \qquad \text{...(1.4)}$$

where the multivariable H-function of several complex variables introduced by Srivastava and Panda ([8] and [9]; see also [10], p.251). The results needed in our investigations are as follows :

The general polynomials $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[x_1,\ldots,x_s]$ occuring in the integral stands for the multivariable polynomials given by Srivastava ([11], p.185, eqn. (7)] defined and represented in the following slightly modified from

$$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[x_1,\ldots,x_s] = \sum_{\alpha_1=0}^{[n_1/m_1]} \cdots\cdot \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}$$

$$A[n_1, \alpha_1;\ldots;\ldots.;n_s, \alpha_s]\,x_1^{\alpha_1}\ldots,x_s^{\alpha_s} \qquad \text{...(1.5)}$$

where $n_i = 0, 1, 2, \ldots; m_i \neq 0\ (i = 1, \ldots, s)$, m_i is an arbitrary positive integer. The coefficients $A[n_i, \alpha_i;\ldots;n_s, \alpha_s]$ being arbitrary constants, real or complex.

If we take $s = 1$ in equation (1.5) and denote $A[n, \alpha]$ thus obtained by $A_{n,\alpha'}$, we arrive at the well-known general class of polynomials $S_n^m[x]$ introduced by Srivastava ([12], p.1, eqn.(1)).

The series representation of Fox's H-function ([3] and [7]; see also [10], p.12) :

$$H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right] = \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G\, \phi(\eta_G) z^{\eta_G}}{G!\, F_g} \qquad \ldots(1.6)$$

where

$$\phi(\eta_G) = \prod_{j=1,\, j\neq g}^{M} \Gamma(f_j - F_j \eta_G) \prod_{j=1}^{N} \Gamma(1 - e_j - E_j \eta_G)$$

$$\left\{ \prod_{j=M+1}^{Q} \Gamma(1 - f_j - F_j \eta_G) \prod_{j=N+1}^{P} \Gamma(e_j - E_j \eta_G) \right\}^{-1}$$

and $\eta_G = (f_g + G)/F_g$ ...(1.7)

The solution of the following differential equation

$$(1 - y^2) \frac{d^2 x}{dy^2} + [(\sigma - \rho) - (\rho + \sigma + 2)y] \frac{dx}{dy} + [(s') + (s'y)^2]x = 0 \ldots(1.8)$$

is the generalized prolate spheroidal wave function ([4], p.107, Eqn. (2.11)), which is denoted as

$$\phi_n^{\rho,\sigma}(s', y) = \sum_{a=0}^{\infty} R_{q,n}^{\rho,\sigma}(s')\, P_{a+n}^{(\rho,\sigma)}(y) \qquad \ldots(1.9)$$

where $s' = 0$, $\xi(0) = (n + a)(\rho + \sigma + n + a + 1)$, $a \geq 0$.

The orthogonality property of the generalized prolate spheroidal wave function ([4], p.107, eqn.(3.1))

$$\int_{-1}^{1} (1 - y)^{\rho} (1 + y)^{\sigma} \phi_v^{\rho,\sigma}(s', y)\, \phi_w^{\rho,\sigma}(s'y)\, dx = N_{v,w}^{\rho,\sigma}\, \delta_{v,w} \ldots(1.10)$$

where

$$N_{v,w}^{\rho,\sigma} = \sum_{a=0}^{\infty} R_{a,v}^{\rho,\sigma}(s')^2$$

$$\frac{2^{\rho+\sigma+1}\Gamma(v + \rho + a + 1)\Gamma(v + \sigma + a + 1)}{(2v + 2a + \rho + \sigma + 1)\Gamma(v + a + 1)\Gamma(v + a + \rho + \sigma + 1)}$$

and $\delta_{v,w}$ is the Kronecker delta.

2. THE MAIN INTEGRAL

$$\int_{-1}^{1} (1 - y)^{\alpha} (1 + y)^{\beta} \phi_w^{\rho,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[z(1 - y)^h (1 + y)^k \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$$

$$S_{n_1,\dots,n_s}^{m_1,\dots,m_s}[(1-y)^{\alpha_1h_1}(1+y)^{\alpha_1k_1},\dots,(1-y)^{\alpha_sh_s}(1+y)^{\alpha_sk_s}]$$

$$H_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}$$

$$\left(z_1(1-y)^{h'}(1+y)^{k'},\dots,z_r(1-y)^{h^{(r)}}(1-y)^{h^{(r)}}(1+y)^{k^{(r)}}\right)dy$$

$$=\sum_{g=1}^{M}\ \sum_{v,G=0}^{\infty}\ \sum_{\alpha_1=0}^{[n_1/m_1]}\dots\sum_{\alpha_s=0}^{[n_s/m_s]}\frac{(-1)^G\phi(\eta_G)}{G!\,F_g}z^{\eta_G}R_{v,w}^{\rho,\sigma}(s',y)$$

$$\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!}\dots\frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}A(n_1,\alpha_1;\dots;n_s,\alpha_s)$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1,\dots,(h_s+k_s)\alpha_s}$$

$$\sum_{u=0}^{v+w}(-v+w)_u(\rho+\sigma+v+w+1)_u\{(\rho+1)_u\,u!\}^{-1}$$

$$H_{A+2,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda+2:(u',v');\dots;(u^{(r)},v^{(r)})}$$

$$\left(\begin{array}{l}[(a):\theta',\dots,\vartheta^{(r)}],[-u-\alpha-h\eta_G-h_1\alpha_1,\dots,h_s\alpha_s;h',\dots,h^{(r)}],\\ [(c):\psi',\dots,\psi^{(r)}];\end{array}\right.$$

$$[-\beta-k\eta_G-k_1\alpha_1,\dots,k_s\alpha_s:k',\dots,k^{(r)}]:$$
$$[-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1\dots-(h_s+k_s)\alpha_s:$$

$$\left.\begin{array}{l}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ h'+k',\dots,h^{(r)}+k^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{array} z_12^{h'+k'},\dots,z_r2^{h^{(r)}+k^{(r)}}\right) \qquad \dots(2.1)$$

where $\eta_i=0,1,2,\dots;m_i\neq 0\ [i=1,\dots s]\ m_i$ is an arbitrary positive integer. The cofficients $A[n_1,\alpha_1;\dots;n_s,\alpha_s]$ being arbitrary constant real or complex.

$$Re(\alpha+hf_l/F_l+\sum_{i=1}^{r}h^{(i)}d_j^{(i)}/\delta_j^{(i)})>-1$$

$$Re(\beta+kf_l/F_l+\sum_{i=1}^{r}k^{(i)}d_j^{(i)}/\delta_j^{(i)})>-1,\ Re(\rho)>-1,\ Re(\sigma)>-1,$$

$$h^{(i)} > 0, k^{(i)} > 0, l = 1, \ldots, M; i = 1, \ldots, r, j = 1, \ldots, u^{(i)}, T_i > 0$$

$$|\arg(z_i)| < T_i \pi/2$$

$$|\arg(z)| < T\pi/2 \text{ and } T = \left(\sum_{1}^{N} E_i - \sum_{N+1}^{P} E_i + \sum_{1}^{M} F_i - \sum_{M+1}^{Q} F_i \right).$$

Evaluation of (2.1). The integral (2.1) can be evaluated in a routine manner with the help of result Chaurasia ([10], p.31 eqn. (2.1) and (1.5)).

3. SOLUTION OF THE PROBLEM

$$U(y, t) = \sum_{g=1}^{M} \sum_{v, G=0}^{\infty} \sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} z^{\eta_G} R_{v,l}^{\rho,\sigma}(s', y)$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1, \ldots, (h_s+k_s)\alpha_s}$$

$$\sum_{l, u=0}^{v} \frac{(-v)_u (\rho+\sigma+v+1)_u}{(\rho+1)_u\, u} \{N_{l,1}^{\rho,\sigma}\}^{-1} \phi_\lambda^{\rho,\sigma}(s', y)$$

$$e^{-v(v+\rho+\sigma+1)t/q}\, H^{0, \lambda+2 : (u', v') ;\ldots; (u^{(r)}, v^{(r)})}_{A+2, C+1 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}]}$$

$$\left(\begin{matrix} [-u-\alpha-h\eta_G-h_1\alpha_1, \ldots, -h_s\alpha_s; h', \ldots, h^{(r)}], [-\beta-k\eta_G-k_1\alpha_1, \ldots, k_s\alpha_s; k', \ldots k^{(r)}], \\ [(c) : \psi', \ldots, \psi^{(r)}], [-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1 \ldots -(h_s+k_s)\alpha_s : \end{matrix} \right.$$

$$\left. \begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ;\ldots; [b^{(r)} : \phi^{(r)}] ; \\ : h'+k', \ldots, h^{(r)}+k^{(r)}] : [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} z_1 2^{h'+k'}, \ldots, z_r 2^{h^{(r)}+k^{(r)}} \right) \quad \ldots(3.1)$$

valid under the same conditions as given in (2.1).

Evaluation of (3.1). Take $t = 0$ then by (1.3), we have

$$F(y) = \sum_{l=0}^{\infty} B_l \phi_l^{\rho,\sigma}(s', y), \qquad \ldots(3.2)$$

where $F(y)$ is given by (1.4).

Equation (3.2) is valid since $F(y)$ is continuous and bounded variation in the interval (-1, 1).

Multiplying both sides of (3.2) by $(1-x)^\rho (1+x)^\sigma \phi_w^{\rho,\sigma}(s', y)$ $\rho > -1, \sigma > -1$ and integrating between -1 to +1 with respect to y and using the orthogonality property (1.9) a lemma ([5], p.57, eqn. (2)) and integral (2.1) after simplification, we arrive at the required result.

4. SPECIAL CASES

(*i*) If we take $s = 1$, $m_1 = 2$, $A_{n_1, \alpha_1} = (-1)^{\alpha_1}$ in (2.1), we obtain

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\, \phi_w^{\rho,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[z(1-y)^{h}(1+y)^{k} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$(1-y)^{h_1 n_1/2}\,(1+y)^{k_1 n_1/2}\, H_{n_1}\left[\frac{1}{2\sqrt{(1-y)^{h_1}(1+y)^{k_1}}} \right]$$

$$H\begin{matrix} 0, \lambda : (u', v') ;\ldots; (u^{(r)}, v^{(r)}) \\ A, C : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(z_1(1-y)^{h'}\,(1+y)^{k'}, \ldots, z_r\,(1-y)^{h^{(r)}}\,(1-y)^{h^{(r)}}\,(1+y)^{k^{(r)}} \right) dy$$

$$\sum_{g=1}^{M} \sum_{v, G=0}^{\infty} \sum_{\alpha_1=0}^{[n_1/2]} \frac{(-1)^{G}\phi\,(\eta_G)}{G!\,F_g}\, z^{\eta_G}\, R_{v, w}^{\rho, \sigma}(s', y)\, \frac{(-n_1)_{2\alpha_1}\,(-1)^{\alpha_1}}{\alpha_1!}$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1} \sum_{u=0}^{v+w} (-v+w)_u(\rho+\sigma+v+w+1)_u$$

$$\{(\rho+1)_u\, u!\}^{-1}\, H\begin{matrix} 0, \lambda+2 : (u', v') ;\ldots; (u^{(r)}, v^{(r)}) \\ A+2, C+1 : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} [-u-\alpha+h\eta_G-h_1\alpha_1 : h', \ldots, h^{(r)}],\ [-\beta-k\eta_G-k_1\alpha_1 ; k', \ldots, k^{(r)}], \\ [(c) : \psi', \ldots, \psi^{(r)}].\ [-1-\alpha-\beta-u-(h+k)\eta_G-(h+k)-(h_1+k_1)\alpha_1 : h'+k', \ldots, h^{(r)}+k^{(r)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}]; \\ : [(d') : \delta'] ;\ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \quad z_1\, 2^{h'+k'}, \ldots, z_r\, 2^{h^{(r)}+k^{(r)}} \right) \quad \ldots(4.1)$$

valid under the same conditions as given by (2.1).

(*ii*) On taking $s = 1$, $m_1 = 1$ and $A_{n_1, \alpha_1} \begin{pmatrix} f_1 + f' \\ n_1 \end{pmatrix} \dfrac{(f' + g' + n_1 + 1)_{\alpha_1}}{(f' + 1)_{\alpha_1}}$

in (2.1), we get

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\, \phi_w^{\rho,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[z(1-y)^{h}(1+y)^{k} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$P_{n_1}^{(f', g')}\,[1 - 2(1-y)^{h_1}\,(1+y)^{k_1}]\, H\begin{matrix} 0, \lambda : (u', v') ;\ldots; (u^{(r)}, v^{(r)}) \\ A, C : [B', D'] ;\ldots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(z_1(1-y)^{h'}(1+y)^{k'}, \ldots, z_r(1-y)^{h^{(r)}}(1-y)^{h^{(r)}}(1+y)^{k^{(r)}} \right) dy$$

$$\sum_{g=1}^{M} \sum_{v,\, G=0}^{\infty} \sum_{\alpha_1=0}^{n_1} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} z^{\eta_G} R_{v,\,w}^{\rho,\,\sigma}(s', y) \frac{(-n_1)_{\alpha_1}}{\alpha_1!} \binom{n_1+f'}{n_1}$$

$$\frac{(f'+g'+n_1+1)_{\alpha_1}}{(f'+1)_{\alpha_1}} \cdot 2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1}$$

$$\sum_{u=0}^{v+w} (-v+w)_u (\rho+\sigma+v+w+1)_u \{(\rho+1)_u\, u!\}^{-1}$$

$$H^{\,0,\,\lambda+2\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A+2,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [-u-\alpha-h\eta_G-h_1\alpha_1\,;\,h',\ldots,h^{(k)}],\ [-\beta-k\eta_G-k_1\alpha_1 : k',\ldots,k^{(r)}] \\ [(c):\psi',\ldots,\psi^{(r)}],\ [-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1 : h'+k',\ldots,h^{(r)}+k^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi']\,;\ldots;\,[b^{(r)}:\phi^{(\rho)}]\,; \\ [(d'):\delta']\,;\ldots;\,[(d^{(r)}):\delta^{(r)}]; \end{matrix} \quad z_1 2^{h'+k'}, \ldots, z_r 2^{h^{(r)}+k^{(r)}} \right) \qquad \ldots(4.2)$$

valid under the same conditions as required by (2.1).

(*iii*) Let $s = 1$, $m_1 = 1$ and $A_{n_1,\,\alpha_1} \binom{n_1+f'}{n_1} \dfrac{1}{(f'+1)_{\alpha_1}}$ in (2.1), we get

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\, \phi_w^{\rho,\,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$L_{n_1}^{(\alpha_1)} (1-y)^{h_1}(1+y)^{k_1} H^{\,0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{\,A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(z_1(1-y)^{h'}(1+y)^{k'}, \ldots, z_r(1-y)^{h^{(r)}}(1-y)^{h^{(r)}}(1+y)^{k^{(r)}} \right) dy$$

$$\sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{\alpha_1=0}^{n_1} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} z^{\eta_G} R_{v,\,w}^{\rho,\,\sigma}(s', y) \frac{(-n_1)_{\alpha_1}\binom{n_1+f'}{n_1}}{\alpha_1!\,(f'+1)_{\alpha_1}}$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1} \sum_{u=0}^{v+w} (-v+w)_u (\rho+\sigma+v+w+1)_u$$

$$\{(\rho+1)_u\, u!\}^{-1}\, H^{\,0,\, \lambda+2\,:\,(u',\, v')\,;\ldots;\,(u^{(r)},\, v^{(r)})}_{\,A+2,\, C+1\,:\,[B',\, D']\,;\ldots;\,[B^{(r)},\, D^{(r)}]}$$

$$\left(\begin{matrix} [-u-\alpha-h\eta_G-h_1\alpha_1 : h',\ldots,h^{(r)}],\ [-\beta-k\eta_G-k_1\alpha_1 : k',\ldots,k^{(r)}],\ [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}],\ [-1-\alpha-\beta-u-(h+k)\eta_G-(h_1+k_1)\alpha_1 : h'+k',\ldots,h^{(r)}+k^{(r)}]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1\, 2^{h'+k'},\ldots, z_r\, 2^{h^{(r)}+k^{(r)}}\right) \qquad \ldots(4.3)$$

valid under the same conditions as required by (2.1).

(*iv*) On taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$ and $D^{(i} = D^{(i)} + 1$, $\forall$ $i = 1, \ldots, r$ in (2.1), we obtain

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta}\phi_w^{\rho,\,\sigma}(s', y)\, H_{P,Q}^{M,N}\left[z(1-y)^h(1+y)^k \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$$

$$S^{\,m_1,\ldots,m_s}_{\,n_1,\ldots,n_s}\,[(1-y)^{\alpha_1 h_1}(1+y)^{\alpha_1 k_1},\ldots,(1-y)^{\alpha_s h_s}(1+y)^{\alpha_s k_s}]$$

$$F^{\,A\,:\,B'\,;\ldots;\,B^{(r)}}_{\,C\,:\,D'\,;\ldots;\,D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(r)}] : [1-(b'):\phi'] ;\ldots; \\ [1-(c):\psi',\ldots,\psi^{(r)}] : [1-(d') ;\ldots; \end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b^{(r)}):\phi^{(r)}] \\ [1-(d^{(r)}):\delta^{(r)}]\end{matrix}\ z_1\,(1-y)^{h'}\,(1+y)^{k'},\ldots,z_r(1-y)^{h^{(r)}}\,(1+y)^{k^{(r)}}\right)dx$$

$$\sum_{g=1}^{M}\ \sum_{v,\,G=0}^{\infty}\ \sum_{\alpha_1=0}^{[n_1/m_1]} \cdots \sum_{\alpha_s=0}^{[n_s/m_s]} \frac{(-1)^G\phi\,(\eta_G)}{G!\,F_g}\, z^{\eta_G}\, R_{v,\,w}^{\rho,\,\sigma}(s', y)$$

$$\frac{(-n_1)_{m_1\alpha_1}}{\alpha_1!} \cdots \frac{(-n_s)_{m_s\alpha_s}}{\alpha_s!}\, A(n_1, \alpha_1 ;\ldots; n_s, \alpha_s)$$

$$2^{1+\alpha+\beta+(h+k)\eta_G+(h_1+k_1)\alpha_1,\ldots,(h_s+k_s)\alpha_s}\ \sum_{u=0}^{v+w}\ (-v+w)_u$$

$$(\rho+\sigma+v+w+1)_u\, \{(\rho+1)_u\, u!\}^{-1}\, F^{\,A+1\,:\,B',\ldots,\,B^{(r)}}_{\,C\,:\,D',\ldots,\,D^{(r)}}$$

$$\left(\begin{matrix}[1+u+\alpha+h\eta_G+h_1\alpha_1+\ldots+h_s\alpha_s;\ h',\ldots,h^{(r)}],[1+\beta+k\eta_G+k_1\alpha_1+\ldots+k_s\alpha_s\ k',\ldots,k^{(r)}] \\ [1-(c):\psi',\ldots,\psi^{(r)}] : \end{matrix}\right.$$

$$\begin{array}{l} [1-(a):\theta',\ldots,\theta^{(r)}]: \\ [2+\alpha+\beta+u+(h+k)\eta_G+(h_1+k_1)\alpha_1,\ldots,(h_s+k_s)\alpha_s:h'+k',\ldots,h^{(r)}+k^{(r)}]: \\ \begin{array}{l}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\end{array} z_1 2^{h'+k'},\ldots,z_r 2^{h'+k'},\ldots,z_r 2^{h^{(r)}+k^{(r)}} \Bigg) \end{array} \quad \ldots(4.4)$$

valid under the same conditions as given in (2.1).

(*v*) If we take $s = 1$ and $m_1 = 0$ in (2.1) it reduces to a known result of Chaurasia ([1], p.31, eqn.(2.1)).

(*vi*) If $m_1 = m_2 = \ldots = m_s = 0$ and $M = 1$, $N = p$, $P = p$, $Q = q + 1$ and $h \to 0$, $k \to 0$ and after a little simplification the result in (2.1) reduces to a known result recently obtained by Chaurasia and Gupta ([2], p.78, eqn. (8)).

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CERTAIN INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS OF THREE AND FOUR VARIABLES

By

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ABSTRACT

In the present paper some integrals involving hypergeometric functions of three and four variables are evaluated. Some particular cases are of interest.

1. INTRODUCTION Let

$$E_\alpha f(\alpha) = f(\alpha+1),\ E_\alpha^m f(\alpha) = f(\alpha+m) \qquad \text{...(1.1)}$$

Recently Agarwal [1] employed the operator E_α to evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)}\, {}_1F_1\left[\begin{matrix}\alpha'; \\ & v \\ \alpha+x;\end{matrix}\right] {}_1F_1\left[\begin{matrix}\beta'; \\ & v \\ \beta-x;\end{matrix}\right] dx;$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\, {}_1F_1\left[\begin{matrix}\alpha'+\beta'; \\ & 2v \\ \alpha+\beta-1;\end{matrix}\right] \qquad \text{...(1.2)}$$

$\mathrm{Re}(\alpha+\beta) > 1$, n is an integer.

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\, {}_1F_1\left[\begin{matrix}\alpha+\beta-1; \\ & v \\ \alpha+x;\end{matrix}\right] {}_1F_1\left[\begin{matrix}\gamma+\delta-1; \\ & v \\ \gamma+x;\end{matrix}\right] dx$$

$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}$$

$$
{}_2F_2\left[\begin{matrix}\dfrac{\alpha+\beta+\gamma+\delta-3}{2}, & \alpha+\beta+\gamma+\delta-3; \\ \beta+\gamma-1, & \delta+\alpha-1;\end{matrix}\; 4v\right] \qquad \ldots(1.3)
$$

$\mathrm{Re}(\alpha+\beta+\gamma+\delta) > 3.$

Ragab and Simary [7] have also evaluated these integrals, by a different technique.

In the present paper, following integrals are evaluated, which involve hypergeometric functions of three and four variables :

$$
(I) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}
$$

$$
H_B(\beta+\gamma, \gamma+\alpha, \alpha+\beta; \gamma, \alpha+x, \beta-x; u, v, w)\,dx
$$

$$
= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} H_A(\beta+\gamma, \gamma+\alpha, \alpha+\beta; \gamma, \alpha+\beta-1; u, 2v, 2w)
$$

$$
\mathrm{Re}(\alpha+\beta) > 1,
$$

$$
(II) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}
$$

$$
F_K(\gamma', \alpha+\beta, \alpha+\beta, \gamma+\beta, \alpha', \gamma+\beta; \gamma, \alpha+x, \beta-x; u, v, w)\,dx
$$

$$
= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_M(\gamma', \alpha+\beta, \alpha+\beta, \gamma+\beta, \alpha', \gamma+\beta, \gamma, \alpha+\beta-1; u, 2v, 2w)
$$

$$
\mathrm{Re}(\alpha+\beta) > 1.
$$

$$
(III) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}
$$

$$
F_A^{(3)}(\alpha', \gamma, \alpha+\beta, \gamma', \alpha+x, \beta-x; u, v, w)\,dx
$$

$$
= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_G(\alpha', \alpha', \alpha', \gamma, \alpha, \beta; \gamma', \alpha+\beta-1, \alpha+\beta-1; u, 2v, 2w)
$$

$$
\mathrm{Re}(\alpha+\beta) > 1.
$$

$$
(IV) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}
$$

$$
F_{29}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \alpha+\delta, \gamma+\beta, \alpha+\delta, \gamma+\beta; \\ \gamma+\delta', \alpha+x, \beta-x, \gamma'+\delta';\end{matrix}\; u, v, w, t\right)dx
$$

$$
= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}
$$

$$
F_{58}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \alpha+\delta, \gamma+\beta, \alpha+\delta, \gamma+\beta; \\ \gamma'+\delta', \alpha+\beta-1, \alpha+\beta-1, \gamma'+\delta';\end{matrix}\; u, 2v, 2w, t\right)
$$

$$
\mathrm{Re}(\alpha+\beta) > 1.
$$

$$(V)\quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$F_{28}^{(4)}\left(\begin{matrix}\gamma+\delta,\gamma+\delta,\alpha+\beta,\alpha+\beta,\gamma+\alpha,\delta+\beta,\gamma+\alpha,\delta+\beta;\\ \alpha+x,\beta-x,\gamma+\delta',\gamma'+\delta';\end{matrix}\,u,v,w,t\right)dx$$

$$=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$.F_{47}^{(4)}\left(\begin{matrix}\gamma+\delta,\gamma+\delta,\alpha+\beta,\alpha+\beta,\gamma+\alpha,\delta+\beta,\gamma+\alpha,\delta+\beta;\\ \gamma'+\delta',\gamma'+\delta',\alpha+\beta-1,\alpha+\beta-1;\end{matrix}\,u,v,2w,2t\right)$$

$$\operatorname{Re}(\alpha+\beta)>1.$$

$$(VI)\quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.F_{27}^{(4)}\left(\begin{matrix}\gamma+\alpha,\gamma+\alpha,\delta+\beta,\delta+\beta,\gamma'+\alpha',\gamma'+\alpha',\delta'+\beta',\delta'+\beta';\\ \gamma+\delta,\alpha+x,\gamma+\delta,\beta-x;\end{matrix}\,u,v,w,t\right)dx$$

$$=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$.F_{47}^{(4)}\left(\begin{matrix}\gamma+\alpha,\gamma+\alpha,\beta+\delta,\beta+\delta,\gamma+\alpha',\gamma'+\alpha',\delta'+\beta',\delta'+\beta';\\ \gamma+\delta,\alpha+\beta-1,\gamma+\delta,\alpha+\beta-1;\end{matrix}\,u,2v,w,2t\right)$$

$$\operatorname{Re}(\alpha+\beta)>1.$$

where H_A, H_B, H_C are Srivastava's functions of three variables [cf. [9]; p.43], $F_G, F_K, F_M, F_A^{(3)}$ are Lauricella's functions of three variables [cf. [9]; p. 42, 33] and $F_{27}^{(4)}, F_{28}^{(4)}, F_{29}^{(4)}, F_{27}^{(4)}, F_{57}^{(4)}, F_{58}^{(4)}$ are Lauricella's functions of four variables [cf. [8]; p. 125-129].

In what follows, we will employ the following results [cf. [2]; p.224].

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad \dots(1.4)$$

2. ANALYSIS OF THE RESULTS

To prove (I), multiply both sides of (1.4) by

$\dfrac{\Gamma(\gamma+\alpha)\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}{\Gamma(\gamma)}$ and apply the operator

$\exp(uE_\alpha E_{\alpha'}+vE_\beta E_{\beta'}+wE_\gamma E_{\gamma'})$; we obtain

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]\;\Gamma(\gamma+\alpha)\Gamma(\beta+\gamma)}{\sin(\pi x)\;\Gamma(\alpha+x)\;\Gamma(\beta-x)\Gamma(\gamma)}$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\beta+\gamma)_{m+p}(\gamma+\alpha)_{m+n}(\alpha+\beta)_{n+p}}{(\gamma)_m\,(\alpha+x)_m\,(\beta-x)_n} \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\,dx$$

$$=\frac{2^{\alpha+\beta-2}\,\Gamma(\gamma+\alpha)\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma)}$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\beta+\gamma)_{m+p}(\gamma+\alpha)_{m+n}(\alpha+\beta)_{n+p}}{(\gamma)_m\,(\alpha+\beta-1)_{m+n}} \frac{u^m}{m!}\frac{(2v)^n}{n!}\frac{(2w)^p}{p!}\,dx$$

$$=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\,H_A\,(\beta+\gamma,\gamma+\alpha,\alpha+\beta;\gamma,\alpha+\beta-1;u,2v,2w)$$

which completes the proof of (*I*).

Now consider (1.4), multiply both the sides by

$\dfrac{\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\Gamma(\gamma'+\delta')}$ and operate both the sides by

$\exp\,(uE_{\gamma}E_{\gamma'}+vE_{\alpha}E_{\alpha'}+wE_{\beta}E_{\beta'}+tE_{\delta}E_{\delta'})$, we get

$$\int_{-\infty}^{\infty} \frac{\sin\,[(2n+1)\pi x]\;\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\sin\,(\pi x)\;\Gamma(\alpha+x)\;\Gamma(\beta-x)\Gamma(\gamma'-\delta')}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\gamma+\alpha)_{m+n}(\beta+\delta)_{p+q}(\gamma+\beta)_{m+p}(\alpha+\delta)_{n+q}}{(\gamma'+\delta')_{m+q}\,(\alpha+x)_n\,(\beta-x)_p} \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\frac{t^q}{q!}dx$$

$$=\frac{2^{\alpha+\beta-2}\,\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma'+\delta')}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\gamma+\alpha)_{m+n}(\beta+\delta)_{p+q}(\gamma+\beta)_{m+p}(\alpha+\delta)_{n+q}}{(\gamma'+\delta')_{m+q}\,(\alpha+\beta-1)_{n+p}} \frac{u^m}{m!}\frac{(2v)^n}{n!}\frac{(2w)^p}{p!}\frac{t}{q}$$

$$=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$F_{58}^{(4)}\left(\begin{matrix}\gamma+\alpha,\gamma+\alpha,\beta+\delta,\beta+\delta,\gamma+\beta,\gamma+\beta,\alpha+\delta,\alpha+\delta;\\ \gamma'+\delta',\alpha+\beta-1,\alpha+\beta-1,\gamma'+\delta';\end{matrix}\;u,v,2w,t\right)$$

which is the required result (*IV*).

By employing the operators

$$\exp\,(uE_{\gamma}E_{\gamma'}+vE_{\alpha}E_{\alpha'}+wE_{\beta}E_{\beta'}),$$

$$\exp\,(uE_{\gamma}E_{\gamma'}+vE_{\alpha}+wE_{\beta}E_{\beta'})E_{\alpha'}],$$

$$\exp\,(uE_{\gamma}E_{\gamma'}+vE_{\delta}E_{\delta'}+wE_{\alpha}E_{\alpha'}+tE_{\beta}E_{\beta'})$$

and $\exp\,(uE_{\gamma}E_{\gamma'}+vE_{\alpha}E_{\alpha'}+wE_{\delta}E_{\delta'}+E_{\beta}E_{\beta'})$ to (1.4).

we can easily obtain results (*II*), (*III*), (*V*) and (*VI*).

3. PARTICULAR CASES

By setting $u = 0$ in (I), following interested results are obtained

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha + x)\Gamma(\beta - x)}$$

$$.\, F_2(\alpha+\beta, \alpha+\beta, \gamma+\alpha, \beta+\gamma; \alpha+x, \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_1(\alpha+\beta, \alpha+\beta, \gamma+\alpha, \beta+\gamma; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.1)$$

For $w = 0$; (II) reduces to the following form

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha + x)\Gamma(\beta - x)}$$

$$.\, F_3(\gamma, \alpha+\beta, \gamma+\beta, \alpha'; \gamma, \alpha+x; u, v)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_3(\gamma', \alpha+\beta, \gamma+\beta, \alpha'; \gamma, \alpha+\beta-1; u, 2v) \quad ...(3.2)$$

For $u = 0$; (II) reduces to the following interesting case

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha + x)\Gamma(\beta - x)}$$

$$.\, F_2(\alpha+\beta, \alpha+\beta, \alpha', \gamma+\beta, \alpha+x; \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_2(\alpha+\beta, \alpha+\beta, \alpha', \gamma+\beta; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w)$$

$$...(3.3)$$

Finally, if we set $u = 0$ in (III), we obtain

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha + x)\Gamma(\beta - x)}$$

$$.\, F_2(\alpha', \alpha', \alpha, \beta; \alpha+x, \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_1(\alpha', \alpha', \alpha, \beta; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.4)$$

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LAPLACE TRANSFORM OF GENERALIZED HYPERGEOMETRIC DISTRIBUTION FUNCTION ASSOCIATED WITH ZONAL POLYNOMIAL

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ABSTRACT

In this paper authors first discuss probability density functions of generalized hypergeometric function of symmetric positive definite matrix associated with zonal polynomial, then establish new result, which has a wide range of application in mathematical Statistics by virtue of Laplace transform of matrix variate.

1. INTRODUCTION. At first Herz (1955) has introduced the theory of matrix argument involving certain special functions such as Bessel function, confluent hypergeometric function etc. Later on James (1961) extended the theory of special functions with matrix argument involving zonal polynomial. James (1961) and Constantine (1963) have given series of papers on multivariate analysis. Mathai (1978) introduced Laplace transform of binomial distribution function for positive definite matrix. Recently Sethi and Mishra (1995) discussed 'Laplace Transform of the binomial distribution function ${}_1F_0\,(\alpha; - ;X)$ associated with zonal polynomial' of symmetric matrix. In this paper generalized hypergeometric distribution function associated with zonal polynomial has been used to define new formula by virtue of Laplace transform of matrix argument. It is very interesting that result earlier by the authors Sethi and Mishra (1995) and Mathai (1991) follows a special case.

All the matrices are assumed to $p \times p$ order symmetric positive definite. Unless otherwise stated. $A > 0$ means A is positive definite.

$R(.)$ means that real part of (.). $\int_0^I (.)\, dA$ means that (.) is integrated out over the $p \times p$ symmetric matrix A, such that $A > 0$ and $I - A > 0$, that in all the eigenvalues of A are between 0 and 1. The notation $\|(.)\|$ stands for the norms of $(.)$. Here since the matrices are symmetric positive definite the largest eigenvalues can be taken as the norms. $|(.)|$ stands for determination of $(.)$, $tr(.)$ denotes the *trace* of $(.)$.

2. ZONAL POLYNOMIALS. Let S be a positive definite symmetric $p \times p$ matrix and $\phi(S)$ a polynomial in the element of S. Then the transformation

$$\phi(S) \to \phi(L^{-1} S L'^{-1}) \quad L \in GL(P) \qquad \dots (2.1)$$

defines a representation of the real linear group $GL(p)$ in the vector space of all polynomials in S. The space V_k of homogeneous polynomial of degree K, in invariant under the given transformation and decomposes into the direct sum of irreducible subspace

$$V_k = \Sigma_k + V_{k,\,k}$$

where $K = (k_1, k_2, \dots, k_p)$; $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$ sums over all partitions of k into not more than p parts.

These subspaces are generated by the zonal polynomial $Z(S)$, being invariant under the orthogonal group i.e.

$$Z_k(HSH') = Z_k(S)\,, \quad H \in 0(p)\,.$$

They are homogeneous symmetric polynomials in the characteristic roots of S.

The fundamental property of the zonal polynomials is given by the following integral,

$$\int_{o(p)} C_k(H'TSH)\,dH = C_k(S)\,C_k(T)/C_k(I) \qquad \dots (2.2)$$

where I is the identity matrix of dH, is invariant Haar measure on the othogonal group.

Hypergeometric function of matrix will be defined as certain series of zonal polynomials. The equivalance of the two sets of functions follows from the important integral identity.

$$\int_{S>0} \exp(-\,tr\,RS)\,|S|^{-t-(p+1)/2}\,C_k(ST)\,dS$$

$$= \Gamma_p(t, k)\,C_k(R^{-1}T)\,|R|^{-t} \qquad \dots (2.3)$$

where $C_k(S)$ in the zonal polynomial corresponding to the partitions $k = (k_1, k_2, \dots, k_p)$; $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$ of the integer k, t is the complex number satisfying $R(t) > \frac{(p-1)}{2}$ and $\Gamma_p(t; k) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(t + k_i - \frac{(i-1)}{2})$, the integration is over the space of positive definite symmetric $p \times p$ matrices.

3. SOLUTION OF LAPLACE TRANSFORM OF THE DISTRIBUTION FUNCTION ${}_pF_q$

Theorem . If X, T and B are real positive definite symmetric matrix of order $p \times p$. If $k = (k_1, k_2, ..., k_p)$; $k_1 \geq k_2 \geq ... \geq k_p \geq 0$ runs over all partitions into not more than p parts, $C_k(S)$ the corresponding zonal polynomial. The Laplace transform of the generalized hypergeometric function ${}_pF_q$ is

$$L_f(T) = \frac{|T+B|^{-t} \, {}_pF_q\,[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; (T+B)^{-1}]}{|B|^{-t} \, {}_pF_q\,[\alpha_1, ... \alpha_p; \beta_1, ..., \beta_q; (B^{-1})]} \quad ... (3.1)$$

here the distribution function $f(x)$ is given as

$$f(x) = \frac{|X|^{t-(p+1)/2} \, e^{-tr(BX)} \, {}_pF_q\,[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X]}{\Gamma_p(t, k) \, |B|^{-t} \, {}_pF_q\,[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}]} \quad ... (3.2)$$

Proof : We consider the integral

$$\int_{x>0} |X|^{t-\frac{(p+1)}{2}} \, e^{-tr(BX)} \, {}_pF_q\,[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X] \, dX \quad ... (3.3)$$

where the generalized hypergeometric function ${}_pF_q$ in terms of zonal polynomial defined as

$$\begin{aligned} &{}_pF_q\,[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X] \\ &= \sum_{k=0}^{\infty} \sum_{k} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \, C_K(X) \end{aligned} \quad ... (3.4)$$

here the zonal polynomial $C_K(X)$ is the component of $e^{tr_{(X)}^{\,k}}$.

Then the equation (3.3) becomes

$$\int_{X>0} |X|^{t-(p+1)/2} \, e^{-tr\,(BX)} \, {}_pF_q\,[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X] \, dX$$

$$= \int_{X>0} |X|^{t-(p+1)/2} \, e^{-tr(BX)} \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \;\; k!} \, C_K(X) \, dX \quad ... (3.5)$$

$$= \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(BX)} C_k(X) dX$$

By the virtue of the integral

$$\int_{X>0} |X|^{t-\frac{(p+1)}{2}} e^{-tr(BX)} C_K(X)\, dX = \Gamma_p(t, k)\, |B|^{-t} C_K(B^{-1})$$

the equation (3.5) becomes

$$= \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \Gamma_p(t, k)\, |B|^{-t} C_K(B^{-1})$$

$$= \Gamma_p(t, k)\, |B|^{-t} \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, K!} C_K(B^{-1})$$

$$= \Gamma_p(t, k)\, |B|^{-t} {}_pF_q [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}] \qquad \text{... (3.6)}$$

Now *p.d.f.* is defined as

$$f(x) = \frac{|X|^{t-(p+1)/2} e^{-tr(BX)} {}_pF_q [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X]}{\Gamma_p(t, k)\, |B|^{-t} {}_pF_q [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}]}$$

for $X = X' > 0$, $B = B' > 0$, $\| X \| < 1$, $\| B \| < 1$ and $R(t) > (p-1)/2$

$= 0$, elsewhere ... (3.7)

$f(X) \geq 0 \;\forall\; X;\; \int f(X)\, dX = 1$

taking Laplace transform on both sides of equation (3.7), we have

$$\int_{X>0} e^{-tr(TX)} f(X)\, dX = \int_{X>0} e^{-tr(TX)} \frac{|X|^{t-(p+1)/2} e^{-tr(BX)}}{\Gamma_p(t, k)\, |B|^{-t}} \times \frac{{}_pF_q [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q X]\, dX}{{}_pF_q (\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1})}.$$

$$L_f(T) = \eta \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(T+B)X} {}_pF_q (\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X)\, dX$$

where $L_f(T) = \int_{x>0} e^{-tr(TX)} f(X)\, dX$

and $\eta = [\Gamma_p(t, k)\, |B|^{-t} {}_pF_q (\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}]^{-1}$

or $$L_f(T) = \eta \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(T+B)X}$$

$$\sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k \, k!} C_K(X) \, dX$$

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k \, k!} \int_{X>0} |X|^{t-(p+1)/2} \times e^{-tr\,(T+B)\,X} C_K(X) \, dX \quad \dots (3.8)$$

Putting

$(T+B)X = Y \qquad \Rightarrow tr(T+B)X = tr(Y)$

$X = (T+B)^{-1} Y \qquad \Rightarrow dX = |T+B|^{-(p+1)/2} \, dY$

$|X| = |T+B|^{-1} \, |Y| \Rightarrow |X|^{t-(p+1)/2} = |T+B|^{-t+(p+1)/2} |Y|^{t-(p+1)/2}$

when $X = 0 \qquad \Rightarrow Y = 0$.

Putting these values in equation (3.8) it becomes

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k \; k!} \int_{Y>0} |T+B|^{-t+(p+1)/2} \times |Y|^{t-(p+1)/2} e^{-tr\,(Y)} C_K\,[(T+B)^{-1} Y] \; |T+B|^{-(p+1)/2} \, dY$$

or $$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k \; k!} \int_{Y>0} |Y|^{-t+(p+1)/2} \times e^{-tr(Y)} C_K\,[(T+B)^{-1} Y] \, dY \quad \dots (3.9)$$

Constantine [6] has proved a integral relating zonal polynomial with gamma function as

$$\int_{X>0} e^{-tr(RX)} |X|^{t-(p+1)/2} C_K(TX) \, dX = \Gamma_p(t,k) \, C_K(R^{-1} T) \, |R|^{-t}$$

This yields the result $R = I$ this equation gives

$$\int_{X>0} |X|^{t-(p+1)/2} \; e^{-tr(X)} \; C_K(XT) \, dX = \Gamma_p(t,k) \, C_K(T) \quad \dots (3.10)$$

using the result (3.10) in equation (3.9), we get

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k \; k!} \; \frac{|T+B|^{-t}}{k!} \Gamma_p(t,k) \, C_k\,[(T+B)^{-1}]$$

Putting the value of η we get

$$L_f(T) = \frac{|T+B|^{-t}\,\Gamma_p(t,k)\sum\limits_{k=0}^{\infty}\sum\limits_{K}\dfrac{(\alpha_1)_k\ldots(\alpha_p)_k\,C_K\,[(T+B)^{-1}]}{(\beta_1)_k\ldots(\beta_q)_k\quad k!}}{\Gamma_p(t,k)\,|B|^{-1}\,{}_pF_q\,[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;B^{-1}]}$$

$$L_f(T) = \frac{|T+B|^{-t}\,{}_pF_q\,[\alpha_1,\ldots,\alpha_p\,;\beta_1,\ldots,\beta_q;(T+B)^{-1}]}{|B|^{-t}\,{}_pF_q\,[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;B^{-1}]} \qquad \ldots (3.11)$$

or $\quad L_f(T) = \left|\dfrac{T+B}{B}\right|^{-t}\chi$

where $\chi = \dfrac{{}_pF_q\,[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;(T+B)^{-1}]}{{}_pF_q\,[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;B^{-1}]}$

or $\quad L_f(T) = |I+B^{-1}\,T|^{-t}$

$\qquad = |I+B^{-1/2}\,T\,B^{-1/2}|^{-t}$

The moment generating function of $f(X)$ is

$$|I+B^{-1}\,T|^{-t} \text{ and } I-B^{-1}\,T>0.$$

4. SPECIAL CASES

(1) When we put $p=0$, $q=0$ in equation (3.11), the equation reduces to exponential series

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\;\frac{{}_0F_0\,[(T+B)^{-1}]}{{}_0F_0\,[(B)^{-1}]}.$$

(2) When $p=1$ and $q=0$ equation (3.11) is reduced to the Laplace transform of binomial function ${}_1F_0\,(\alpha\,;-X)$ as

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\;\frac{{}_1F_0\,[\alpha\,;-\,;(T+B)^{-1}]}{{}_1F_0\,[\alpha\,;-\,;(B^{-1})]}$$

(3) When $p=1$ and $q=1$, equation (3.11) is reduced to the Laplace transform of confluent hypergeometric function ${}_1F_1\,(\alpha;\gamma\,;X)$ as

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\;\frac{{}_1F_1\,[\alpha;\gamma;(T+B)^{-1}]}{{}_1F_1\,[\alpha;\gamma,(B^{-1})]}$$

(4) When $p=2$ and $q=1$ equation (3.11) reduces to the Laplace transform of Gaussian hypergeometric function ${}_2F_1\,(\alpha,\beta;\gamma,X)$ as

$$L_f(T) = \frac{|T+B|^{-t} \ {}_2F_1\,[\alpha, \beta; \gamma; (T+B)^{-1}]}{|B|^{-t} \ {}_2F_1\,[\alpha, \beta; \gamma; (B^{-1})]}$$

When $K = 0$ in all the investigations, we have probability density for gamma function due to Mathai and Saxena [4]

$$f(X) = \frac{|X|^{t-(p+1)/2} e^{-tr(BX)}}{\Gamma_p(t) \ |B|^{-t}}$$

where $X = X' > 0$ and $B = \beta' > 0$ and its Laplace transform is given as

$$L_f(T) = |I + B^{-1} T|^{-t} = |I + B^{-1/2} TB^{-1/2}|^{-t}$$

The moment generating function of $f(X)$ is

$$|I + B^{-1}T|^{-t}; I - B^{-1} T > 0.$$

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EFFECT OF SUSPENDED PARTICLES ON THERMAL INSTABILITY OF A ROTATING MAXWELLIAN VISCOELATIC FLUID IN POROUS MEDIUM

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ABSTRACT

The thermal instability of a rotating Maxwellian viscoelastic fluid permeated with suspended partiles, in porous medium, is considered. The presence of each-rotation, suspended particles and viscoelasticity brings oscillatory modes in the system which were nonexistent in their absence. For stationary convection, the Maxwellian viscoelastic fluid behaves like a Newtonian fluid and the medium permeability succeeds in stabilizing the thermal instability of a fluid-particle layer for certain wave numbers in the presence of rotation which were unstable in the absence of rotation. However, the suspended particles and the rotation are found to have destabilizing and stabilizing effects respectively on the thermal instability problem for stationary convection.

1. INTRODUCTION. A detailed account of the thermal instability in Newtonian fluids has been given by Chandrasekhar [1]. Lapwood [2] has studied the stability of convective flow in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [3]. When the fluid slowly percolates through the pores of the rock, the gross effect is represented by the well-known Darcy's law.

Chandra [4] observed in an air layer that convection occured at much lower gradients than predicted and appeared as irregular strips of elongated cells with fluid rising at the centre if the layer depth was less than 7 mm and called this motion "columnar instability". However, for layers deeper than 10 mm, a Bénard type cellular convection was observed. Chandra [4] added an aerosol to mark the flow pattern. Thus there is a decades-old contradiction between the theory and the experiment. Scanlon and Segel [5] have considered the effect of suspended particles on the onset of thermal convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure fluid was supplemented by that of the particles. The effect of suspended particles was thus found to destablize the layer.

In all the above studies, the fluid has been considered to be Newtonian. The stability of a horizontal layer of Maxwell's viscoelastic fluid heated from below has been investigated by Vest and Arpaci [6]. The nature of the instability and some factors may have different effects on viscoelastic fluids as compared to the Newtonian fluids. For example, Bhatia and Steiner [7] have considered the effect of a uniform rotation on the thermal instability of a Maxwell fluid and have found that rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid. Experimental demonstration by Toms and strawbridge [8] has revealed that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the the theoretical model of Oldroyd fluid. The thermal instability of an Oldroydian viscoelastic fluid has been considered in the presence of rotation (Eltayeb [9], Sharma [10]).

The present paper attempts to study the thermal instability of rotating Maxwellian viscoelastic fluid, permeated with suspended particles, in porous medium. The knowledge regarding fluid-particle mixtures is not commensurate with their scientific and industrial importance. The analysis would be relevant to the stability of some polymer solutions and the problem finds its usefulness in several geophysical situations (e.g. convection in petroleum deposits below the surface of Earth where the Earth's rotation plays a significant role on convection and in extraction of energy) and in chemical technology. These aspects form the motivation for the present study.

2. PERTURBATION EQUATIONS. Consider an infinite horizontal layer of a Maxwellian viscoelastic fluid permeated with suspended particles and bounded by the planes $z = 0$ and $z = d$ in porous medium. This layer is heated from below such that a steady adverse temperature gradient $\beta (= |\frac{dT}{dz}|)$ is maintained. The system is acted on by gravity force $\mathbf{g}(0, 0, -g)$ and uniform rotation $\vec{\Omega}(0, 0, \Omega)$. Let $T_{ij}, z_{ij}, e_{ij}, \delta_{ij}, \mu, \lambda, p, u_i, x_i$ and $\frac{d}{dt}$ denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the Kronecker delta, the viscosity, the stress relaxation time, the isotropic pressure, the velocity vector, the position vector and the mobile operator. Then the Maxwellian viscoelastic fluid is described by the constitutive relations

$$T_{ij} = -p\delta_{ij} + z_{ij},$$

$$\left(1 + \lambda\frac{d}{dt}\right) z_{ij} = 2\mu e_{ij},$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \ldots(1)$$

When the fluid slowly percolates through the pores of the rock, the gross effect is represnetd by the usual Darcy's law. As a consequence, the resistance term $-\left(\frac{\mu}{k_1}\right)\boldsymbol{v}$ will replace the usual viscous term in the equations of motion. Here k_1 is the permeability of the medium and $\boldsymbol{v}$ is the filter velocity of the fluid, the porous medium being homogeneous and isotropic.

Let ρ, μ, p, $\vec{u}(u, v, w)$ and $\vec{\Omega}(0, 0, \Omega)$ denote respectively, the density, the viscosity, the pressure, the velocity of pure fluid and uniform rotation; $v(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and the number density of the particles respecitvely. $k = 6\pi\rho\upsilon\eta$, where η is the particle radius is the Stoke's drag coefficient $v = (l, r, s)$, $\bar{x} = (x, y, z)$, $\bar{\lambda} = (0, 0, 1)$ and ε is the medium porosity. Then the equations of motion and continuity for the rotating Maxwellian viscoelastic fluid permeated with suspended particles in a porous medium are

$$\left(1 + \lambda\frac{d}{dt}\right)\frac{\rho}{\varepsilon}\left[\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{\varepsilon}(\vec{\boldsymbol{u}}.\nabla)\vec{\boldsymbol{u}}\right] = \left(1 + \lambda\frac{d}{dt}\right)\left[-\nabla p - \rho g\bar{\lambda} + \frac{kN}{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}) + \frac{2\rho}{\varepsilon}(\boldsymbol{u} \times \vec{\Omega})\right] - \frac{\mu}{k_1}\boldsymbol{u} \qquad \ldots(2)$$

$$\nabla . \boldsymbol{u} = 0. \qquad \ldots(3)$$

In the equations of motion (2), the presence of partilces adds an extra force term, proportional to the velocity difference between particles and fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion of the particles. The distances between particles are assumed quite large compared with their diameter so that interparticle reactions are ignored. The effects of pressure, gravity, Darcian force and Coriolis force on the suspended particles are negligibly small and therefore ignored. If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions are

$$mN\left[\frac{\partial \boldsymbol{v}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\,.\,\nabla)\boldsymbol{v}\right]=kN\,(\boldsymbol{u}-\boldsymbol{v}), \qquad ...(4)$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla.\,(N\boldsymbol{v})=0. \qquad ...(5)$$

Let c, c_{pt}, T and q denote respectively the heat capacity of the fluid, the heat capacity of the particles, the temperature and the "effective thermal conductivity" of the pure fluid. When the fluid and the particles are in thermal equilibrium, the equation of heat conduction gives

$$[\rho c\varepsilon+\rho_s c_s(1-\varepsilon)]\frac{\partial T}{\partial t}+\rho c(\boldsymbol{u}.\nabla)T+mNc_{pt}\left(\varepsilon\frac{\partial}{\partial t}+\boldsymbol{v}\,.\,\nabla\right)T$$

$$=q\nabla^2T, \qquad ...(6)$$

where ρ_s, c_s stand for the density and the heat capacity of the solid matrix, respectively.

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution N_0. The initial state

$$\boldsymbol{u}=0,\ \boldsymbol{v}=0,\ T=-\beta z,\ N_0=\text{constant} \qquad ...(7)$$

is an exact solution to the governing equations. The equatuion of state for the fluid is

$$\rho=\rho_0\,[1-\alpha(T-T_0)], \qquad ...(8)$$

where ρ_0, T_0 are respectively, the mean density and the temperature of the clean fluid at the bottom surface $z=0$ and α is the coefficient of thermal expansion.

Let $\delta\rho, \delta p, \theta, \boldsymbol{u}(u, v, w), \boldsymbol{v}(l, r, s)$ and N denote respectively perturbations in density ρ, pressure p, temperature T, fluid velocity (zero initially), particle velocity (zero initially) and number denisty N_0. Then the linearized perturbation equations of the fluid-particle layer are

$$\left(1+\lambda\frac{d}{dt}\right)\frac{\rho_0}{\varepsilon}\frac{\partial \boldsymbol{u}}{\partial t}=\left(1+\lambda\frac{d}{dt}\right)\left[-\nabla\delta p+g\rho_0\alpha\theta\vec{\lambda}+\frac{kN_0}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\right.$$

$$+ \frac{2\rho_0}{\varepsilon} (\boldsymbol{u} + \vec{\Omega}) \Big] - \frac{\mu}{k_1} \boldsymbol{u} \qquad ...(9)$$

$$\nabla . \boldsymbol{u} = 0 \qquad ...(10)$$

$$mN_0 \frac{\partial \boldsymbol{v}}{\partial t} = kn_0(\boldsymbol{u} - \boldsymbol{v}), \qquad ...(11)$$

$$\varepsilon \frac{\partial M}{\partial t} + \nabla . \boldsymbol{v} = 0 \qquad ...(12)$$

and

$$(E + h\varepsilon) \frac{\partial \theta}{\partial t} + (\boldsymbol{u} . \nabla) T + h(\boldsymbol{v} . \nabla) T = \frac{q}{\rho_0 c} \nabla^2 \theta \qquad ...(13)$$

where

$$M = \frac{N}{N_0}, E = \varepsilon + (1 - \varepsilon) \frac{\rho_s c_s}{\rho c},$$

$$h = \frac{mN_0}{\rho_0} \frac{c_{pt}}{c} = f \frac{c_{pt}}{c} \text{ and } f = \frac{mN_0}{\rho_0}.$$

Now introducing the non-dimensional quantities defined by

$$z = z^* d, t = \frac{d^2}{x} t^*, u = \frac{x}{d} u^*, p = \frac{\rho v x}{d^2} p^*, \theta = (\beta d) \theta^*, \Omega = \frac{x}{d^2} \Omega^* \quad ..(14)$$

and omitting the asterisks for simplicity, (9)-(13) in the non-dimensional forms become

$$(1 + \lambda' \frac{d}{dt}) N_p^{-1} \frac{\partial \boldsymbol{u}}{\partial t} = (1 + \lambda' \frac{d}{dt}) \Big[- \nabla \delta p + N_R \theta \vec{\lambda} + \omega(\boldsymbol{v} - \boldsymbol{u})$$

$$+ 2N_p^{-1} (\boldsymbol{u} \times \vec{\Omega}) \Big] - p^{-1} \boldsymbol{u}, \qquad ...(15)$$

$$\nabla . \boldsymbol{u} = 0, \qquad ...(16)$$

$$\left(z \frac{\partial}{\partial t} + 1 \right) \vec{v} = \vec{u}, \qquad ...(17)$$

$$(E + h\varepsilon) \frac{\partial \theta}{\partial t} = (w + hs) + \nabla^2 \theta, \qquad ...(18)$$

where $N_p = \frac{\varepsilon v}{x}$ is the modified Prandtl number, $N_R = \frac{g \alpha \beta d^4}{\nu x}$ is the Rayleigh number, $\omega = \frac{k N_0 d^2}{\rho_0 v \varepsilon}$, $f = \frac{m N_0}{\rho_0} = z w N_p$ is the mass fraction, $z = \frac{m x}{k d^2}$, $p = \frac{k_1}{d^2}$ and $\lambda' = \frac{\lambda x}{d^2}$. Also w, s are the vertical fluid and particle velocities, v is the kinematic viscosity of the fluid and x is the thermal diffusivity.

Let us consider the case of two free surfaces having uniform temperatures. The scaled boundary conditions appropriate for the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \text{ at } z = 0 \text{ and } 1. \qquad ...(19)$$

Eliminating $\boldsymbol{v}$ from (15), (17) and (18), the fluid and heat equations become

$$[L_1(1 + \lambda' \frac{d}{dt}) + p^{-1} L_2]\zeta = (1 + \lambda' \frac{d}{dt})(2L_2 N_p^{-1} \Omega D w), \qquad ...(20)$$

$$[L_1(1 + \lambda' \frac{d}{dt}) + p^{-1} L_2]\nabla^2 w = (1 + \lambda' \frac{d}{dt})L_2(N_R \nabla_1^2 \theta - 2N_p^{-1} \Omega D\zeta] \qquad ...(21)$$

and $$L_2[c(E + h\varepsilon)\frac{\partial}{dt} - \nabla^2]\theta = [z\frac{\partial}{\partial t} + H]w, \qquad ...(22)$$

where we have taken twice the curl of fluid equations and taking z-components only, and $\zeta = (\nabla \times \boldsymbol{u})_z$ is the z-components of vorticity.

$$L_1 = N_p^{-1}\left(z\frac{\partial^2}{\partial t^2} + F\frac{\partial}{\partial t}\right), L_2 = z\frac{\partial}{\partial t} + 1, F = f + 1,$$

$$H = h + 1, \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Following the normal mode analysis we assume that the solutions of (20)-(22) are given by

$$[w, \zeta, \theta] = [W(z), Z(z), \Theta(z)] \exp(ik_x x + ik_y y + nt), \qquad ...(23)$$

where n is the growth rate and $k = (k_x^2 + k_y^2)^{1/2}$ is the wave number of the disturbance.

Equations (20)-(22), with the help of expression (23), take the forms

$$[(1 + \lambda' n)L_1 + p^{-1} L_2]z = 2(1 + \lambda' n)L_2 N_p^{-1} \Omega DW, \qquad ...(24)$$

$$[(1 + \lambda' n)L_1 + p^{-1} L_2](D^2 - k^2)W = -(1 + \lambda' n)L_2 N_k k^2 \Theta$$
$$- 2(1 + \lambda' n)L_2 N_p^{-1} \Omega Dz \qquad ...(25)$$

$$L_2 [(E - h\varepsilon)n - (d^2 - k^2)\Theta = (zn + H)W \qquad ...(26)$$

Eliminating z and Θ between (24)-(26), we obtain

$$[(1+\lambda' n)L_1 + p^{-1}L_2]^2\ [(E+h\varepsilon)n - (D^2-k^2)][(D^2-k^2)W$$

$$= -(1+\lambda' n)N_R k^2[(1+\lambda' n)L_1 + p^{-1}L_2](zn+H)W$$

$$-4(1+\lambda' n)^2 L_2^2 N_p^{-2}\Omega^2\ [(E+h\varepsilon)n - (D^2-k^2)]D^2W \qquad \text{...(27)}$$

where $L_1 = N_p^{-1}(zn^2 + F_n)$, $L_2 = (1+zn)$ and $D = \frac{d}{dz}$.

Here we consider the case of two free boundaries. The dimensionless boundary conditions appropriate for the problem are

$$W = D^2W = Dz = \Theta = 0 \text{ at } z = 0 \text{ and } 1. \qquad \text{...(28)}$$

The solution of (27) satisfying (28) is given by

$$W = W_0 \sin \pi z, \qquad \text{...(29)}$$

where W_0 is a contant. Substituting (29) in (27), we obtain the dispersion relations

$$N_R = \frac{[(E+h\varepsilon)n + (\pi^2+k^2)] \times [\{(1+\lambda' n)L_1 + p^{-1}L_2\}^2(\pi^2+k^2) + 4\pi^2 L_2^2\Omega^2 N_p^{-2}(1+\lambda' n)^2]}{k^2(1+\lambda' n)[(1+\lambda' n)L_1 + p^{-1}L_2][zn+H]} \qquad \text{...(30)}$$

3. THE STATIONARY CONVECTION. When the instability sets in as stationary convection, the marginal state will be characterized by $n = 0$. Putting $n = 0$, the dispersion relation (30) reduces to

$$N_R = \frac{(\pi^2+k^2)^2}{Hpk^2} + 4N_p^{-2}\Omega^2\pi^2(\pi^2+k^2)\frac{p}{Hk^2} \qquad \text{...(31)}$$

Thus for stationary convection, the stress relaxation time parameter vanishes with n and the Maxwell fluid behaves like a Newtonian fluid.

Equation (31) can be written as

$$N_R = \frac{(\pi^2+k^2)^2}{Hpk^2} + 4\Omega^2\pi^2(\pi^2+k^2)\frac{p}{Hk^2N_p^2} \qquad \text{...(32)}$$

To study the effects of suspended particles, medium permeability and rotation, we examine the natures of $\frac{dN_R}{dH}$, $\frac{dN_R}{dP}$ and $\frac{dN_R}{d\Omega}$ respectively. Equation (32) gives

$$\frac{dN_R}{dH} = -\left[\frac{(\pi^2+k^2)^2}{H^2k^2p} + \frac{4\Omega^2\pi^2(\pi^2+k^2)p}{H^2k^2N_p^2}\right], \qquad ...(33)$$

which is negative. The effect of suspended particles is, thus, destabilizing on thermal instability of fluid-particle layer in the presence of uniform rotation in porous medium.

From (32), it follows that

$$\frac{dN_R}{dp} = \left[\frac{(\pi^2+k^2)^2}{Hk^2p^2} - \frac{4\Omega^2\pi^2(\pi^2+k^2)}{Hk^2N_p^2}\right], \qquad ...(34)$$

which is, positive if

$$\left(\frac{\pi^2+k^2}{p^2}\right) < \frac{4\Omega^2\pi^2}{N_p^2}$$

and is negative if

$$\left(\frac{\pi^2+k^2}{p^2}\right) > \frac{4\Omega^2\pi^2}{N_p^2}.$$

Thus, the medium permeability has been stabilizing and destabilizing effects depending upon the values of the various parameters. In the absence of rotation, the medium permeability has a destabilizing effect since for this case

$$\frac{dN_R}{dp} = -\frac{(\pi^2+k^2)^2}{Hp^2k^2}, \qquad ...(35)$$

which is always negative. The medium permeability, thus, succeeds in stabilizing the thermal instability of fluid-particle layer for certain wave numbers in the presence of rotation, which were unstable in the absence of rotation.

Equation (32) also yields

$$\frac{dN_R}{d\Omega} = \frac{8\Omega\pi^2\,(\pi^2+k^2)p}{Hk^2N_p^2} \qquad ...(36)$$

which is always positive. This shows that the rotation has a stabilizing effect on the system.

4. STABILITY OF THE SYSTEM AND OSCILLATORY MODES

Multiplying equation (25) by W^*, the complex conjugate of W, integrating over the range of z and making use of (24), (26) together with the boundary conditions (28), we obtain

$$L_2^*[zn^* + H][1 + \lambda' n^*][1 + \lambda' n)L_1 + p^{-1}L_2]I_1 = [(1 + \lambda' n^*)(1 + \lambda' n)$$
$$L_2 L_2^{2*} N_R k^2][(E + h\varepsilon)n^* I_2 + I_3] - L_2[zn^* + H][1 + \lambda' n]$$
$$[(1 + \lambda' n^*)L_1^* + p^{-1}L_2^*]I_4, \qquad \ldots(37)$$

where

$$I_1 = \int_0^1 (|DW|^2 + k^2|W|^2)\,dz,$$
$$I_2 = \int_0^1 |\Theta|^2\,dz,$$
$$I_3 = \int_0^1 (|D\Theta|^2 + k^2|\Theta|^2)\,dz,$$
$$I_3 = \int_0^1 |z|^2\,dz, \qquad \ldots(38)$$

which are all positive definite. Putting $n = in_0$, where n_0 is real, in (37) and equating the imaginary parts on both sides, we obtain

$$n_0\,[n_0^4\{N_p^{-1}z^2\lambda'^2(1 + H - F)\}I_1 + \{N_R k^2(E + h\varepsilon)\lambda'^2 z^2\}I_2$$
$$+ \{N_R k^2 \lambda'^2 z^3\}I_3 + \{N_p^{-1}z^2\lambda'^2(1 - H - F)\}I_4]$$
$$+ n_0^2\,[\{H\lambda' + Hz + z) < N_p^{-1}\,(z + \lambda' F) > - (H\lambda' + \lambda' + z)$$
$$< N_p^{-1}\,Fz + z^2 p^{-1} > - N_p^{-1}\lambda' zH + z^2\lambda' p^{-1}\}I_1$$
$$+ \{N_R k^2\,(E + h\varepsilon)(z^2 + \lambda'^2)\}I_2 + \{N_R k^2\,(z^3 + \lambda'^2 z)\}I_3$$
$$+ \{N_p^{-1}z^2\,(1 - F - H) - N_p^{-1}\,HF\lambda'^2 + Hz^2\lambda' p^{-1} - z^3 p^{-1}\}I_4]$$
$$+ \{N_R k^2 z\}I_3 + \{H\lambda' p^{-1} - N_p^{-1}HF - zp^{-1}\}\,I_4] = 0. \qquad \ldots(39)$$

Equation (39) implies that $n_0 = 0$ or $n_0 \neq 0$, which means that the modes may be oscillatory or non-oscillatory. In the absence of rotation, suspended particles and for Newtonian fluid ($\lambda = 0$), (39) reduces to

$$n_0\left[\frac{I_1}{N_p} + N_k k^2 E I_2\right] = 0, \qquad \ldots(40)$$

and terms in brackets are positive definite. Thus $n_0 = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for a porous medium in the absence of rotation, suspended particles and for Newtonian fluid. This result is true for the porous as well as non-porous (Chandrasekhar [1]) medium. The presence of each-rotation, suspended particles and viscoelasticity, brings oscillatory modes (as n_0 may not be zero) which were non-existent in their absence.

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SPECIAL, UNION AND HYPERASYMPTOTIC CURVES OF A TACHIBANA HYPERSURFACE

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ABSTRACT

Special, union and hyperasymptotic curves of a Riemannian hypersurface have been studied by Singh [1]. The object of this paper is to be investigate these curves in a Tachibana hypersurface.

1. INTRODUCTION. In an $(n + 1)$ dimensional complex space C_{n+1} referred to an allowable coordinate system

$$(z^i, z^{\bar{\imath}}) \equiv (z^1, z^2, \ldots, z^{n+1}, z^{\bar{1}}, z^{\bar{2}}, \ldots, z^{\overline{n+1}})^* ,$$

let us introduce the metric defined by the positive Hermitian form

$$ds^2 = 2g_{i\bar{\jmath}}(z^i, z^{\bar{\imath}})\, dz^i\, dz^{\bar{\jmath}} . \qquad \ldots (1.1)$$

If the tensor $g_{i\bar{\jmath}}$ also satisfies the Kaehler's condition

$$\frac{\partial g_{i\bar{\jmath}}}{\partial z^{k}} = \frac{\partial g_{i\bar{k}}}{\partial z^{\bar{\jmath}}} , \qquad \ldots (1.2)$$

then the complex space with metric satisfying the condition (1.2) is called Kaehler space. We always assume the self adjointness of the indices [2].

If a Kaehler space satisfies the conditions

$$F^{h}_{j,\,i} = 0,\ F^{\bar{h}}_{\bar{\jmath},\,\bar{\imath}} = 0, \qquad \ldots (1.3a)$$

$$F_i = -F^{k}_{i,\,j} = 0,\ F_{\bar{\imath}} = -F^{\bar{k}}_{\bar{\imath},\,\bar{\jmath}} = 0. \qquad \ldots (1.3b)$$

where the comma (,) followed by an index denotes the covarient differentiation w.r.t. the Christoffel's symbols formed with metric tensor, then the space is called an $(n + 1)$ dimensional Tachibana space and is denoted by T^{c}_{n+1}.

The angle θ between two self adjoint vectors $\bar{u}$ and $\bar{v}$, whose components are $(r^i, r^{\bar{\imath}})$ and $(s^i, s^{\bar{\imath}})$ is given by

$$\cos\theta = \frac{g_{i\bar{j}}\, r^i s^{\bar{j}} + g_{ij}\, r^i s^j}{2\sqrt{(g_{i\bar{j}}\, r^i r^j)}\,\sqrt{(g_{\bar{i}\bar{j}}\, s^i s^j)}} \qquad \dots (1.4)$$

Let us consider an analytic hypersurface T_n^c of T_{n+1}^c. If $(u^\alpha, u^{\bar{\alpha}}) \equiv (u^1, \dots, u^n, u^{\bar{1}}, \dots, u^{\bar{n}})$ denote the coordinates of a point in T_n^c, the equations of the analytic hypersurface T_n^c may be written in the form

$$z^i = z^i\,(u^\alpha),\; z^{\bar{i}} = z^{\bar{i}}\,(u^{\bar{\alpha}}) \qquad \dots (1.5)$$

We quote below some fundamental formulae from [3] which will be used in the later part of this paper. Suppose that $g_{\alpha\beta}$ is the fundamental metric tensor of T_n^c, then we have

$$g_{\alpha\beta} = g_{i\bar{j}}\, B^i_\alpha\, B^{\bar{j}}_\beta \qquad \dots (1.6)$$

where $B^i_\alpha = \dfrac{\partial z^i}{\partial u^\alpha}$, $B^{\bar{i}}_\beta = \dfrac{\partial z^{\bar{i}}}{\partial u^\beta}$.

If $(N^i, N^{\bar{i}})$ be the components of unit normal vector to the hypersurface, then

$$2g_{i\bar{j}}\, N^i N^{\bar{j}} = 1 \qquad \dots (1.7)$$

and $$g_{i\bar{j}}\, N^i B^{\bar{j}}_\beta = 0,\; g_{i\bar{j}} N^{\bar{j}}\, B^i_\alpha = 0\,. \qquad \dots (1.8)$$

The unit vector $(\xi^i, \xi^{\bar{i}})$ orthogonal to $\left(\dfrac{dz^i}{ds}, \dfrac{dz^{\bar{i}}}{ds}\right)$ is given by

$$g_{i\bar{j}} \frac{dz^i}{ds}\, \xi^{\bar{j}} + g_{\bar{i}j} \frac{dz^{\bar{i}}}{ds}\, \xi^j = 0 \qquad \dots (1.9)$$

and $$2\, g_{i\bar{j}}\, \xi^i \xi^{\bar{j}} = 1\,.$$

Consider a curve $C : z^i = z^i(s)$, $z^{\bar{i}} = z^{\bar{i}}\,(s)$ (where s is real) of T_n^c. The components $\dfrac{dz^i}{ds}\left(\text{or } \dfrac{dz^{\bar{i}}}{ds}\right)$ and $\dfrac{du^\alpha}{ds}\left(\text{or } \dfrac{du^{\bar{\alpha}}}{ds}\right)$ of the unit tangent vectors of C with respect to the enveloping space and the hypersurface are related by

$$\frac{dz^i}{ds} = B^i_\alpha\, \frac{du^\alpha}{ds}\,, \qquad \dots (1.11a)$$

$$\frac{dz^{\bar{i}}}{ds} = B^{\bar{i}}_{\bar{\alpha}}\, \frac{du^{\bar{\alpha}}}{ds} \qquad \dots (1.11b)$$

If $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vectors with respect to T^c_{n+1} and T^c_n respectively, then

$$q^i = B^i_\alpha p^\alpha + K_n N^i \qquad \text{... (1.12a)}$$

$$q^{\bar{i}} = B^{\bar{i}}_{\bar{\alpha}} p^{\bar{\alpha}} + \bar{K}_n N^{\bar{i}} . \qquad \text{... (1.12b)}$$

where K_n is the component of the normal curvature of T^c_n in the direction of the curve C and $\bar{K}_n$ is its complex conjugate, and

$$\left.\begin{array}{lll} q^i = dz^i/ds, & \beta\, du^\beta/ds + dz^i/ds, & \bar{\beta}\, du^\beta/ds \\ q^{\bar{i}} = dz^{\bar{i}}/ds, & \beta\, du^\beta/ds + dz^{\bar{i}}/ds, & \bar{\beta}\, du^\beta/ds \\ p^\alpha = du^\alpha/ds, & \beta\, du^\beta/ds + du^\alpha/ds, & \bar{\beta}\, du^\beta/ds \\ p^{\bar{\alpha}} = du^{\bar{\alpha}}/ds, & \beta\, du^\beta/ds + du^{\bar{\alpha}}/ds, & \bar{\beta}\, du^{\bar{\beta}}/ds \end{array}\right\} \text{... (1.13)}$$

$$B^i_{\alpha,\beta} = \Omega_{\alpha\beta} N^i, \qquad B^{\bar{i}}_{\bar{\alpha},\bar{\beta}} = \Omega_{\bar{\alpha}\bar{\beta}} N^{\bar{i}}$$

$$K_n = \Omega_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}, \qquad \bar{K}_n = \Omega_{\bar{\alpha}\bar{\beta}} \frac{du^{\bar{\alpha}}}{ds} \frac{du^{\bar{\beta}}}{ds}.$$

$\Omega_{\alpha\beta}$ and $\Omega_{\bar{\alpha}\bar{\beta}}$ are the components of the second fundamental tensors of the hypersurface, $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vector of the curve with respect to T^c_{n+1} and T^c_n respectively.

2. Union and Special Curves in T^c_n. The first two of Frenet's formulae of a curve in T^c_{n+1} are given by

$$\frac{\delta\, \eta^i_{(0)}}{\delta s} = K_{(1)}\, \eta^i_{(1)} \text{ and } \frac{\delta \eta^i_{(1)}}{\delta s} = -K_{(1)}\, \eta^i_{(0)} + K_{(2)}\, \eta^i_{(2)} \qquad \text{... (2.1)}$$

and their conjugates, where

$(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)}) = (dz^i/ds, dz^{\bar{i}}/ds)$, $(\eta^i_{(1)}, \eta^{\bar{i}}_{(1)})$ and $(\eta^i_{(2)}, \eta^{\bar{i}}_{(2)})$ are the components of unit tangent, unit principal normal vector and unit first binormal vector, $K_{(1)}$ and $K_{(2)}$ are the first and second curvatures of the curve in the embedding space, $\delta/\delta s$ is the usual covariant differentiation.

Consider two congruences of curves λ and $\bar{\mu}$ given at any point of T^c_n by

$$\lambda^i = \gamma^\alpha B^i_\alpha + cN^i \qquad \text{... (2.2a)}$$

its conjugate,

$$\mu^i = s^\alpha B^i_\alpha + DN^i \qquad \text{... (2.2b)}$$

and its conjugate.

Let Γ be a special curve relative to the congruence $\bar{\lambda}$ and an union curve relative to the congruence $\bar{\mu}$. A curve of the congruence is said to be a special curve relative to the congruence $\bar{\lambda}$ if the vector $(\lambda^i, \lambda^{\bar{i}})$ lies in a variety spanned by the vectors $(p^{\alpha}B^i_{\alpha}, p^{\bar{\alpha}}\, B^{\bar{i}}_{\bar{\alpha}})$ and $(q^i, q^{\bar{i}})$ and a curve of the congruence is said to be union curve relative to the congruence $\bar{\mu}$ if the vector $(\mu^i, \mu^{\bar{i}})$ lies in a variety spanned by the vectors $(dz^i/ds, dz^{\bar{i}}/ds)$ and $(q^i, q^{\bar{i}})$. In other words

$$(\lambda^i, \lambda^{\bar{i}}) = u(p^{\alpha}\, B^i_{\alpha}, p^{\bar{\alpha}}\, B^{\bar{i}}_{\bar{\alpha}}) + w(q^i, q^{\bar{i}})$$

and $$(\mu^i, \mu^{\bar{i}}) = x(dz^i/ds, dz^{\bar{i}}/ds) + y(q^i, q^{\bar{i}})$$

which yield

$$\lambda^i = up^{\alpha}\, B^i_{\alpha} + wq^i \text{ and its conjugate,} \qquad \text{... (2.3a)}$$

$$\mu^i = x\, dz^i/ds + yq^i \text{ and its conjugate.} \qquad \text{... (2.3b)}$$

From equations (1.12a), (2.2a), (2.3a), (2.3b) and their conjugates one obtains

$$r^{\alpha} = (u + w)\, p^{\alpha}, C = wK_n \qquad \text{... (2.4)}$$

$$s^{\alpha} = x\, dx^{\alpha}/ds + yp^{\alpha}, D = yK_n \qquad \text{... (2.5)}$$

and their conjugate relations.

We define

$$R^2 = 2g_{\alpha\bar{\beta}}\, \gamma^{\alpha}\, \gamma^{\bar{\beta}}, S^2 = 2g_{\alpha\bar{\beta}}\, s^{\alpha}\, s^{\bar{\beta}}, k^2_{(1)} = 2g_{\alpha\bar{\beta}}\, p^{\alpha}p^{\bar{\beta}} \qquad \text{... (2.6)}$$

where $k_{(1)}$ is the first curvature of the curve in T^c_n. Using (1.9), (2.4), its conjugate, (2.5) and its conjugate, we obtain from these equations

$$K_n\, S \cos\theta = Dk_{(1)} \qquad \text{... (2.7)}$$

and its conjugate.

From this equation and the equations

$$1 = 2\, g_{i\bar{j}}\, \mu^i\, \mu^{\bar{j}} = S^2 + D\bar{D} \qquad \text{... (2.8)}$$

$$K^2_{(1)} = 2g_{i\bar{j}}\, q^i\, q^{\bar{j}} \qquad \text{... (2.9)}$$

where $$K^2_{(1)} = k^2_{(1)} + K_n\, \bar{K}_n$$

We obtain

$$|K_n| = \{eK_{(1)}\, (1 - S^2)^{1/2}\}/(1 - S^2 \sin^2\theta)^{1/2} \qquad \text{... (2.10)}$$

and $$k_{(1)} = \{eK_{(1)}\, S\cos\theta\}/(1 - S^2\sin^2\theta)^{1/2} \qquad \text{... (2.11)}$$

where $e = \pm 1$ in order that $e \cos\theta$ be non-negative.

Theorem 2.1 . If a special curve relative to a fixed congruence $\bar\lambda$ is an union curve relative to another fixed congruence $\bar\mu$, then the modulus of the normal and the first curvatures with respect to T^c_n at a given point of the curve are proportional to its first curvature with respect to T^c_{n+1}.

Theorem 2.2 . If the components of the vector fields $\bar\lambda$ and $\bar\mu$ tangent to the hypersurface are in the same direction, then the ratio of the two first curvatures is equal to the magnitude of the tangential (to the hypersurface) component of $\bar\mu$.

Proof . $\theta = 0$ proves both the theorems.

3. HYPERASYMPTOTIC CURVES IN T^c_n . A curve of the hypersurface is said to be an hyperasymptotic curve relative to the congruence $\bar\mu$, if the vector $(\mu^i, \mu^{\bar i})$ lies in the variety spanned by the vectors $(\eta^i_{(0)}, \eta^{\bar i}_{(0)})$ and $(\eta^i_{(2)}, \eta^{\bar i}_{(2)})$.

In other words

$$(\mu^i, \mu^{\bar i}) = X_{(1)}\,(\eta^i_{(0)}, \eta^{\bar i}_{(0)}) + Y_{(1)}\,(\eta^i_{(2)}, \eta^{\bar i}_{(2)})$$

which yields

$$\mu^i = X_{(1)}\,\eta^i_{(0)} + Y_{(1)}\eta^i_{(2)} \text{ and its conjugate.} \qquad \dots (3.1)$$

From the equation (2.1) and its conjugate we deduce

$$\frac{\delta q^i}{\delta s} = -K^2_{(1)}\,\eta^i_{(1)} + \frac{d}{ds}\log K_{(1)}\,q^i + K_{(1)}\,K_{(2)}\,\eta^i_{(2)} \qquad \dots (3.2)$$

and its conjugate.

Another expression for $\left(\frac{\delta q^i}{\delta s}, \frac{\delta q^{\bar i}}{\delta s}\right)$ will be obtained from the equation (1.12 a, b) after using

$$\frac{\delta N^i}{\delta s} = -\frac{1}{2}\,\Omega_{\bar\beta\,\bar\gamma}\,g^{\beta\alpha}\,B^i_\alpha\,\frac{dx^{\bar\gamma}}{ds} \text{ and its conjugate.} \qquad \dots (3.3)$$

Equations (1.12a), (2.1), (3.1), (3.2) and their conjugates yield

$$s^\alpha = X_{(1)}\,\frac{dx^\alpha}{ds} + z\left(\frac{\delta p^\alpha}{\delta s} - K_{n/2}\,\Omega_{\bar\beta\,\bar\gamma}\,g^{\bar\beta\alpha}\,\frac{dx^{\bar\gamma}}{ds} - p^\alpha\,\frac{d}{ds}\log K_{(1)} + K^2_{(1)}\,\frac{dx^\alpha}{ds}\right) \qquad \dots (3.4)$$

$$D = z\left(\Omega_{\alpha\beta}\, p^{\alpha} \frac{dx}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds} \log k_{(1)} \right) \qquad \ldots (3.5)$$

and their conjugates, where $z = Y_{(1)}/K_{(1)}\, K_{(2)}$.

Let $(\xi^{\alpha}_{(0)}, \xi^{u}_{(0)}, \xi^{\alpha}_{(1)}, \xi^{\overline{\alpha}}_{(1)})$ and $(\xi^{\alpha}_{(2)}, \xi^{\overline{\alpha}}_{(2)})$ be the unit tangent, unit principal, normal, unit first binormal vectors, $k_{(1)}$ and $k_{(2)}$ be the first and second curvatures of the curve with respect to the hypersurface. We obtain from Frenet's formulae with respect to T^{c}_{n}.

$$\frac{\delta p^{\alpha}}{\delta s} = -k^{2}_{(1)} \frac{dx^{\alpha}}{ds} + p^{\alpha} \frac{d}{ds} \log k_{(1)} + k_{(1)} k_{(2)}\, \xi^{\alpha}_{(2)} \qquad \ldots (3.6)$$

and its conjugate.

Equations (3.4), (3.5), (3.6) their conjugates and the definition

$$\cos\phi \left\{ q_{\alpha\overline{\beta}}\, s^{\alpha} \frac{dx^{\overline{\beta}}}{ds} + g_{\overline{\alpha}\beta}\, s^{\overline{\alpha}} \frac{dx^{\overline{\beta}}}{ds} \right\} \quad S$$

yield

$$S \cos\phi = X(1) \qquad \ldots (3.7)$$

and $$\left(s^{\alpha} - S\cos\phi \frac{dx^{\alpha}}{ds} \right)\left[\Omega_{\gamma\beta}\, p^{\gamma} \frac{dx^{\beta}}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds} \log k_{(2)} \right]$$

$$= D\left[K_n \overline{K}_n \frac{dX^{\alpha}}{ds} + p^{\alpha} \frac{d}{ds} \log \frac{k_{(1)}}{K_{(1)}} + k_{(1)}\, k_{(2)}\, \xi^{\alpha}_{(2)} - K_n / 2\, \Omega_{\overline{\beta}\,\overline{\gamma}}\, g^{\overline{\alpha\beta}} \frac{dx^{\overline{\gamma}}}{ds} \right]$$

and its conjugate.

A hyperasymptotic curve relative to $\overline{\mu}$ is characterised by this equation.

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SOME DOUBLE INTEGRALS AND LAPLACE TRANSFORMS ASSOCIATED WITH THEM

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ABSTRACT

In the present paper, we derive two infinite double integrals involving the exponential function, the product of general polynomials $S_V^U[x]$, the polynomial set $S_n^{\alpha, \beta, 0}[x]$ and the H-function of two variables. Due to the presence of generalized polynomials and functions, the results obtained here are quite general in nature and are capable of yielding a large number of known and new integrals and double Laplace transform of certain functions. To illustrate we have obtained here two particular cases of our result. One of them gives the double Laplace transform of product of two generalizations of Hermite polynomials given by Gould and Hopper [4] and the other is an integral involving the product of Laguerre polynomials $L_n^{(\alpha)}(x)$ and a generalization of Laguerre polynomials $L_n^{(\alpha)}(x, r, \beta)$ introduced by Singh and Srivastava [11].

1. INTRODUCTION

(i) The two dimensional Laplace transform of a function $f(x_1, x_2)$ will be defined as

$$L\{f(x_1, x_2) : s_1, s_2\} = \int_0^\infty \int_0^\infty e^{-s_1 x_1 - s_2 x_2} f(x_1, x_2)\, dx_1\, dx_2 \qquad \text{... (1.1)}$$

(ii) The general class of polynomials $S_V^U[x]$ introduced by Srivastava [12, p. 1, eq. (1)]

$$S_V^U[x] = \sum_{K=0}^{[V/U]} \frac{(-V)_{UK} A_{V,K}\, x^K}{K!} \quad (K = 0, 1, 2, \ldots) \qquad \text{... (1.2)}$$

where U is an arbitrary positive integer and the coefficients $A_{V,K}$ $(V, K \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{V,K}$, the polynomials $S_V^U[x]$ can easily be reduced to the classical orthogonal polynomials including for example, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the generalized Hermite polynomials $g_V^U(x, h)$ and several others (see, for details [14, p.158-161]).

(iii) The generalized polynomial set $S_n^{\alpha,\beta,\tau}[x]$ is defined by following Rodrigues type formula [9,p.64,eq.(2.1.8)]

$$S_n^{\alpha,\beta,\tau}[x : r, s, q, A, B, m, k, l]$$
$$= (Ax+B)^{-\alpha}(1-\tau x^r)^{-\beta/\tau}\, T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}(1-\tau x^r)^{\beta/\tau+sn}\right] \qquad \ldots (1.3)$$

with the differential operator $T_{K,l}$ being defined as

$$T_{k,l} \equiv x^l\left(k + x\frac{d}{dx}\right) \qquad \ldots (1.4)$$

The explicit form of this generalization polynomial set [9,p.71, eq.(2.3.4)] is given by

$$S_n^{\alpha,\beta,\tau}[x : r, s, q, A, B, m, k, l] = B^{qn}\, x^{l(m+n)}\,(1-\tau x^r)^{sn}\, l^{m+n}$$

$$\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{j=0}^{m+n}\sum_{i=0}^{j}\frac{(-1)^j(-j)_i(\alpha)_j(-v)_u(-\alpha-qn)_i}{u!\,v!\,i!\,j!\,(1-\alpha-j)_i}$$

$$\left(-\frac{\beta}{\tau}-sn\right)_v\left(\frac{i+k+ru}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^v\left(\frac{Ax}{B}\right)^i \qquad \ldots (1.5)$$

It may be pointed out here that the polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [1], Dhillon [3], Gould- Hopper [4], Krall and Frink [7], Singh [10] and Singh and Srivastava [11] etc. Some of the special cases of (1.4) are given by Raizada in a tabular form [9,p.65]. We shall require the following special case of (1.4).

If we take $A = 1$, $B = 0$ and let $\tau \to 0$ in (1.4) and use the well known confluence principle

$$\lim_{|b|\to\infty}(b)_n(x/b)^n = x^n \qquad \ldots (1.6)$$

therein, we arrive at the following polynomial set

$$S_n^{\alpha,\beta,0}[x] = S_n^{\alpha,\beta,0}\,[x : r, q, l, 0, m, k, l]$$

$$= x^{an+l(m+n)} l^{m+n} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{(-v)_u \left(\dfrac{\alpha + an + k + ru}{l}\right)_{m+n} (\beta x^r)^v}{u!\, v!} \qquad \ldots (1.7)$$

(iv) The H-function of two variables defined by Mittal and Gupta [8,p.117, eq.(1.1.1)], possesses the following integral representation

$$H[x, y] = H^{O,N\,:\,M_1,N_1\,:\,M_2,N_2}_{P,Q\,:\,P_1,Q_1\,:\,P_2,Q_2}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\, \begin{matrix} (a_j : \alpha_j', \alpha_j'')_{1,p} : (c_j', \gamma_j')_{1,P_1} : (c_j'', \gamma_j'')_{1,P_2} \\ (b_j : \beta_j', \beta_j'')_{1,Q} : (d_j', \delta_j')_{1,Q_1} : (d_j'', \delta_j'')_{1,Q_2} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(\xi, \eta)\, \phi_1(\xi)\, \phi_2(\eta)\, x^{\xi} y^{\eta}\, d\xi\, d\eta, \quad \omega = \sqrt{-1} \qquad \ldots (1.8)$$

where

$$\phi(\xi, \eta) = \frac{\prod\limits_{j=1}^{N} \Gamma\,(1 - a_j + \alpha_j'\xi + \alpha_j''\eta)}{\prod\limits_{j=N+1}^{p} \Gamma\,(a_j - \alpha_j'\xi + \alpha_j''\eta) \prod\limits_{j=1}^{Q} \Gamma\,(1 - b_j + \beta_j'\xi + \beta_j''\eta)} \qquad \ldots (1.9)$$

$$\phi_1(\xi) = \frac{\prod\limits_{j=1}^{M_1} \Gamma\,(d_j' - \delta_j'\xi) \prod\limits_{j=1}^{N_1} \Gamma(1 - c_j' + \gamma_j'\xi)}{\prod\limits_{j=M_1+1}^{Q_1} \Gamma\,(1 - d_j' + \delta_j'\xi) \prod\limits_{j=N_1+1}^{P_1} \Gamma(c_j' - \gamma_j'\xi)} \qquad \ldots (1.10)$$

and

$$\phi_2(\eta) = \frac{\prod\limits_{j=1}^{M_2} \Gamma\,(d_j'' - \delta_j'')\eta \prod\limits_{j=1}^{N_2} \Gamma\,(1 - c_j'' + \gamma_j''\eta)}{\prod\limits_{j=M_2+1}^{Q_2} \Gamma\,(1 - d_j'' + \delta_j''\eta) \prod\limits_{j=N_2+1}^{P_2} \Gamma(c_j'' - \gamma_j''\eta)} \qquad \ldots (1.11)$$

For the convergence, existence conditions and other details of the H-function of two variables, we refer to the book [13,p.83, eq.(6.1.5), (6.1.6)].

2. FIRST INTEGRAL

$$\int_0^{\infty}\int_0^{\infty} (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1} \exp\,[-s_1\,(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]$$

$$S_V^U\,[\lambda_1 x_1 + \lambda_2 x_2]\;\; S_n^{\alpha,\beta,0}\,[\mu_1 x_1 + \mu_2 x_2]\; dx_1\, dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{[V/U]} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \theta(K, u, v) \frac{\Gamma(\sigma_1 + K)\, \Gamma(\sigma_2 + qn + l(m+n) + vr)}{s_1^{\sigma_1 + k}\, s_2^{\sigma_2 + qn + (m+n) + vr}} \qquad \ldots (2.1)$$

where

$$\theta(K, u, v) = \frac{(-V)_{UK} A_{V,K}\, l^{m+n} (-v)_u\, \beta^v}{K!\, u!\, v!} \left(\frac{\alpha + qn + k + ru}{l} \right)_{m+n} \qquad \ldots (2.2)$$

and

$$R = \begin{vmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \end{vmatrix} \neq 0, \ \mathrm{Re}(\sigma_1) > 0, \ \mathrm{Re}(s_i) > 0 \ (i = 1, 2,) \qquad \ldots (2.3)$$

Proof : To prove (2.1), we make use of the known double integral [15,p.241, eq.(7)]

$$\int_0^\infty \int_0^\infty F(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 x_1 + \mu_2 x_2)\, dx_1\, dx_2 = \frac{1}{R} \int_0^\infty \int_0^\infty F(u_1, u_2)\, du_1\, du_2 \qquad \ldots (2.4)$$

where R is defined as in (2.3).

If we take

$$F(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 x_1 + \mu_2 x_2) = f_1(\lambda_1 x_1 + \lambda_2 x_2)\, f_2(\mu_1 x_1 + \mu_2 x_2)$$

then $\int_0^\infty \int_0^\infty f_1(\lambda_1 x_1 + \lambda_2 x_2)\, f_2(\mu_1 x_1 + \mu_2 x_2)\, dx_1\, dx_2$

$$= \frac{1}{R} \int_0^\infty f_1(u_1)\, dx_1 \int_0^\infty f_2(u_2) du_2 \qquad \ldots (2.5)$$

Consider

$$f_1(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} \exp[-s_1 (\lambda_1 x_1 + \lambda_2 x_2)]\, S_V^U [\lambda_1 x_1 + \lambda_2 x_2]$$

and $f_2(\mu_1 x_1 + \mu_2 x_2) = (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1} \exp[-s_2(\mu_1 x_1 + \mu_2 x_2)]\, S_n^{\alpha, \beta, 0} [\mu_1 x_1 + \mu_2 x_2]$

Then, from (2.5), we have

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1}$$
$$\exp[-s_1 (\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]$$
$$S_V^U [\lambda_1 x_1 + \lambda_2 x_2]\ S_n^{\alpha, \beta, 0} [\mu_1 x_1 + \mu_2 x_2]\ dx_1\, dx_2$$
$$= \frac{1}{R} \int_0^\infty u_1^{\sigma_1 - 1} e^{-s_1 u_1} S_V^U [u_1]\, du_1 \int_0^\infty u_2^{\sigma_2 - 1} e^{-s_2 u_2}\, S_n^{\alpha, \beta, 0} [u_2]\, du_2 \qquad \ldots (2.6)$$

On expressing the general class of polynomials $S_V^U [u_1]$ and generalized polynomial set $S_n^{\alpha, \beta, 0} [\mu_2]$ occurring on the right hand side of (2.6) in series form with the help of (1.2) and (1.7) respectively, interchanging the order of integrals and summation in the result thus obtained and evaluating the u_1 and u_2 integrals with the help of a known formula [5,p.317,eqn.(3.381.4)], we arrive at the desired result.

3. SECOND INTEGRAL

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1}$$

$$\exp\left[- s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)\right] S_V^U \left[\lambda_1 x_1 + \lambda_2 x_2\right]$$

$$S_n^{\alpha, \beta, 0} \left[\mu_1 x_1 + \mu_2 x_2\right] H^{O, N : M_1, N_1 : M_2, N_2}_{P, Q : P_1, Q_1 : P_2, Q_2} \left[\begin{matrix} z_1 (\lambda_1 x_1 + \lambda_2 x_2)^{\rho_1} \\ z_2 (\mu_1 x_1 + \mu_2 x_2)^{\rho_2} \end{matrix} \right|$$

$$\left. \begin{matrix} (a_j : \alpha_j', \alpha_j'')_{1, P} : (c_j', \gamma_j')_{1, P_1} : (c_j'', \gamma_j'')_{1, P_2} \\ (b_j : \beta_j', \beta_j'')_{1, Q} : (d_j', \delta_j')_{1, Q_1} : (d_j'', \delta_j'')_{1, Q_2} \end{matrix} \right] dx_1\, dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{[V/U]} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{\theta(K, u, v)}{s_1^{\sigma_1 + K}\, s_2^{\sigma_2 + qn + l(m+n) + vr}} H^{O, N : M_1, N_1 + 1 : M_2, N_2 + 1}_{P, Q : P_1 + 1, Q_1 : P_2 + 1, Q_2}$$

$$\left[\begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2} \end{matrix} \right| \begin{matrix} (a_j : \alpha_j', \alpha_j'')_{1, P} : (1 - \sigma_1 - K, \rho_1), (c_j', \gamma_j')_{1, P_1} ; \\ (b_j : \beta_j', \beta_j'')_{1, Q} : (d_j', \delta_j')_{1, Q_1} ; \end{matrix}$$

$$\left. \begin{matrix} (1 - \sigma_2 - qn - l(m+n) - vr, \rho_2), (c_j'', \gamma_j'')_{1, P_2} \\ (d_j'', \delta_j'')_{1, Q_2} \end{matrix} \right] \qquad \text{... (3.1)}$$

where θ and R are defined as in (2.2) and (2.3) respectively and the following conditions are satisfied :

(i) $\rho_i \geq 0$; $\text{Re}(s_1) > 0$, $(i = 1, 2)$

(ii) $\text{Re}(\sigma_i) + \rho_i \min\limits_{1 \leq j \leq M_i} \text{Re}\,(d_j^{i}/\delta_j^{i}) > 0 \quad (i = 1, 2)$

(iii) The H-function of two variables occurring in (3.1) satisfies the conditions corresponding appropriately to those given in the book [6,p.83,eqns.(6.1.5)-(6.1.6)]. ... (3.2)

Proof : To prove (3.1) we first express the H-function of two varaibels in terms of its Mellin-Barnes contour integrals with the help of (1.8) and interchange the order of ξ, η and x_1, x_2 integrals (which is permissible under the conditions stated with the integral (3.1)), to get

$$\frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(\xi, \eta)\, \phi_1(\xi)\, \phi_2(\eta)\, z_1^{\xi} z_2^{\eta}\, d\xi\, d\eta$$

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2' x_2)^{\sigma_1 + \rho_1 \xi - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 + \rho_2 \eta - 1}$$

$$\exp\left[-s_1(\lambda_1 x_1+\lambda_2 x_2)-s_2(\mu_1 x_1+\mu_2 x_2)\right] S_V^U\left[\lambda_1 x_1+\lambda_2 x_2\right]$$

$$S_n^{\alpha,\beta,0}\left[\mu_1 x_1+\mu_2 x_2\right] dx_1\, dx_2 \qquad \text{... (3.3)}$$

Now evaluating the above x_1, x_2 double integral with the help of (2.1) and reinterpreting the result, thus obtained in terms of the H-function of two variables, we easily arrive at the desired result after a little simplification.

4. SPECIAL CASES

(i) If we take $\lambda_1=\mu_2=1$, $\lambda_2=\mu_1=0$ in (3.1) and reduce the polynomial $S_V^U[x_1]$ in terms of Gould and Hopper polynomials $g_V^U[x_1, h]$ [4,p.58,eq.(6.2)] for which [see 14, p.161, eq.(1.15)] and the generalized polynomials set $S_n^{\alpha,\beta,0}[x_2]$ in terms of Gould and Hopper polynomials $H_n^{(r)}[x_2, \alpha, \beta]$ [u, p.52] for which [see, 9, p.65], we arrive at the following double Laplace transform :

$$L\left\{x_1^{\sigma_1-1} x_2^{\sigma_2-1} g_V^U[x_1, h]\, H_n^{(r)}[x_2, \alpha, \beta]\, H[z_1 x_1^{\rho_1}, z_2 x_2^{\rho_2}] : s_1, s_2\right\}$$

$$= \sum_{K=0}^{[V/U]} \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(UK)! \binom{V}{UK} h^K (-1)^n (-v)_u (-\alpha-ur)_n \beta^v}{K!\, u!\, v!\, s_1^{\sigma'_1} s_2^{\sigma_2-n+vr}}$$

$$H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1:P_2+1,Q_2}\left[\begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2}\end{matrix}\,\middle|\, \begin{matrix}(a_j:\alpha'_j,\alpha''_j)_{1,P}:(1-\sigma'_1,\rho_1),(c'_j,\gamma'_j)_{1,P_1}; \\ (b_j:\beta'_j,\beta''_j)_{1,Q}:(d'_j,\delta'_j)_{1,Q_1};\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\sigma_2+n-vr,\rho_2),(c''_j,\gamma''_j)_{1,P_2} \\ (d''_j,\delta''_j)_{1,Q_2}\end{matrix}\right] \qquad \text{... (4.1)}$$

where $\sigma'_1=\sigma_1+(V-UK)$ ... (4.2)

and the conditions easily obtainable from those stated with (3.1) are satisfied.

(ii) If we take $\lambda_1=\mu_2=1$, $\lambda_2=\mu_1=0$ in (3.1) and reduce the H-function of two variables in terms of product of a Whittaker and a modified Bessel's function [13, p.90, eq. (6.4.15), p.18, eqs. (2.6.7), (2.6.6)], the polynomials $S_V^U[x_1]$ in terms of Hermite polynomials $H_n(1/\sqrt{x_1})$ with the help of [14,p.158,eq.(1.4)] and the polynomial set $S_n^{\alpha,\beta,0}[x_2]$ in terms of Laguerre polynomials $L_n^{(\alpha)}[x_2]$ by taking $\beta=r=1$ in [9, p.65, eq. (2.1.15)], we arrive at a result given by Gupta and Agrawal [6, p.303, eq. (3.2)].

(iii) If in (3.1), we reduce the polynomials $S_V^U\,[\lambda_1 x_1 + \lambda_2 x_2]$ to Laguerre polynomials $L_V^{(p)}\,[\lambda_1 x_1 + \lambda_2 x_2]$ using [14, p.159, eq.11.8] and the generalized polynomial set $s_n^{\alpha,\,\beta,\,0}\,[\mu_1 x_1 + \mu_2 x_2]$ to polynomials $L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2, r, \beta]$ introduced by Singh and Srivastava [11] by using [9, p.65, eq. (2.1.15)], we arrive at the following integral

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1}$$

$$\exp\,[-s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]$$

$$L_V^{(p)}\,[\lambda_1 x_1 + \lambda_2 x_2]\; L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2, r, \beta]$$

$$\cdot H\left[\begin{matrix} z_1\,(\lambda_1 x_1 + \lambda_2 x_2)^{\rho_1} \\ z_2\,(\mu_1 x_1 + \mu_2 x_2)^{\rho_2} \end{matrix}\right] dx_1\, dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{V} \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(-V)_k\,(-v)_u\,\beta^v\,(-\alpha - n - ru)_n}{K!\,u!\,v!\,(1+p)_K} \binom{V+p}{V}$$

$$H^{O,\,N\,:\,M_1,\,N_1+1\,:\,M_2,\,N_2+1}_{P,\,Q\,:\,P_1+1,\,Q_1\,:\,P_2+1,\,Q_2}\left[\begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2} \end{matrix}\,\middle|\, \begin{matrix} (a_j : \alpha_j', \alpha_j'')_{1,\,P} : (1 - \sigma_1 - K, \rho_1), \\ (b_j : \beta_j', \beta_j'')_{1,\,Q} : (d_j', \delta_j')_{1,\,Q_1} \end{matrix}\right.$$

$$\left.\begin{matrix} (c_j', \gamma_j')_{1,\,P_1}\,;\,(1 - \sigma_2 - vr, \rho_2), (c_j'', \gamma_j'')_{1,\,P_2} \\ ;\,(d_j'', \delta_j'')_{1,\,Q_2} \end{matrix}\right] \qquad \ldots (4.3)$$

where R is defined as in (2.3) and the conditions easily derivable from (3.1) are satisfied.

Further, if we take $\sigma_1 = \sigma_2 = 1 = s_1 = s_2$ in (4.3), reduce the polynomials $L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2, r, \beta]$ to Laguerre polynomials $L_n^{(\alpha)}\,[\mu_1 x_1 + \mu_2 x_2]$ by taking $\beta = r = 1$ and the H-function of two variables into unity [By first reducing the H-function of two variables into product of two Fox's H-functions and then each of Fox's H-functions into exponential function with the help of results [13,p.90, eq.(6.4.15) and p.18, eq.(2.6.2)]. Next taking the limit as $z_1, z_2 \to 0$ and suitably specializing the parameters], we arrive at a known result given by Dhawan [2, p.417, eq. (2.2)] after a little simplification.

A number of other results involving products of several simpler functions and simple polynomials, obtained as special cases of H-function of two variables and the general polynomials $S_U^V\,[x]$ and the polynomial set $s_n^{\alpha,\,\beta,\,0}\,[x]$ can be obtained as special cases of our main result (3.1) but we do not record them here on account of lack of space.

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SOME PROBABILITY DISTRIBUTIONS AND EXPECTATIONS ASSOCIATED WITH MULTIAVRIATE BETA AND GAMMA DISTRIBUTIONS INVOLVING MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT

In the present paper, we discuss some multivariate beta and gamma distributions and make their applications to derive their expectations in terms of multiple hypergeometric functions of several variables.

1. Introduction. Exton [6, p.222] studied many special multivariate distributions having expectations in terms of Lauricella's multiple hypergeometric functions [8]. Motivated by this work, here in the present paper, we discuss some multivariate beta and gamma distributions and make their applications to derive the expectations of different functions in terms of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{C}$ of Chandel [1], ${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [5,6], ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ of Chandel and Gupta [3] and ${}^{(k)}F^{(n)}_{CD}$ of Karlsson [7]. Finally we also discuss their special cases.

2. Expectations of different functions related to multivariate beta distributions. The expectation for the function $f(x_1, \ldots, x_n)$ having multivariate density function $f(x_1, \ldots, x_n)$ is defined as

$$(2.1) < g\,(x_1\,, \ldots, x_n) > = \int_{-\infty}^{\infty} \ldots (n) \ldots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) g\,(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

We consider the density function

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$$(2.2)\ f_1(x_1, \dots, x_n) = x_1^{b_1-1} \,..\, x_n^{\,b_n-1} (1-x_1-\dots-x_n)^{c-b_1-\dots-b_k-1}$$
$$.\ (1-x_{k+1}-\dots-x_n)^{c'-b_{k+1}-\dots-b_n-1}$$
$$.\ \frac{\Gamma(c)\Gamma(c')}{\Gamma(b_1)\dots\Gamma(b_n)\Gamma(c-b_1-\dots-b_k)\Gamma(c'-b_{k+1}-\dots-b_n)},$$

provided that $0 \le x_1, \dots, 0 \le x_n$, $x_1 + \dots + x_k \le 1$, $x_{k+1} + \dots + x_n \le 1$ and real parts of c, c' , $c-b_1 - \dots - b_k$ and $c' - b_{k+1} - \dots - b_n$ are positive and $f_1 = 0$ elsewhere.

Another density function is

$$(2.3)\ f_2(x_1, \dots, x_n) = x_1^{b_1-1} \dots x_n^{\,b_n-1} (1-x_1-\dots-x_k)^{c-b_1-\dots-b_k-1}$$
$$.\ (1-x_{k+1})^{c_{k+1}-b_{k+1}-1} \dots (1-x_n)^{c_n-b_n-1}$$
$$.\ \frac{\Gamma(c)\ \Gamma(a_{k+1}) \dots \Gamma(a_n)}{\Gamma(b_1)\dots\Gamma(b_n)\Gamma(c-b_1-\dots-b_k)\Gamma(c_{k+1}-b_{k+1}) \dots \Gamma(c_n-b_n)}$$

where $x_1 + \dots + x_k \le 1$, $0 \le x_r \le 1$, $r=1,\dots,n$

$\mathrm{Re}(c) > \mathrm{Re}(b_1 + \dots + b_k)$, $\mathrm{Re}(b_j) > 0$, j = 1, ..., k,

$\mathrm{Re}(c_i) > \mathrm{Re}(b_i) > 0$, $i=k+1, \dots, n$ and f_2 is zero elsewhere.

Corresponding to density function f_1 defined by (2.2) consider the function

$$(2.4)\qquad g_1(x_1, \dots, x_n) = (1-x_1\alpha_1 - \dots - x_n\alpha_n)^{-a}.$$

Now putting the values of f_1 and g_1 from (2.2) and (2.4) respectively in (2.1) and making an appeal to the result due to Exton [6, p.93. (3.4.2.4)] we obtain the expectation of g_1 corresponding to function f_1 :

$$(2.5)\quad < E_{1,1} > {}^{(k)}_{(1)}E_D^{(n)}(a, b_1, \dots, b_n\,; c, c'\,; \alpha_1, \dots, \alpha_n),$$

where ${}^{(k)}_{(1)}E_D^{(n)}$ is multiple hypergeometric function related to Lauricella's $F_D^{(n)}$ introduced by Exton [5].

Further putting the values of f_2 and g_1 from (2.3) and (2.4) respectively in (2.1) and making an appeal to the result due to Karlsson [7.(3.2)], we derive following expectation value of g_1 corresponding to the function f_2

(2.6) $< E_{1,2} > {}^{(k)}F_{AD}^{(n)}\,(a,b_1,\ldots,b_n\,;c,c_{k+1},\ldots,c_n\,;\alpha_1,\ldots,\alpha_n)$

where ${}^{(k)}F_{AD}^{(n)}$ is one of intermediate Lauricella's multiple hypergeometric function introduced by Chandel and Gupta [3].

Now we consider the density function

$$(2.7)\; f_3\,(x_1,\ldots,x_n) = x_1^{b_1-1}\ldots x_n^{b_n-1}\,(1-x_1-\ldots-x_n)^{c-b_1-\ldots-b_n-1} \cdot \frac{\Gamma(c)}{\Gamma(b_1)\ldots\Gamma(b_n)\Gamma(c-b_1-\ldots-b_n)}$$

provided that $0 \le x_1,\ldots,0 \le x_n\,,\ x_1+\ldots+x_n \le 1$ and all the real parts of c and $c-b_1-\ldots-b_n$ are positive, also $f_3 = 0$ else where.

Consider

$$(2.8)\; g_2\,(x_1,\ldots,x_n) = (1-\alpha_1x_1-\ldots-\alpha_kx_k)^{-a}\,(1-\alpha_{k+1}x_{k+1}-\ldots-\alpha_nx_n)^{-a'}.$$

Then putting these values of f_3 and g_2 from (2.7) and (2.8) respectively in (2.1) and making an appeal to the result due to Exton [6, p.93. (3.4.2.5)], we derive the following expectation of g_2 corresponding to function f_3 :

$$(2.9)\quad < E_{2,3} > = {}^{(k)}_{(2)}E_D^{(n)}\,(a,a',b_1,\ldots,b_n\,;c;\,\alpha_1,\ldots,\alpha_n)$$

where ${}^{(k)}_{(2)}E_D^{(n)}$ is another multiple hypergeometric function related to Lauricella's $F_D^{(n)}$ introduced by Exton [5].

Further consider the function

$$(2.10)\; g_3\,(x_1,\ldots,x_n) = (1-\alpha_1x_1-\ldots-\alpha_kx_k)^{-a}\,(1-\alpha_{k+1}x_{k+1})^{-a_{k+1}}\ldots(1-\alpha_nx_n)^{-a_n}.$$

Now putting the value of f_3 and g_3 from (2.7) and (2.10) respectively in (2.1) and making an appeal to the result due to Karlsson [7,(3.1)], we get the expectation value of g_3 corresponding to the function f_3

$$(2.11) < E_{3,3} > = {}^{(k)}F_{BD}^{(n)}\,[a,a_{k+1},\ldots a_n\,,b_1,\ldots,b_n\,;c\,;\alpha_1\ldots,\alpha_n\,].$$

where ${}^{(k)}F_{BD}^{(n)}$ is one of intermediate Lauricella's multiple hypergeometric functions due to Chandel and Gupta [3].

3. Expectations of different multiple hypergeometric functions related to multivariate gamma distributions. In this section, we discuss some density functions associated with gamma distributions and derive some expectations involving multiple hypergeometric functions.

Consider the density function

$$(3.1) \quad f(x,y,z) = \frac{1}{\Gamma(a)\Gamma(a')\Gamma(b)}\, e^{-x-y-z}\, x^{a-1}\, y^{a'-1}\, z^{b-1},$$

provided that $0 \le x,y,z < \infty$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(a')>0$,$\mathrm{Re}(b)>0$ and $f(x,y,z)=0$ elsewhere.

Consider another function

$$(3.2) \quad g(x,y,z) = {}_0F_1\,(-\,;c_1\,;x_1\,xz) \ldots {}_0F_1\,(-\,;c_k\,;x_k\,xz\,)$$
$${}_0F_1\,(-\,;c_{k+1}\,;x_{k+1}\,yz) \ldots {}_0F_1\,(-\,;c_n\,;x_n\,yz\,).$$

Now making an appeal to the result due to Exton [6, p.96 (3.4.4.6)], we derive the following expectations value of g(x, y, z) corresponding to the function f(x, y, z) :

$$(3.3) \quad < g(x, y, z) > = {}^{(k)}_{(1)}E_C^{(n)}\,(a,a'\,,b\,;c_1\,,\ldots,\,c_n\,;x_1\,,\,\ldots\,,x_n\,),$$

where ${}^{(k)}_{(1)}E_C^{(n)}$ is multiple hypergeometric function related to Lauricella's $F_C^{(n)}$ introduced by Chandel [1].

Also consider the density function

$$(3.4)\, f_4(y,x_1\,,\ldots,x_n) = \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_n)}\, e^{-y-x_1-\ldots-x_n}\, y^{a-1}\, x_1^{b_1-1}\ldots x_n^{b_n-1},$$

provided that $0 < y\,,x_1\,,\ldots,\,x_n < \infty$,$\mathrm{Re}(a)$, $\mathrm{Re}(b_1)$,..., $\mathrm{Re}\,(b_n) > 0$ and

$f_4 = 0$ elsewhere.

Further consider

$$(3.5) \quad f_4(y,x_1\,,\,\ldots\,,x_n) = {}_0F_1[-\,,c\,;\alpha_1xy + \ldots + \alpha_kx_k\,y]$$
$${}_0F_1[-\,,c'\,;\alpha_{k+1}x_{k+1}y + \ldots + \alpha_nx_n\,y].$$

Now putting the values of f_4 and g_4 from (3.4) and (3.5) respectively in (2.1) and making an appeal to the result due to Chandel and Gupta [2, (2.2)], we obtain the following expectation of g_4 corresponding to the function f_4 :

$$(3.6) < E_{4,4} > {}^{(k)}_{(1)}E_D^{(n)}\,(a,b_1\,,\ldots,\,b_n\,;c,c'\,;\alpha_1\,,\,\ldots\,,\alpha_n)\,.$$

Now we consider the density function

$$(3.7)\quad f_5(z,y,x_1,\ldots,x_n)=\frac{1}{\Gamma(a)\Gamma(a')\Gamma(b_1)\ldots\Gamma(b_n)}\,e^{-z-y-x_1-\ldots-x_n}\,z^{a-1}\,y^{a'-1}\,x_1^{b_1-1}\ldots x_n^{b_n-1},$$

where $0 < z,y,x_1,\ldots,x_n < \infty$ and all Re (a) Re (a'), Re(b_1),...,Re$(b_n) > 0$ also $f_5 = 0$ elsewhere.

Consider the density function

$$(3.8)\; g_5(z,y,x_1,\ldots,x_n) = {}_0F_1[-,c;(\alpha_1x_1+\ldots+\alpha_kx_k)z + (\alpha_{k+1}x_{k+1}+\ldots+\alpha_nx_n)y].$$

Now putting the values of f_5 and g_5 from (3.7) and (3.8) respectively in (2.1) and then making an appeal to the result due to Chandel and Gupta [2, (2.3)] we get the following expectation of g_5 corresponding to the function f_5 :

$$(3.9)\; < E_{5,5} > {}^{(k)}_{(2)}E_D^{(n)}\,(a,a',b_1,\ldots,b_n;c,;\alpha_1,\ldots,\alpha_n).$$

Further consider the density function

$$(3.10)\; f_6(x,x_{k+1},\ldots,x_n,y_1,\ldots,y_n) = \frac{1}{\Gamma(a)\Gamma(a_{k+1})\ldots\Gamma(a_n)\Gamma(b_1)\ldots\Gamma(b_n)}\, x^{a-1}\,x_{k+1}^{a_{k+1}-1}\ldots x_n^{a_n-1}\,y_1^{b_1-1}\ldots y_n^{b_n-1}\,e^{-(x+x_{k+1}+\ldots+x_n+y_1+\ldots+y_n)},$$

where $0 \le x,x_{k+1},\ldots,x_n,y_1,\ldots,y_n < \infty$ and Re(a), Re (a_{k+1}), ..., Re(a_n), Re (b_1), ..., Re$(b_n) > 0$, also $f_6 = 0$ elsewhere.

Consider another density function

$$(3.11)\; g_6(x,x_{k+1},\ldots,x_n,y_1,\ldots,y_n) = {}_0F_1[-,c;\alpha_1x_1x+\ldots+\alpha_kx_kx+\alpha_{k+1}x_{k+1}y_{k+1}+\ldots+\alpha_nx_ny_n].$$

Now putting the values of f_6 and g_6 from (3.10) and (3.11) respectively in (2.1) and making an appeal to the result due to Chandel and Gupta [3,(5.5)], we obtain the following expectation of g_6 having density function f_6 :

$$(3.12)\; < E_{6,6} > {}^{(k)}F_{BD}^{(n)}\,[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;\alpha_1,\ldots,\alpha_n].$$

Further consider density function

$$(3.13)\ f_7(z,y,x_1\,,\,\dots\,,x_k) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(b_1)\dots\Gamma(b_k)}\, e^{-z-y-x_1-\dots-x_k}$$

$$.\, z^{a-1}\, y^{b-1}\, x_1^{b_1-1} \dots x_k^{b_k-1},$$

provided that $0 \le z,y,x_1\,,\dots,\,x_k < \infty$ and $\mathrm{Re}(a)$, $\mathrm{Re}(b)$, $\mathrm{Re}(b_1)$, ..., $\mathrm{Re}(b_k) > 0$, also $f_7 = 0$ elsewhere and

$$(3.14)\ g_7\,(z,y,x_1\,,\,\dots\,,x_k) = {}_0F_1(-\,,\,c\,;\,\alpha_1 x_1\, z + \dots + \alpha_k x_k\, z)$$

$${}_0F_1\,(-\,;\,c_{k+1}\,;\,zy\alpha_{k+1}) \dots {}_0F_1(-\,;\,c_n\,;\,zy\alpha_n).$$

Now putting the values of f_7 and g_7 from (3.13) and (3.14) respectively in (2.1) and applying the result due to Chandel and Vishwakarma [4, p.178(3.9)], we derive corresponding expectation of g_7 :

$$(3.15)\ < E_{7,7} > {}^{(k)}F_{CD}^{(n)}\,(a,b\,,\,b_1\,,\dots,\,b_k\,;\,c,c_{k+1}\,,\dots,\,c_n\,;\ \alpha_1\,,\,\dots\,,\,\alpha_n)$$

where ${}^{(k)}F_{CD}^{(n)}$ is fourth possible intermediate Lauricella's multiple hypergeometric function due to Karlsson [8].

4. Special Cases.

(4.1) For $k = 0$, from (2.5) and (2.9), we deduce separately the result due to Exton [6, (7.2.1.5)].

(4.2) For $k = 0$, from (2.6), we derive the result due to Exton [6, (7.2.3.1)],

while

(4.3) for $k = n$, from (2.6), we obtain the result due to Exton [6,(7.2.1.5)]

(4.4) for $k = 0$, (2.11) gives the result due to Exton [6, (7.2.1.6)],

while

(4.5) for $k = n$, (2.11) gives the result due to Exton [6, (7.2.1.5)].

Further

(4.6) for $k = 0$, from (3.6) and (3.9), we deduce the result due to Exton [6, (7.2.3.2)]

Also

(4.7) for $k = n$, from (3.12) and (3.15), we derive the result due to Exton [6, (7.2.3.2)].

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ON A UNIFIED FINITE INTEGRAL

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In the present paper, we first evaluate a unified finite integral involving the product of a general class of polynomials $S_V^U[x]$, the generalized polynomial set $S_n^{\alpha,\beta,\tau}[x]$ and the multivariable H-function. Being of very general nature, due to the presence of these polynomials and function, our result provides interesting unification and extension of a large number of new and known results obtained by several authors and hitherto lying scattered in the litrature. For the sake of illustration, we have evaluated here three new integrals as special cases of our main integral. Further, a reference to 12 integrals which also follow as simple special cases of our results has also been given.

1. INTRODUCTION

(*i*) Srivastava [3, eq. (1), p. 1] has introduced the following general class of polynomials :

$$S_V^U[x] \quad \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}\, A(V,K)\, x^K}{K!}, \quad V = 0, 1, 2, \ldots \qquad \ldots (1.1)$$

where U is an arbitrary positive integer and the coefficients $A(V,K)(V, K \geq 0)$ are arbitrary constants real or complex. On suitably specializing the coefficients $A(V,K)$, $S_V^U[x]$ yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials, Gould-Hopper polynomials, Brafman polynomials and several others [5, pp. 158-161].

(*ii*) The generalized polynomial set is defined by the following Rodrigues type formula [12, eq. (2.1.8), p.64].

$$S_n^{\alpha,\beta,\tau}[x : r, s, q, A, B, m, k, l]$$

$$= (Ax + B)^{-\alpha} (1 - \tau x)^{-\beta/\tau} T_{k,l}^{m+n} [(Ax + B)^{\alpha + qn} (1 - \tau x^r)^{\beta/\tau + sn}] \qquad \text{... (1.2)}$$

with the differential operator $T_{k,l}$ being defined as

$$T_{k,l} \equiv x^l \left(k + x \frac{d}{dx} \right) \qquad \text{... (1.3)}$$

The explicit form of this generalized polynomial set [12, eq. (2.3.4), p. 71] is

$$S_n^{\alpha, \beta, \tau} [x : r, s, q, A, B, m, k, l]$$

$$= B^{qn} x^{l(m+n)} (1 - \tau x^r)^{sn} l^{m+n} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \frac{(-1)^\delta (-\delta)_p}{p!\, \delta!} \qquad \text{... (1.4)}$$

$$\frac{(\alpha)_\delta (-\nu)_e (-\alpha-qn)_p}{e!\, \nu! (1+\alpha-\delta)_p} \cdot (-\beta/\tau - \lambda_n)_\nu \left(\frac{p+k+re}{l} \right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r} \right)^\nu \left(\frac{Ax}{B} \right)^\delta$$

It may be pointed out here that the polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [11], Dhillon [14], Gould and Hopper [7], Krall and Frink [2], Singh [1], Singh and Srivastava [10] etc. Some of the special cases of (1.3) are given by Raizada in a tabular form [12, p. 65].

(*iii*) The multivariable *H*-function introduced and studied by Srivastava and Panda [6], occurring in this paper will be defined and represented as follows [4, pp. 251-252, eqns. (C.1)-(C.3)].

$$H_{P,Q:P_1,Q_1;\ldots;P_t,Q_t}^{O,N:M_1,N_1;\ldots;M_t,N_t} \left[\begin{matrix} z_1 \\ \cdot \\ \vdots \\ z_t \end{matrix} \middle| \begin{matrix} (a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)})_{1,P} : \\ (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)})_{1,Q} : \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)})_{1,P_1}; \ldots; (c_j^{(t)}, \gamma_j^{(t)})_{1,P_t} \\ (d_j^{(1)}, \delta_j^{(1)})_{1,Q_1}; \ldots; (d_j^{(t)}, \delta_j^{(t)})_{1,Q_t} \end{matrix} \right]$$

$$= \frac{1}{(2\pi\omega)^t} \int_{L_1} \cdots \int_{L_t} \phi_1(\xi_1) \ldots \phi_t(\xi_t)\, \psi(\xi_1, \ldots, \xi_t)$$

$$z_1^{\xi_1} \ldots z_t^{\xi_t}\, d\xi_1 \ldots d\xi_t \qquad \text{... (1.5)}$$

where $\omega = \sqrt{-1}$

$$\phi_i(\xi_i) = \frac{\prod\limits_{j=1}^{M_i} \Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \prod\limits_{j=1}^{N_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)}\,\xi_i)}{\prod\limits_{j=M_i+1}^{Q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)}\,\xi_i) \prod\limits_{j=N_i+1}^{P_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)}\,\xi_i)}$$

$$(i = 1, \ldots, t) \quad \ldots (1.6)$$

and

$$\psi(\xi_1, \ldots, \xi_t) = \frac{\prod\limits_{j=1}^{N} \Gamma\left(1 - a_j + \sum\limits_{i=1}^{t} \alpha_j^{(i)}\,\xi_i\right)}{\prod\limits_{j=N+1}^{P} \Gamma\left(a_j - \sum\limits_{i=1}^{t} \alpha_j^{(i)}\,\xi_i\right) \prod\limits_{j=1}^{Q} \Gamma\left(1 - b_j + \sum\limits_{i=1}^{t} \beta_j^{(i)}\,\xi_i\right)} \quad \ldots (1.7)$$

All the Greek letters occuring on the left hand side of (1.5) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable H-function will, however, be meaningful even if some of these quantities are zero. For the convergence and existence conditions of the multivariable H-function we refer to the book by Srivastava et al. [4, eqns. (C.4)-(C.8), pp. 252-253]. Throughout the paper it is assumed that this function satisfies the above cited conditions.

2. MAIN INTEGRAL

$$\int_0^a x^{\rho-1} (a-x)^{\sigma-1} (c+bx)^{-\eta} S_n^{\alpha,\beta,\tau} [vx^{\mu} (a-x)^{\nu} (c+bx)^{-\lambda}]$$

$$S_V^U [zx^{\zeta} (a-x)^{\xi} (c+bx)^{-\gamma}] \, H\, [z_1 x^{P_1} (a-x)^{q_1} (c+bx)^{-r_1} \ldots$$

$$z_t\, x^{P_t} (a-x)^{q_t} (c+bx)^{-r_t}]\, dx$$

$$= \frac{a^{\rho+\sigma-1}}{c^{\eta}} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \theta(k, e, v, \delta, p)$$

$$H^{O,N+3:M_1,N_1;\ldots;M_t,N_t:1,1;1,0}_{P+3,Q+2:P_1,Q_1;\ldots;P_t,Q_t:1,1;0,1} \left[z_1 \frac{a^{P_1+q_1}}{c^{r_1}}, \ldots, z_t \frac{a^{P_t+q_t}}{c^{r_t}}, \right.$$

$$\left. \frac{-\tau y^r a^{(\mu+\nu)r}}{c^{\lambda r}}, \frac{ab}{c} \;\middle|\; \begin{matrix} C : E\,; (1-v+sn, 1) : - \\ D : F\,; (0, 1) \qquad : (0, 1) \end{matrix} \right] \quad \ldots (2.1)$$

where

$$\theta(K, e, v, \delta, p) = B^{qn}\, l^{m+n}\, \frac{a^{l(\mu+\nu)(m+n)}\, (-1)^{\delta}\, (-\delta)_p\, (\alpha)_\delta}{c^{\lambda l(m+n)}\, p!\, \delta!\, e!\, v!\, K!}$$

$$\frac{(-v)_e\, (-\alpha-qn)_p\, (-V)_{UK}\, A(V, K)\, z^{K}\, z^{K(\zeta+\xi)}}{(1-\alpha-\delta)_p\, \Gamma(v-sn)\, c^{\gamma K}}\, (-\beta/\tau - sn)_v$$

$$\left(\frac{p+k+re}{l}\right)_{m+n} + \left(\frac{-\tau a^{(\mu+\nu)}}{c^{\lambda r}}\right)^{v} \left(\frac{A a^{(\mu+\nu)}}{B c^{\lambda}}\right)^{\delta} \qquad \ldots (2.2)$$

$$C = (1-\rho-\mu R-\zeta K : p_1, \ldots, p_t, \mu r, 1), (1-\sigma-\nu R-\xi K : q_1, \ldots, q_t, \nu r, 0),$$
$$(1-\eta-\lambda R-\gamma K : r_1, \ldots, r_t, \lambda r, 1), (a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0, 0)_{1, P}$$

$$D = (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0, 0)_{1, Q}, (1-\eta-\lambda R-\gamma K : r_1, \ldots, r_t, \lambda r, 0),$$
$$(1-\rho-\sigma-(\mu+\nu)R-(\zeta+\xi)K : p_1+q_1, \ldots, p_t+q_t, (\mu+\nu)r, 1)$$

$$E = (c_j^{(1)}, \gamma_j^{(1)})_{1, P_1}; \cdots; (c_j^{(t)}, \gamma_j^{(t)})_{1, P_t}$$

$$F = (d_j^{(1)}, \delta_j^{(1)})_{1, Q_1}; \cdots; (d_j^{(t)}, \delta_j^{(t)})_{1, Q_t} \qquad \ldots (2.3)$$

$$R = l(m+n) + rv + \delta \qquad \ldots (2.4)$$

and the following conditions are satisfied :

(*i*) a, b, c are positive numbers such that $\left|\frac{ab}{c}\right| < 1$,

$\mathrm{Re}(\eta) > 0, \min\{\mu, \nu, \xi, \lambda, \zeta, \gamma, p_i, q_i, r_i\} \geq 0, i = 1, \ldots, t$

(not all zero simultaneously)

(*ii*) $$\mathrm{Re}(\rho) + \sum_{i=1}^{t} p_i \min_{1 \leq j \leq M_i} \left\{ \mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) \right\} > 0,$$

$$\mathrm{Re}(\sigma) + \sum_{i=1}^{t} q_i \min_{1 \leq j \leq M_i} \left\{ \mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) \right\} > 0,$$

$$(i = 1, \ldots, t) \qquad \ldots (2.5)$$

Proof . To establish the integral (2.1), we first express the polynomials $S_V^U[x]$ and $S_n^{\alpha, \beta, \tau}[x]$ occurring in the left-hand side of (2.1) in their respective series forms with the help of equations (1.1) and (1.4) and then interchange the order of summations and integration (which is permissible under the conditions stated with

(2.1)). The left-hand side of the integral (2.1) (say I), then takes the following form :

$$I = B^{qn}\, l^{m+n} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \sum_{u=0}^{\infty} \frac{(-1)^{\delta}\,(-\delta)_p\,(-v)_e\,(\alpha)_{\delta}}{p!\,\delta!\,e!\,v!\,K!} \cdot$$

$$\frac{(-1)^{v}\,(-\alpha-qn)_p\, y^{R+ur}\,(-V)_{KU}\,(v-sn)_u\,(\tau)^{u+v}\,A(V,K)\,z^{K}}{u!\,(1-\alpha-\delta)_p}$$

$$(-\beta/\tau - sn)_v \left(\frac{p+k+re}{l}\right)_{m+n} \left(\frac{A}{B}\right)^{\delta} \int_0^a x^{\rho+\mu(R+ur)+\zeta K-1}$$

$$(a-x)^{\sigma+\nu(R+ur)+\xi K-1}\,(c+bx)^{-\eta-\lambda(R+ur)-\gamma K}$$

$$H\,[z_1\,x^{p_1}\,(a-x)^{q_1}\,(c+bx)^{-r_1}, \ldots, z_t x^{p_t}\,(a-x)^{q_t}\,(c+bx)^{-r_t}\,]\,dx \qquad \ldots (2.6)$$

Evaluating the x-integral occurring in the above equation (2.6) with the help of [9, eq. (5.2.6), p. 244] and expressing the result thus obtained in terms of H-function of $(t + 2)$ variables, we arrive at the desired result (2.1).

3. SPECIAL CASES

(i) If we take $A = 1$, $B = q = s = m = k = 0$, $l = -1$ and let $\tau \to 0$ in the main integral (2.1), the polynomial set $S_n^{\alpha, \beta, \tau}[x]$ reduces into Gould and Hopper polymials $H_n^{(r)}[x, \alpha, \beta]$ [7] and we arrive at the following interesting integral after a little simplification :

$$\int_0^a x^{\rho-1}\,(a-x)^{\sigma-1}\,(c+bx)^{-\eta}\,H_n^{(r)}\,[yx^{\mu}\,(a-x)^{\nu}\,(c+bx)^{-\lambda}, \alpha, \beta]$$

$$S_V^U\,[zx^{\zeta}\,(a-x)^{\xi}\,(c+bx)^{-\gamma}]\,H\left[\frac{z_1 x^{P_1}\,(a-x)^{q_1}}{(c+bx)^{r_1}}, \ldots, \frac{z_t\,x^{p_t}\,(a-x)^{q_t}}{(c+bx)^{r_t}}\right] dx$$

$$= \frac{a^{\rho+\sigma-1}}{c^{\eta}} \sum_{v=0}^{n} \sum_{e=0}^{v} \sum_{K=0}^{[V/U]} \theta_1\,(K, e, v)$$

$$H^{O,\,N+3\,:\,M_1,\,N_1\,;\ldots;\,M_t,\,N_t\,;\,1,\,0}_{P+3,\,Q+2\,:\,P_1,\,Q_1\,;\ldots;\,P_t,\,Q_t\,;\,0,\,1}\left[\begin{matrix}\dfrac{z_1 a^{p_1+q_1}}{c^{r_1}}, \ldots, \dfrac{z_t a^{p_t+q_t}}{c^{r_t}}, \dfrac{ab}{c}\end{matrix}\,\middle|\,\begin{matrix}C_1 : E : - \\ D_1 : F : (0,1)\end{matrix}\right] \qquad \ldots (3.1)$$

where

$$\theta_1(K, e, v) = \frac{(-1)^n (-v)_e (-\alpha - re)_n \beta^v a^{(\mu+\nu)R_1 + K(\zeta+\xi)} (-V)_{UK} A(V, K) z^K}{e!\, v!\, K!\, c^{\lambda R_1 + \gamma K}} \quad \ldots (3.2)$$

$$C_1 = (1 - \rho - \mu R_1 - \zeta K : p_1, \ldots, p_t, 1), (1 - \sigma - \nu R_1 - \xi K : q_1, \ldots, q_t, 0),$$
$$(1 - \eta - \lambda R_1 - \gamma K : r_1, \ldots, r_t, 1), (a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0)_{1, P}$$

$$D_1 = (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0)_{1, Q}, (1 - \eta - \lambda R_1 - \gamma K : r_1, \ldots, r_t, 0),$$
$$(1 - \rho - \sigma - (\mu+\nu)R_1 - (\zeta+\xi)K : p_1 + q_1, \ldots, p_t + q_t, 1) \quad \ldots (3.3)$$

$$R_1 = rv - \eta$$

provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

(*ii*) Taking $\eta = \rho + \sigma$, $\lambda = \mu + \nu$, $\gamma = \zeta + \xi$, $r_1 = p_i + q_i$ $(i = 1, \ldots, t)$ and replacing x by $ax/(1+x)$ in (2.1), after making a little simplifications, we arrive at the following infinite form of the integral which itself is quite general in nature.

$$\int_0^\infty x^{\rho-1} (1+dx)^{-\rho-\sigma} S_n^{\alpha, \beta, \tau} [y_1 x^\mu (1+dx)^{-\mu-\nu}]$$
$$S_V^U [z' x^\zeta (1+dx)^{-\zeta-\xi}]\, H [z_1' x^{p_1} (1+dx)^{-p_1-q_1}, \ldots, z_t' x^{p_t} (1+dx)^{-p_t-q_t}]\, dx$$
$$= d^{-\rho} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \theta_2 (K, e, v, \delta, p)$$
$$H^{O, N+2 : M_1, N_1; \ldots; M_t, N_t; 1, 1}_{P+2, Q+1 : P_1, Q_1; \ldots; P_t, Q_t; 1, 1} \left[z_1' d^{-p_1}, \ldots, z_t' d^{-p_t}, -\tau y_1 d^{-\mu r} \,\middle|\, \begin{matrix} C_2 : E : (1 - v + sn, 1) \\ D_2 : F : (0, 1) \end{matrix} \right] \quad \ldots (3.5)$$

where

$$d = \frac{c + ab}{c}, y_1' = y \left(\frac{a}{c} \right)^{\mu+\nu}, z' = z \left(\frac{a}{c} \right)^{\zeta+\xi},$$

$$z_i' = z_i \left(\frac{a}{c} \right)^{p_i + q_i} \quad \text{for } i = 1, \ldots, t \quad \ldots (3.6)$$

$$\theta_2 (K, e, v, \delta, p) = B^{qn} l^{m+n} \frac{(-1)^\delta (-\delta)_p (-v)_e (\alpha)_\delta (-\alpha - qn)_p}{p!\, \delta!\, e!\, v!\, K!\, \Gamma(v - sn)}$$

$$\frac{(-V)_{UK}\, A(V, K)\, (z')^{K}\, (-\beta/\tau - sn)_{v}\, (-\tau)^{V}}{(1-\alpha-\delta)_{p}} \left(\frac{a}{c}\right)^{(\mu+\nu)R} \left(\frac{A}{B}\right)^{\delta} .$$

$$\left(\frac{p+k+re}{l}\right)_{m+n} \quad \ldots (3.7)$$

$$C_2 = (1-\rho-\mu R-\zeta K : p_1, \ldots, p_t, \mu r), (1-\sigma-\nu R-\xi K : q_1, \ldots, q_t, \nu r)$$

$$(a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0)_{1, P}$$

$$D_2 = (b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0)_{1, Q}, (1-\rho-\sigma-(\mu+\nu)R-(\zeta+\xi)K :$$

$$p_1+q_1, \ldots, p_t+q_t, (\mu+\nu)r \quad \ldots (3.8)$$

$$R = l(m+n) + rv + \delta \quad \ldots (3.9)$$

provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

Further, if we take $A = 1, B = 0 = q$, $m = -n$, let $\tau \to 0$ and $V = 0$, both the polynomials $S_n^{\alpha, \beta, \tau}[x]$ and $S_V^U[x]$ reduce into unity. Again, we put $t = 3$ and reduce the H-function of 3-variables thus obtained into the product of H-function of one variable and H-function of two variables by taking $\alpha_j^{(1)} = \beta_j^{(1)} = 0$. Finally taking $\rho + \sigma = \eta$, $p_i + q_i = r_i$ $(i = 1, 2, 3)$, $N_i = 0$, $p_1 = 1$, $r_1 = 0$, $r_2 = p_2$, $r_3 = p_3$. We arrive at a known result [4, eq. (8.2.4), p. 136] after making some suitable adjustments and simplifications on the right-hand side.

(*iii*) If we take $\eta = \rho + \sigma$, $\lambda = \mu + \nu$, $\gamma = \zeta + \xi$, $r_i = p_i + q_i$ $(i = 1, \ldots, t)$, $c = a$, $b = (f-g)/(h-f)$ and replace x by $a\,(f-x)/(f-g)$ in the main integral (2.1), we arrive at the following form of the integral after a little simplification

$$\int_g^{\pounds} (f-x)^{\rho-1} (x-g)^{\sigma-1} (x-h)^{-\rho-\sigma} S_n^{\alpha, \beta, \tau}\left[\frac{y(f-x)^{\mu}(x-g)^{\nu}}{(x-h)^{\mu+\nu}}\right]$$

$$S_V^U\left[z\,\frac{(f-x)^{\zeta}(x-g)^{\xi}}{(x-h)^{\zeta+\xi}}\right] H\left[z_1 \frac{(f-x)^{p_1}(x-g)^{q_1}}{(x-h)^{p_1+q_1}}, \ldots, z_t \frac{(f-x)^{p_t} x-g)^{q_t}}{(x-h)^{p_t+q_t}}\right] dx$$

$$= \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \sum_{K=0}^{[V/U]} \theta_3(K, e, v, \delta, p)$$

$$H^{O, N+2 : M_1, N_1 ; \ldots ; M_t, N_t ; 1, 1}_{P+2, Q+1 : P_1, Q_1 ; \ldots ; P_t, Q_t ; 1, 1}$$

$$\left[z_1\left(\frac{f-g}{f-h}\right)^{p_1}\left(\frac{f-g}{g-h}\right)^{q_1},\ldots,z_t\left(\frac{f-g}{f-h}\right)^{p_t}\left(\frac{f-g}{g-h}\right)^{q_t},-\tau y^r\left(\frac{f-g}{f-h}\right)^{\mu r}\left(\frac{f-g}{g-h}\right)^{r}\right.$$

$$\left|\begin{matrix} C_3 : E : (1-v+sn,1) \\ D_3 : F : (0,1) \end{matrix}\right] \quad \ldots (3.10)$$

where

$$Y = y\left(\frac{f-h}{f-g}\right)^{\mu+\nu}, Z = z\left(\frac{f-h}{f-h}\right)^{\zeta+\xi}, Z_i = z_i\left(\frac{f-h}{f-g}\right)^{p_i+q_i}$$

$$(i = 1, \ldots, t) \quad \ldots (3.11)$$

$$\theta_3(K, e, v, \delta, p) = B^{qn}\, l^{m+n}\,\frac{(-1)^{\delta}(-\delta)_p(\alpha)_{\delta}(-v)_e(-\alpha-qn)_p}{p!\,\delta!\,e!\,v!\,K!\,(1-\alpha-\delta)_p}$$

$$\frac{(-\beta/\tau-sn)_v(-\tau)^v(-V)_{UK}A(V,K)\,Z^k}{\Gamma(v-sn)}\left(\frac{A}{B}\right)^{\delta}\left(\frac{f-g}{f-h}\right)^{\zeta}\left(\frac{f-g}{g-h}\right)^{\xi}$$

$$\left(\frac{p+k+re}{l}\right)_{m+n} \quad \ldots (3.12)$$

C_3, D_3, R can be obtained from the equations (2.3) and (2.4) respectively. Provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

Now, if we take $A = 1, B = 0 = q, m = -n$ and let $\tau \to 0$ in (3.10) the polynomial set $S_n^{\alpha,\beta,\tau}[x]$ reduces into unity. Further, change t to $(t+1)$ and reduce the H-function of $(t+1)$ variables into the product of H-function of one variable H-function of t-variables by taking $\alpha_j^{(t+1)} = 0 = \beta_j^{(t+1)}$ therein, we arrive at an integral obtained by Chaurasia and Sharma [15, eq. (2.1), p. 42] after a little simplification.

(*iv*) If we take $A = 1, B = 0 = q, m = -n$ and let $\tau \to 0$ in the main integral (2.1), the polynomial set $S_n^{\alpha,\beta,\tau}[x]$ reduces into unity and we arrive at an integral essentially similar to the one obtained by Agrawal [13, eq. (4.5.1), p. 122].

Further, if we put $\eta = 0$, $\gamma = 0$, $r_i = 0$ $(i = 1, \ldots, t)$ change t to $(t+1)$ and reduce the H-function of $(t+1)$ variables into the product of H-function of one variable and the H-function of t-variables by taking $\alpha_j^{(t+1)} = 0 = \beta_j^{(t+1)}$, we arrive at another known integral given by Chaurasia and Sharma [15, eq. (2.2), p. 43] after a little simplification. Further, if we take $t = 2$ and $V = 0$ reduce the polynomial $S_0^U[x]$ into unity, we easily arrive at another known integral [4, eq. (8.2.7), p. 138].

The results obtained earlier by Gupta et al. [8, eq. (3.1), p. 69], Srivastava and Singh [5, eq. (2.2), p. 166], Prasad and Singh [16, p.126] and Chaurasia and Sharma [15, eq. (2.3), p. 44 and eq. (2.4), p. 45] can also be obtained as simple special cases of the result referred in the above paragraphs. However, we omit the details here.

(*v*) If in the main result (2.1) we reduce both the polynomials $S_V^U[x]$ and $S_n^{\alpha,\beta,\tau}[x]$ into unity by the substitutions indicated in special case (*ii*), we arrive at an integral same in essence obtained by Garg [9, eq. (5.2.6), p. 244]. Further, if we take $t=2$, $\eta=\rho+\sigma$, $r_i=p_i+q_i\,(i=1,2)$, c=a, b = $\frac{c-d}{l+d}$ and replace x by ax, we get a known integral [4, eq. (8.4.4), p.144] after a little simplification.

Several other interesting results involving a large variety of polynomials, obtained as special cases of $S_V^U[x]$ and $S_n^{\alpha,\beta,\tau}[x]$, and various simpler functions, which are particular cases of the multivariable H-function, can also be obtained as special cases of our main result (2.1) but we do not record them here for lack of space.

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QUASI-HADAMARD PRODUCT OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

We consider the class $F_p^c(\lambda, \alpha)$ of certain analytic functions. It is shown that if $f \in F_p^{c_1}(\lambda_1, \alpha_1)$, $0 \le \lambda_1 < 1$, and $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, $\lambda_2 \ge 1$, then the quasi-Hadamard product $f*g$ belongs to the class $F_p^{c_1+c_2}(\lambda_1, \alpha_1)$ or $F_p^{c_1+c_2}(\lambda_1, \alpha_2)$ according as $\alpha_1 \le \alpha_2$ or $\alpha_1 \ge \alpha_2$, respectively. This result fills a gap in a recent result of Subhas S. Bhoosnurmath and S.R. Swamy [Pure Appl. Math. Sci. **35** (1992), 71-77].

1. INTRODUCTION

Let T_p denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \; p \in N, \; a_{p+n} \ge 0,$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

If $f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ and $g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$ be any two functions in T_p, then their quasi-Hadamard product, denoted by $f*g$, is defined by (see [3])

$$(f*g)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}$$

A function f in T_p belongs to the class $F_p^c(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^c (p + n\lambda)\, a_{p+n} \le p - \alpha,$$

where $c \ge 0$, $\lambda \ge 0$ and $0 \le \alpha < p$.

It is easy to establish the following containment relations :

(*i*) $F_p^c(\lambda, \alpha_2) \subset F_p^c(\lambda, \alpha_1)$, if $\alpha_1 < \alpha_2$.

(*ii*) $F_p^c(\lambda_2, \alpha) \subset F_p^c(\lambda_1, \alpha)$, if $\lambda_1 < \lambda_2$.

(*iii*) $F_p^{c_2}(\lambda, \alpha) \subset F_p^{c_1}(\lambda, \alpha)$, if $c_1 < c_2$.

Bhoosnurmath and Swamy [1] introduced the above class and proved the following result :

Theorem. *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *then the quasi-Hadamard product f*g belongs to* $F_p^{c_0}(\lambda_0, \alpha_0)$, *where*

$$F_p^{c_0}(\lambda_0, \alpha_0) \equiv F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2+1}(\lambda_2, \alpha_2), \textit{ if } \lambda_1, \lambda_2 \geq 1$$

and

$$F_p^{c_0}(\lambda_0, \alpha_0) \equiv F_p^{c_1+C_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2), \textit{ if } 0 \leq \lambda_1, \lambda_2 < 1.$$

The technique used by Bhoosnurmath and Swamy [1] is the same as employed by the author [2].

We note that their is a gap in the above result, *i.e.*, the case when any one of λ_1 and λ_2 is less than 1 and the other is greater than or equal to 1. It is the purpose of this paper to fill up this gap.

2. MAIN RESULT

Theorem. *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *where* $0 \leq \lambda_1 < 1$ *and* $\lambda_2 \geq 1$, *then the quasi-Hadamard product f*g belongs to* $F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Proof. $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ implies $f \in F_p^{c_1}(0, 0)$ so that

$$\sum_{n=1} \left(\frac{n+p}{p}\right)^{c_1} a_{p+n} \leq 1.$$

Therefore

$$\left(\frac{n+p}{p}\right)^{c_1} a_{p+n} \leq 1, \quad \text{for all } n = 1, 2, 3, \ldots \qquad \ldots (2.1)$$

Further, $g \in F_p^{c_2}(\lambda_2, \alpha_2)$ implies that $g \in F_p^{c_2}(1, 0)$ so that

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_2+1} b_{p+n} \leq 1.$$

Therefore

$$\left(\frac{n+p}{p}\right)^{c_2+1} b_{p+n} \le 1, \text{ for all } n = 1, 2, 3, \ldots \qquad \ldots (2.2)$$

Using (2.2) we get

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1+c_2+1} (p+n\lambda_1)\, a_{p+n}\, b_{p+n}$$

$$\le \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1} (p+n\lambda_1)\, a_{p+n}$$

$$\le p - \alpha_1, \quad \text{since } f \in F_p^{c_1}(\lambda_1, \alpha_1).$$

This shows that $f*g \in F_p^{c_1+c_2+1}(\lambda_1, \alpha_1)$.

Similarly, by using (2.1) we get

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1+c_2} (p+n\lambda_2)\, a_{p+n}\, b_{p+n}$$

$$\le \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_2} (p+n\lambda_2)\, b_{p+n}$$

$$\le p - \alpha_2, \quad \text{since } g \in F_p^{c_2}(\lambda_1, \lambda_2).$$

This shows that $f*g \in F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Hence $f*g \in F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Corollary. *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *where* $0 \le \lambda_1 < 1$ *and* $\lambda_2 \ge 1$, *then the quasi-Hadamard product* $f*g$ *belongs to* $F_p^{c_1+c_2}(\lambda_1, \alpha_1)$ *or* $F_p^{c_1+c_2}(\lambda_1, \alpha_2)$ *according as* $\alpha_1 \le \alpha_2$ *or* $\alpha_1 \ge \alpha_2$.

Proof. We have

$$F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$$

$$\subset F_p^{c_1+c_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$$

$$\subset F_p^{c_1+c_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_1, \alpha_2)$$

$$= F_p^{c_1+c_2}(\lambda_1, \alpha_1) \text{ or } F_p^{c_1+c_2}(\lambda_1, \alpha_2)$$

$$\text{according as } \alpha_1 \le \alpha_2 \text{ or } \alpha_1 \ge \alpha_2.$$

The result follows now from the above theorem.

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ON A SPECIAL KAEHLERIAN RECURRENT SPACE OF SECOND ORDER

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1. INTRODUCTION - An $n(=2m)$ dimensional Kaehlerian space K_n is a Riemannian space, which admits a structure field F_i^h, satisfying the relations (Yano 1965) :

$$F_i^h F_h^j = -\delta_i^j, \qquad \text{... (1.1)}$$

$$F_{ij} = -F_{ji}, \ (F_{ij} = F_i^a g_{aj}) \qquad \text{... (1.2)}$$

and $$\nabla_j F_i^h = 0, \qquad \text{... (1.3)}$$

where ∇ denotes covariant differentiation with respect to the linear symmetric connection of the space.

The Riemannian curvature tensor, which we denote by R_{ijk}^h is given by

$$R_{ijk}^h = \partial_i \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} h \\ ia \end{matrix} \right\} \left\{ \begin{matrix} a \\ jk \end{matrix} \right\} - \left\{ \begin{matrix} h \\ ja \end{matrix} \right\} \left\{ \begin{matrix} a \\ ik \end{matrix} \right\}.$$

whereas the Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$.

Let us, now, consider an affinely connected n-dimensional Kaehlerian recurrent space of second order, whose curvature tensor R_{ijk}^h satisfies the following condition :

$$\nabla_n \nabla_m R_{ijk}^h = \lambda_{mn} R_{ijk}^h, \qquad \text{... (1.4)}$$

where λ_{mn} is a non-symmetric, in general and non-vanishing covariant tensor. We shall assume to put the following two conditions in our space :

$$\nabla_j v^i = \phi_j v^i \qquad \dots (1.5)$$

and $$R_{jk} = \phi_j \alpha_k , \qquad \dots (1.6)$$

where α_k means a suitable covariant tensor, R_{kl}, we have

$$\nabla_n \nabla_m R_{ij} = \lambda_{mn} R_{ij} \qquad \dots (1.7)$$

Making use of (1.6) in (1.7), we have

$$\nabla_n \alpha_j \nabla_m \phi_i + \alpha_j \nabla_n \nabla_m \phi_i + \nabla_n \phi_i \nabla_m \alpha_j + \phi_i \nabla_n \nabla_m \alpha_j = \alpha_{mn} \phi_i \alpha_j \quad \dots (1.8)$$

In fact, when the space under consideration admits an affine motion

$$\bar{x}^i = x^i + v^i \delta t.$$

Characterized by (1.5), we have a resolved form of Ricci-tensor of the form (1.6). Here, we assume the existence of recurrent cavariant vector v^i given by (1.5) and in addition, the resolvability of R_{ij}.

Making a commutator on the indices m and n in (1.8), we obtain

$$-\alpha_j \phi_a R^a_{imn} - \phi_i \alpha_a R^a_{jmn} = A_{mn} \phi_i \alpha_j \qquad \dots (1.9)$$

where $$A_{mn} \overset{\text{def}}{=\!=} \lambda_{mn} - \lambda_{nm}.$$

Multiplying (1.9) by v^i and summing over i and making use of (1.8), we have

$$\phi\,(A_{mn} \alpha_j + \alpha_a R^a_{jmn} + \alpha_j \Omega_{mn}) = 0,$$

where $$\phi \overset{\text{def}}{=\!=} \phi_i v^i.$$

In the present paper, we have to discuss the next two cases :

$$A_{mn} \alpha_j + \alpha_a R^a_{jmn} + \alpha_j \Omega_{mn} = 0 \qquad \dots (1.10a)$$

and $$\phi = 0. \qquad \dots (1.10b)$$

2. THE CASE OF $\alpha = 0$. Putting $\alpha = \alpha_m v^m$, the condition (1.10a) yields this case. We shall show this fact. Making use of (1.8) in (1.10a), we have

$$\alpha\,(A_{mn} + 2\Omega_{mn}) = 0.$$

Consequently, in the present case (1.10 a), we have to consider two cases. The first is a case of $\alpha = 0$ and the second is that of

$$A_{mn} + 2\Omega_{mn} = 0.$$

In the latter case, multiplying this condition by v^h and making use of (1.8), we have

$$A_{mn}\, v^h + 2R^h_{amn}\, v^a = 0,$$

from which, by contraction on h and n, it follows that

$$A_{mn}\, v^n + 2R_{am}\, v^a = 0,$$

or

$$A_{mn}\, v^n + 2\phi\alpha_m = 0,$$

i.e., the second case means the first case and (1.10 a) may be replaced by $\alpha = 0$. By this reason, there exists only one case of $\alpha \neq 0$.

From (1.4), we have

$$A_{mn}\, R^h_{ijk} = R^h_{amn}\, R^a_{ijk} - R^a_{imn}\, R^h_{ajk} + R^a_{jmn}\, R^h_{ika} - R^a_{kmn}\, R^h_{ija} \quad \dots (2.1)$$

Contracting the indices h and k in (2.1), we have

$$A_{mn}\, R_{ij} = -R^a_{imn}\, R_{aj} - R^a_{jmn}\, R_{ia}\,. \qquad \dots (2.2)$$

Multiplying (2.2) by v^i and summing on i, because of $v^i R_{ij} = v^i \phi_i\, \alpha_j = \phi\alpha_j$ and $\phi = 0$, we have

$$\alpha_a\, R^a_{jmn} = -(A_{mn} + \Omega_{mn})\alpha_j\,, \qquad \dots (2.3)$$

where we have used (1.8) introducing (2.3) into the left hand side of (1.9), we have

$$\alpha_k\, (\phi_a\, R^a_{jmn} - \phi_j\, \Omega_{mn}) = 0.$$

Thus, there exist the following two cases to be discussed :

$$\alpha_k = 0 \qquad \dots (2.4a)$$

$$\phi_a\, R^a_{jmn} = \phi_j\, \Omega_{mn} \qquad \dots (2.4b)$$

In case of (2.4b), let us remember the first formula due to L.Bianchi, then we obtain

$$\phi_j\, \Omega_{mn} + \phi_m\, \Omega_{nj} + \phi_j\, \Omega_{jm} = 0. \qquad \dots (2.5)$$

On the other hand, as we have

$$\Omega_{mn} = -\,\Omega_{nm} \text{ and } \phi = \phi_j\, v^j\,,$$

we derive $\quad \Omega_{jk} v^i = -\,R_{ak} v^a = -\,\phi\alpha_k\,.$

from (1.8) by contraction on h and j.

Multiplying (2.5) by v^j and summing on j, we obtain

$$\Omega_{mn} = \alpha_m \phi_n - \alpha_n\, \phi_m\,. \qquad \text{... (2.6)}$$

Remembering the definition of Ω_{mn} and (1.6), we can write down an interesting formula :

$$R_{jk} - R_{kj} = \nabla_j\, \phi_k - \nabla_k\, \phi_j\,.$$

As we have (2.4b) and (1.8), we see that

$$\phi R^a_{jmn}\, v^h = \phi_j\, \Omega_{mn}\, v^h \text{ and } R^h_{amn}\, v^a = \Omega_{mn} v^h.$$

Hence, we have

$$\phi_a\, R^a_{jmn}\, v^h = \phi_j\, R^h_{amn}\, v^a$$

and this means that

$$R^a_{jmn}\, \nabla_a\, v^h - R^h_{amn}\, \nabla_j\, v^a = 0,$$

i.e., we have actually

$$\nabla_n\, \nabla_m\, (\nabla_j\, v^i) - \nabla_m\, \nabla_n\, (\nabla_j\, v^i) = 0.$$

Consequently, we can imagine the existence of a gradient vector ρ_k and we are able to put

$$\nabla_k\, \nabla_j\, v^h = \rho_k\, \nabla_j\, v^h \qquad \text{... (2.7)}$$

From (1.5) and (2.7), we can deduce the equation :

$$\nabla_k\, \phi_j + \phi_j\, \phi_k = \rho_k\, \phi_j\,.$$

Multiplying the above equation by v^j, we have the following :

$$\begin{aligned}
\phi\rho_k &= v^j\, \nabla_k \phi_j + \phi\phi_k \\
&= \nabla_k\, (v^j\, \phi_j) - \phi_j\, \nabla_k\, v^j + \phi\phi_k \\
&= \nabla_k\, \phi - \phi_j\, \phi_k\, v^j + \phi\phi_k \\
&= \nabla_k\, \phi - \phi\phi_k + \phi\phi_k = \nabla_k \phi\,.
\end{aligned}$$

In this way, the existence of ϕ_j is examined and we have here a characteristic condition on $\nabla_j\, v^h$.

$$\nabla_k \nabla_j v^h = \rho_k \nabla_j v^h,\ \rho_k = \frac{\nabla_k \phi}{\phi}.$$

On the other hand, in case of (2.4a), from (1.6), we obtain $R_{jk} = 0$.

Summerizing the above all conditions, we have the following :

Theorem (2.1) . In a n-dimensional Kaehlerian recurrence space of second order, admitting a contravariant vector v^h, characterized by

$$\nabla_j v^i = \phi_j v^i$$

and having a disjointed Ricci tensor of the form $R_{jk} = \phi_j \alpha_k$, there exist a case of $\alpha_k v^m = 0$.

In this case, if $\alpha_m = 0$, then we have the vanishing of Ricci tensor R_{jk} and if $\alpha_m \neq 0$, we have

$$R_{kl} - R_{lk} = \nabla_l \phi_k - \nabla_k \phi_l.$$

The mixed tensor $\nabla_j v^i$ itself is a recurrent one characterized by

$$\nabla_l \nabla_k v^i = \rho_l \nabla_k v^i$$

for a definite gradient vector $\rho_l = \nabla_l \phi/\phi$.

Remark . The latter part of the above theorem does not contain a case of ϕ_j being a gradient vector.

Such a case is a showing but worthless one, in fact, if ϕ_j will be a gradient vector, we have

$$\alpha_l \phi_k - \alpha_k \phi_l = \nabla_k \phi_l - \nabla_l \phi_k = 0,$$

from which we have

$$\lambda \phi_k v^k = \alpha_k v^k = \alpha = 0,$$

this is $\lambda\phi = 0$ from which being $\phi \neq 0$, it follows that $\lambda = 0$.

Hence, we have $\alpha_k = 0$.

This a contradiction, for we have $\nabla_l \phi_k + \phi_k \phi_l = 0$ at this moment and the case of ϕ_k being a gradient vector occurs.

3. The Case of $\phi = 0$. Let us consider the case of (1.10b). Then using the analogous method used in §2, multiplying (2.2) by v^j, we obtain the following formula :

$$\phi_a R^a_{jmn} = -(A_{mn} + \Omega_{mn})\,\phi_j \qquad \dots (2.8)$$

Substituting (2.8) into the left hand side of (1.9), we get

$$\phi_j (\alpha_a R^a_{kmn} - \alpha_k \Omega_{mn}) = 0.$$

Hence, we have here two cases to be discussed. They are

$$\phi_j = 0 \qquad \dots (2.9a)$$

and

$$\alpha_a R^a_{kmn} = \alpha_k \Omega_{mn} \qquad \dots (2.9b)$$

The case of (2.9a) yields one of $\nabla_i v^h = 0$ and $R_{ij} = 0$.

The case of (2.9b) may be treated as follows :

We have

$$\alpha_j \Omega_{mn} + \alpha_m \Omega_{nj} + \alpha_n \Omega_{jm} = 0,$$

from which by contraction of v^j, we have

$$\alpha\Omega_{mn} = 0,$$

because of $\Omega_{jk} v^j = -R_{ak} v^a = -\phi_a \alpha_k v^a = -\phi\alpha_k$.

From (1.8), by contraction on the indices h and j, and $\Omega_{mn} = -\Omega_{nm}$ and $\phi = \phi_j v^j$.

In this way, we obtain $\Omega_{mn} = 0$, say $\nabla_n \phi_m = \nabla_m \phi_n$.

Thus, we can state here the following :

Theorem 2.2 . When $\phi = 0$ in our space, there exist two cases. One of them is a case of satisfying $\nabla_j v^i = 0$ and $R_{jk} = 0$ and the other is a case of $\nabla_l \phi_k = \nabla_k \phi_i$.

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ON THE SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

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Let $\phi_{ij}(x, y)$, $(i, j = 0, 1, 2, \ldots)$ be a sequence of orthonormal functions in the rectangle R $[a \le x \le b, c \le y \le d]$ i.e.

$$\int_R\!\!\int \phi_{ij}(x, y)\, \phi_{k,l}(x, y)\, dx\, dy = \begin{cases} 1 & \text{for } k = i, l = j \\ 0 & \text{for } k \ne i, l \ne j \end{cases}$$

Consider the double orthogonal series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\, \phi_{ij}(x, y) \qquad \ldots (1.1)$$

where $\{a_{ij}\}$ is an arbitrary sequence of real numbers.

The mnth partial sum of the series (1.1) is given by

$$s_{mn}(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}\, \phi_{ij}(x, y).$$

The series (1.1) is said to be $(R, \lambda, \mu, 1, 1)$ summable to a function $s(x, y)$, if

$$\lim_{m, n \to} \sigma_{mn}(\lambda, \mu, x, y) = s(x, y)$$

where

$$\sigma_{mn}(\lambda, \mu, x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{m+1}}\right)\left(1 - \frac{\mu_l}{\mu_{n+1}}\right) a_{kl}\, \phi_{kl}(x, y)$$

The mnth Euler mean of the series (1.1) is given by

$$\tau_{mn}(x, y) = \frac{1}{2^{m+n}} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} s_{kl}(x, y).$$

The series (1.1) is said to be $(E,1,1)$ summable to a function $s(x, y)$, if

$$\lim_{m, n \to \infty} \tau_{mn}(x, y) = s(x, y).$$

The convergence and Casáro summability of single orthogonal series have been studied by Alexits[2], Kaczmarz[3], Menchoff[6], Rademacher[10], Tandori[12] and Zygmund[13] and those of double orthogonal series by Agnew[1], Fedulov[4], Pandzakidze[8], Mitchell[7], Patel[9] and Sapre[11]. Dealing with (C,1,1) summability of double orthogonal series (1.1) Patel[9] has proved the following theorem :

Theorem. If

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty \qquad \text{... (1.2)}$$

Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} (s_{mn}(x, y) - \sigma_{mn}(x, y))^2, p > 1$$

is convergent almost everywhere in R, where $\sigma_{mn}(x, y)$ is the $(C,1,1)$ mean of the series (1.1) We extend in this paper the above result to Riesz and Euler summability as follows :

Theorem 1. Under the condition (1.2)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2, p > 1$$

converges almost everywhere in R.

Theorem 2. Under the condition (1.2)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} [s_{mn}(x, y) - \tau_{mn}(x, y)]^2, p > 1$$

converges almost everywhere in R.

In order to prove the above theorems, we need the following lemmas :

Lemma 1 [5]. Writing

$$W_{nk} = \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1},$$

we have

$$W_{nk} < 0 \left[\frac{n}{3} \right] + 2 \le k \le n.$$

Lemma 2 [5].

$$\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i}^2 \le A \frac{k^2}{n^2} \text{ for } 1 \le k \le \left[\frac{n}{3} \right] + 1$$

2. Proof of Theorem 1. We have

$$\begin{aligned}
& s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y) \\
&= \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl}\, \phi_{k\lambda}(x, y) - \sum_{k=0}^{m} \sum_{l=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{m+1}}\right)\left(1 - \frac{\mu_1}{\mu_{n+1}}\right) a_{kl}\, \phi_{kl}(x, y) \\
&= \sum_{k=0}^{m} \sum_{l=0}^{n} \left(\frac{\lambda_k}{\lambda_{m+1}} + \frac{\mu_l}{\mu_{n+1}} - \frac{\lambda_k \mu_l}{\lambda_{m+1} \mu_{n+1}} \right) a_{kl}\, \phi_{kl}(x, y) \\
&= \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k}{\lambda_{n+1}} a_{kl}\, \phi_{kl}(x, y) + \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\mu_1}{\mu_{n+1}} a_{kl}\, \phi_{kl}(x, y) \\
&\qquad - \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k \mu_i}{\lambda_{m+1} \mu_{n+1}} a_{kl}\, \phi_{kl}(x, y).
\end{aligned}$$

Now, from the inequality

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 \le 3(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$$

and the orthonormality property of the functions $\phi_{ij}(x, y)$, we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \int_R\!\!\int [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2 \, dx\, dy$$

$$\leq 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k^2}{\lambda_{m+1}^2}\, a_{kl}^2 +$$

$$+ 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\mu_l^2}{\mu_{n+1}^2}\, a_{kl}^2$$

$$+ 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k^2\,\mu_l^2}{\lambda_{m+1}^2\,\mu_{n+1}^2}\, a_{kl}^2$$

$= 3\,\Sigma_1 + 3\,\Sigma_2 + 3\,\Sigma_3$, say.

Now,

$$\sum\nolimits_1 = \sum_{m=1}^{\infty} \frac{1}{m\,(\log m)^p\,\lambda_{m+1}^2} \sum_{k=1}^{m} \lambda_k^2 \sum_{n=1}^{\infty} \frac{1}{n\,(\log n)^p} \sum_{l=1}^{n} a_{kl}^2$$

$$= \sum_{k=1}^{\infty} \lambda_k^2 \sum_{m=k}^{\infty} \frac{1}{m\,(\log m)^p\,\lambda_{m+1}^2} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$\leq \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \frac{1}{m\,(\log m)^p} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty\,.$$

The convergence of Σ_2 and Σ_3 follows on the same line.

Hence by B.Levy's theorem, it follows that the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p}\, [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2$$

converges almost everywhere in R.

This completes the proof of the theorem.

3. Proof of Theorem 2. We have

$s_{m,n}(x, y) - \tau_{m,n}(x, y)$

$$2^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl}\,\phi_{kl}(x, y) \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j}$$

$$-\frac{1}{2^{m+n}}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\sum_{k=0}^{i}\sum_{l=0}^{j}a_{kl}\,\phi_{kl}(x,y)$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\,\phi_{kl}(x,y)\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}$$

$$-\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\,\phi_{kl}(x,y)\sum_{i=k}^{m}\sum_{j=l}^{n}\binom{m}{i}\binom{n}{j}$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}\left[\sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{l-1}\binom{n}{j}+\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}\right]a_{kl}\,\phi_{kl}(x,y)$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}2^{m}\sum_{j=0}^{l-1}\binom{n}{j}a_{kl}\,\phi_{kl}(x,y)$$

$$+\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\,\phi_{kl}(x,y)\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}.$$

Now, from the inequality

$$(\alpha_1+\alpha_2)^2\le 2\,(\alpha_1^2+\alpha_2^2)$$

and the orthonormality property of the functions $\{\phi_{ij} : (x, y)\}$, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^{p}}\iint_{R}[s_{mn}(x,y)-\tau_{mn}(x,y)]^{2}\,dx\,dy$$

$$\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^{p}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}^{2}\left\{\frac{1}{2^{n}}\sum_{j=0}^{l-1}\binom{n}{j}\right\}^{2}$$

$$+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^{p}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}^{2}\left\{\frac{1}{2^{m+n}}\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}\right\}^{2}$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^{p}}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}^{2}\,\frac{l^{2}}{n^{2}}$$

$$+O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^{p}}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}^{2}\,\frac{k^{2}}{m^{2}}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^2$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m\,(\log m)^p} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty.$$

Hence, it follows by B.Levy's theorem that the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \; [s_{mn}(x, y) - \tau_{mn}(x, y)]^2$$

converges almost everywhere.

With this the theorem is proved.

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FUNCTIONS OF CLASS *Lip* (α, *p*) AND THEIR *F*(*a*, *q*) MEAN

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ABSTRACT

In the present paper some results of Mohapatra, Holland and Sahney [6] on Taylor mean are extended to general class of $F(a, q)$ transform.

1. INTRODUCTION

Let $f \in L[-\pi, \pi]$ and be periodic with period 2π outside this range. Let the Fourier series associated with $f(x)$ be given by

$$f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \qquad \ldots (1.1)$$

We write

$$\varphi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \qquad \ldots (1.2)$$

The family $F(a, q)$ of summability methods was introduced by Meir[5]. The summability matrix $\{c_{pk}\}$ belongs to $F(a, q)$ if it satisfies the following conditions :

p is a discrete or continuous parameter; $q = q(p)$ is a positive increasing function which tends to infinity as $p \to \infty$, 'a' is a positive constant; for every fixed $\delta : \frac{1}{2} < \delta < \frac{2}{3}$.

$$c_p k = \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right)\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \qquad \ldots (1.3)$$

as $p \to \infty$ uniformly in k for $|k - q| \le q^{\delta}$; and

$$\sum_{|k-q| > q^{\delta}} (k+1)\, c_p k = O\{\exp(-q^{\mu})\} \qquad \ldots (1.4)$$

where μ is some positive number independent of p.

Let

$$l_p(f;x) = \sum_{k=0}^{\infty} c_{pk}\, s_k(f;x) \qquad \dots (1.5)$$

denote the $F(a,q)$ mean of the Fourier series (1.1) of f, where $s_k(f;x)$ is the kth partial sum of (1.1).

The family $F(a,q)$ contains the summability methods of generalised Borel, Euler, Taylor, S_α and Valiron.

It is known (see Kuttner, Rajagopal and Rangachari[4]) that

$$\sum_{k=0}^{\infty} c_{pk} = 1 + O(q^{-1/2}) \qquad \dots (1.6)$$

The summability methods of Euler, Taylor, S_α and Borel satisfy (1.6) in the stronger form

$$\sum_{k=0}^{\infty} c_{pk} = 1 \qquad \dots (1.7)$$

Norms considered in this chapter are Lp $(p \geq 1)$ norms and throughout the paper norms are taken with respect to variable x. The classes *Lip* α, *Lip* (α, p) $(p \geq 1)$ are as defined by Hardy and Littlewood [1, p.612] and Zygmund [7], p. 42, 45. The class *Lip* (α, p) reduces to *Lip* α with $p = \infty$.

For $p \geq 1$, $\delta > 0$, let $\omega(\delta, g)$ and $\omega_p(\delta, g)$ denote the modulus of continuity and integral modulus of continuity respectively of an appropriate function g.

2. In 1985 Mohapatra, Holland and Sahney [6] studied order of approximation by Taylor mean $T_n^r(f;x)$ of the Fourier series of $f(x)$ when $f(x)$ belongs to the class *Lip* (α, p) and proved the following theorems :

THEOREM A. If $f \in Lip\,(\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$, then

$$\| T_n^r(f;x) - f(x) \| = O(n^{-\alpha\beta}) \qquad \dots (2.1)$$

for $0 < \beta < 1/2$.

THEOREM B. If $f \in Lp$, $(p > 1)$, then for $0 < \beta < 1$

$$\|T_n^r (f\,;x) - f(x)\| = O(\omega_p (1/n\,;f)) + O\left(\int_{a_n}^{(a_n)^\beta} \frac{\omega_p (t\,;f)}{t}\, dt \right) + O\,(n^\beta \exp(-An^{1-2\beta})) \quad \ldots (2.2)$$

where $a_n = \pi \left(n + \frac{1}{2} + \frac{n+1}{1-r}\, r \right)^{-1}$.

THEOREM C. If $f \in Lip\ \alpha$, $0 < \alpha \le 1$, $p > 1$, $\alpha p > 1$

$$T_n^r (f\,;x) - f(x) = O(n^{-(\alpha - 1/p)\beta})\ (0 < \beta < 1/2) \quad \ldots (2.3)$$

uniformly in x almost everywhere.

Since $F(a, q)$ method includes Taylor method, it is natural to ask as to what be the result if we apply $F(a, q)$ mean in place of Taylor mean in the above theorems. In answer to this question our aim in the present paper is to extend the above results to the family $F(a, q)$.

We prove the following results :

THEOREM 1. Let $[q]$ denote the integral part of $q = q(p)$ and $m = [q] + 1$. If $f \in Lp$, $(p > 1)$ and

$$\frac{\omega_p (t, f)}{t^{1/2}} \text{ is decreasing function of } t \text{ in } 0 \le t \le \pi \quad \ldots (2.4)$$

then, for $0 < \beta < 1$,

$$\|\, t_p (f\,;x) - f(x)\,\| = O(\omega_p^{(1/m,\,f)}) + O\left(\int_{\pi/m}^{\pi/m^\beta} \frac{\omega_p (t\,;f)}{t}\, dt \right) + O\,(m^\beta \exp(-Am^{1-2\beta})) \quad \ldots (2.5)$$

where $A = \frac{\pi^2}{4a}$, a constant.

THEOREM 2. If $f \in Lip\ (\alpha, p)$, $0 < \alpha \le 1$, $p > 1$, then

$$\|\, t_p (f\,;x) - f(x)\,\| = O(m^{-\alpha\beta}) \quad \ldots (2.6)$$

for $0 < \beta < 1/2$, where m is as in the Theorem 1.

THEOREM 3. If $f \in Lip\ (\alpha, p)$. $0 < \alpha \le 1$, $p > 1$. $\alpha p > 1$

$$t_p (f\,;x) - f(x) = O(m^{-(\alpha - 1/p)\beta})\ (0 < \beta < 1/2) \quad \ldots (2.7)$$

uniformly in x almost everywhere.

3. We shall need the following lemmas :

LEMMA 1. If $h(x, t)$ is a function of two variables defined for $0 \le t \le \pi$, $0 \le x \le 2\pi$, then

$$\| \int h(x, t)\, dt \| \le \int \| h(x, t)\, dt \|_p \, dt \ (p > 1).$$

This is due to Hardy, Littlewood and Polya[2].

LEMMA 2. If $q = q(p)$ is an integer valued function of p, then for $\frac{1}{2} < \delta < \frac{2}{3}$, we have

$$\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-1)^2) \sin(k+1/2)\, t\, dt$$

$$= \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{qt^2}{4a}\right) \sin(q+1/2)\, t\, dt + O(q \exp(-aq^{2\delta-1}))$$

PROOF OF LEMMA 2. From the proof of Lemma 3.2 of Ikeno[3], we have

$$\sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)\, t$$

$$= \exp\left(-\frac{qt^2}{4a}\right) \sin(q+1/2)\, t\, dt + O(q \exp(-aq^{2\delta-1})\, |t|)$$

Hence the Lemma 2 follows in view of boundedness of $\varphi_x(t)$ and since $\sin t/2 > t/\pi$ $(0 < t < \pi)$.

LEMMA 3. If m is as defined in the Theorem 1, we have

$$\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)\, t\, dt$$

$$= \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^{\delta}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt + O(\omega_p(1/m, f))$$

PROOF OF LEMMA 3. We estimate the difference

$$\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$- \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^{\delta}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt$$

$$= \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{m \le k \le m + m^\delta} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2)\right] \sin(k+1/2)\, t\, dt$$

$$+ \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{m - m^\delta \le k < m} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2)\right] \sin(k+1/2)t\, dt$$

$$- \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{q + q^\delta < k \le m + m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$+ \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{q - q^\delta \le k \le m - m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$= D_1 + D_2 + D_3 + D_4 \qquad \dots (3.1)$$

From Ikeno[3], p.261, 262, we have

$$\sum_{q + q^\delta < k \le m + m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t = O(\sqrt{q} \exp(-aq^{2\delta - 1}))$$

and also

$$\sum_{q - q^\delta < k \le m - m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t = O(\sqrt{q} \exp(-aq^{2\delta - 1}))$$

Thus, using Lemma 1

$$\| D_3 \| \le \int_0^\pi \frac{\| \varphi_x(t) \|}{t} \sum_{q + q^\delta < k \le m + m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, |\sin(k + 1/2)t|\, dt$$

$$= O(\sqrt{q} \exp(-aq^{2\delta - 1})). \qquad \dots (3.2)$$

Similarly,

$$\| D_4 \| = O(\sqrt{q} \exp(-aq^{2\delta - 1})) \qquad \dots (3.3)$$

In case where $q < m \le k \le m + m^\delta$, we have

$$O \le \frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}}.$$

Therefore, from Ikeno [3, p.250]

$$\left| \sqrt{\frac{a}{\pi q}}\, \exp\left(-aq^{-1}(k-q)^2\right) - \sqrt{\frac{a}{\pi m}}\, \exp\left(-am^{-1}(k-m)^2\right) \right|$$

$$= O\left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right)\left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m}\right)\right\} \qquad \dots (3.4)$$

Using (3.3) and Lemma 1, we obtain

$$\| D_1 \| = O\left[\int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{m \le k \le m + m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m}\right)\right\} |\sin(k + 1/2)t|\, dt \right]$$

$$= O\left[\int_0^{\pi/m} \frac{\| \varphi_x(t) \|}{t} \sum_{m \le k \le m + m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m}\right)\right\} (|k-m| + m + 1/2)t\, dt \right]$$

$$+ O\left[\int_{\pi/m}^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{m \le k \le m + m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m}\right)\right\} dt \right]$$

$$= O\left\{ \sqrt{m} \int_0^{\pi/m} \| \varphi_x(t) \|\, dt \right\} + O\left\{ \frac{1}{\sqrt{m}} \int_{\pi/m}^{\pi} \frac{\| \varphi_x(t) \|}{t}\, dt \right\}$$

$$= O\left\{ \sqrt{m} \int_0^{\pi/m} \omega_p(t, f)\, dt \right\} + O\left\{ \frac{1}{\sqrt{m}} \int_{\pi/m}^{\pi} \frac{\omega_p(t, f)}{t}\, dt \right\}$$

$$= O\left(\omega_p(1/m.\, f)\right) \qquad \dots (3.5)$$

Also in case where $k \le [q] \le q < m$, we get

$$\frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}} \le 0.$$

Hence from Ikeno [3, p.260] the following estimate results :

$$\left| \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) - \sqrt{\frac{a}{\pi m}} \exp\left(-am^{-1}(k-m)^2\right) \right|$$

$$= O\left(\frac{1}{\sqrt{[q]}} \exp\left(-a[q]^{-1}(k-[q])^2\left(\frac{(k-[q])^2}{[q]^2} + \frac{|k-[q]|}{[q]} + \frac{1}{[q]} \right)\right)\right)$$

Proceeding as in the estimation of D_1 and using the above inequality we get

$$\| D_2 \| = \omega_p\,(1/m, f). \qquad \dots (3.6)$$

Thus, Lemma 3 follows from (3.1) to (3.6).

LEMMA 4. Suppose that $f \in Lip\,(\alpha, p)$, where $p > 1$, $0 < \alpha \leq 1$, $\alpha p > 1$. Then f is equal to a function $g \in Lip\,(\alpha - 1/p)$ almost everywhere.

This is due to Hardy and Littlewood[1], Theorem 5(ii), p.627.

4. PROOF OF THEOREM 1. Since

$$s_k\,(f\,;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sin\,(k+1/2)t\,dt$$

we have

$$t_p\,(f\,;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x\,(t)}{\sin t/2} \sum_{k=0}^{\infty} c_p\,k \sin\,(k+1/2)t\,dt + O(q^{-1/2})$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x\,(t)}{\sin t/2}\left[\left(\sum_{|k-q| \leq q^{\delta}} + \sum_{|k-q| > q^{\delta}} \right) c_p\,k \sin\,(k+1/2)t \right] dt + O(q^{-1/2})$$

$$= S_1 + S_2 + O(q^{-1/2}). \qquad \dots (4.1)$$

By the Minkowski Inequality

$$\| t_p\,(f\,;x) - f(x) \| \leq \| S_1 \| + \| S_2 \| + O(q^{-1/2}) \qquad \dots (4.2)$$

By (1.4), Lemma 1 and the fact that $\sin t/2 \geq t/\pi\ (O < t \leq \pi)$, we obtain

$$\| S_2 \| \leq \int_0^{\pi} \frac{\| \varphi_x\,(t) \|}{t} \sum_{|k-q| > q^{\delta}} c_p\,k\,(k+1/2)t\,dt = O(\exp\,(-q^{\mu})).$$

$$\dots (4.3)$$

Now making use of (1.3), we can write

$$S_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}}\ \exp\left(-aq^{-1}(k-q)^2\right)$$

$$\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \sin(k+1/2)t\,dt$$

$$= S_3 + S_4 + S_5. \qquad \dots (4.4)$$

Using Lemma 1, we estimate S_4 as follows :

$$\| S_4 \| \le \int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| \le q^{\delta}} \left\{\sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) O\left(\frac{|k-q|+1}{q}\right)\right\}$$

$$|\sin(k+½)t|\,dt$$

$$= \int_0^{\pi/q} \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| \le q^{\delta}} \left\{\sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) O\left(\frac{|k-q|+1}{q}\right)\right\}$$

$$(|k-q| + q + ½)t\,dt$$

$$+ \int_{\pi/q}^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}}\ \exp\left(-aq^{-1}(k-q)^2\right) O\left(\frac{|k-q|+1}{q}\right) dt$$

$$= O\left(\sqrt{q}\int_0^{\pi/q} \| \varphi_x(t) \|\,dt\right) + O\left(\frac{1}{\sqrt{q}}\int_{\pi/q}^{\pi} \frac{\| \varphi_x(t) \|}{t}\,dt\right)$$

$$= IO\left(\omega_p(1/q, f)\right). \qquad \dots (4.5)$$

Similarly,

$$\| S_5 \| = O\left(\omega_p(1/q, f)\right). \qquad \dots (4.6)$$

Applying Lemma 2 and Lemma 3 and noting that $m = m(p)$ is an integer valued function of p we get

$$S_3 = \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a}\right) \sin(m+½)t\,dt + O\left(\omega_p(1/m, f)\right)$$

$$\dots (4.7)$$

Now we write

$$I_1 = \left(\int_0^{\pi/m} + \int_{\pi/m}^{\pi/m\,\beta} + \int_{\pi/m\,\beta}^{\pi}\right) \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a}\right) \sin(m+½)t\,dt$$

$$= I_{1.2} + I_{1.2} + I_{1.3} \qquad \dots (4.8)$$

Since $\sin t/2 \le t/\pi \ (0 < t \le \pi)$, we have by Lemma 1

$$\| I_{1.1} \| \le \pi \int_0^{\pi/m} \frac{\| \varphi_x(t) \|}{t} \exp\left(-\frac{mt^2}{4a} \right) \sin (m + ½)t \, dt$$

$$\le O\left(\int_0^{\pi/m} \frac{\omega_p(t, f)}{t} (m + ½)t \, dt \right)$$

$$= O(\omega_p(1/m, f)). \qquad \dots (4.9)$$

Further, by Lemma 1

$$\| I_{1.3} \| \le O\left(\int_{\pi/m\,\beta}^{\pi} \frac{\omega_p(t, f)}{t} \exp\left(-\frac{mt^2}{4a} \right) dt \right)$$

$$= O\,\{ m^{\beta} \exp(-Am^{1-2\beta}) \}. \qquad \dots (4.10)$$

Finally, by using Lemma 1 and the fact that $|\sin x| \le 1$ for all x, we obtain

$$\| I_{1.2} \| \le \int_{\pi/m}^{\pi/m\,\beta} \frac{\| \varphi_x(t) \|}{t} \exp\left(-\frac{mt^2}{4a} \right) dt$$

$$= \int_{\pi/m}^{\pi/m\,\beta} \frac{\omega_p(t, f)}{t} \exp\left(-\frac{mt^2}{4a} \right) dt$$

$$= O\left(\int_{\pi/m}^{\pi/m\,\beta} \frac{\omega_p(t, f)}{t} \, dt \right). \qquad \dots (4.11)$$

Estimates (4.1), (4.2), ..., (4.11) togther yield the proof of the Theorem 1.

PROOF OF THE THEOREM 2. For $\beta < \frac{1}{2}$, we have

$$m^{\beta} \exp(-Am^{1-2\beta}) = \omega_p(1/m, f). \qquad \dots (4.12)$$

Since $f \in Lip(\alpha, p)$, $0 < \alpha \le 1$, $p > 1$,

$$\omega_p(1/m, f) = O(m^{-\alpha}). \qquad \dots (4.13)$$

Now to deduce the Theorem 2 from Theorem 1 it suffices to show that

$$\int_{\pi/m}^{\pi/m\,\beta} \frac{\omega_p(t, f)}{t} \, dt = O(m^{-\alpha\beta}). \qquad \dots (4.14)$$

Above equation follows from the fact that

$$\int_{\pi/m}^{\pi/m} t^{\alpha - 1} \, dt = O(m^{-\alpha\beta}) + O(m^{-\alpha}).$$

PROOF OF THE THEOREM 3. Since $f \in Lip\,(\alpha, p)$, $0 \leq \alpha < 1$, $p > 1$, $\alpha p > 1$ from Lemma 4 we see that there exists a function $g \in Lip\,(\alpha - 1/p)$, which is equal to f almost everywhere. Hence, we have almost everywhere

$$\varphi_x(t) = O\,(t^{\alpha - 1/p}). \qquad \ldots (4.15)$$

Now in order to prove the Theorem 3 we modify estimates of D_1, D_2, D_3, D_4 in the proof of Lemma 3.

Using arguments similar to those used in estimating D_3

$$D_3 = O\left[\int_0^{\pi} \frac{|\varphi_x(t)|}{t} \sum_{q+q^{\delta} < k \leq m+m^{\delta}} \sqrt{\frac{a}{\pi q}}\,\exp(-aq^{-1}(k-q)^2)\,|\sin(k+1/2)t|\,dt\right]$$

$$= O\,(\sqrt{q}\,\exp(-aq^{2\delta-1})). \qquad \ldots (4.16)$$

Similarly,

$$D_4 = O\,(\sqrt{q}\,\exp(-aq^{2\delta-1})). \qquad \ldots (4.17)$$

As in the proof of Lemma 3, we get

$$D_1 = O\left\{\sqrt{m}\int_0^{\pi/m} |\varphi_x(t)|\,dt\right\} + O\left\{\frac{1}{\sqrt{m}}\int_{\pi/m}^{\pi} \frac{|\varphi_x(t)|}{t}\,dt\right\}$$

$$= O(m^{-\alpha + 1/p}). \qquad \ldots (4.18)$$

In a similar manner, we can obtain

$$D_2 = O(m^{-\alpha + 1/p}). \qquad \ldots (4.19)$$

Thus, combining (3.1), (4.16), (4.17) and (4.19), we have

$$\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \leq q^{\delta}} \sqrt{\frac{a}{\pi q}}\,\exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t\,dt$$

$$= \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \leq m^{\delta}} \sqrt{\frac{a}{\pi m}}\,\exp(-am^{-1}(k-m)^2)\sin(k+1/2)t\,dt$$

$$+ O\,(m^{-\alpha + 1/p}) \qquad \ldots (4.20)$$

In the notations of the Theorem 1.

$$t_p\,(f\,;x) - f(x) \leq S_1 + S_2 + O(q^{-1/2}). \qquad \ldots (4.21)$$

Using (1.4), we get

$$S_2 = O\left(\int_0^{\pi} \frac{|\varphi_x(t)|}{t} \sum_{|k-q|>q^{\delta}} c_p\, k\,(k+1/2)t\, dt \right)$$

$$= O\,(\exp\,(-q^{\mu})). \qquad \ldots (4.22)$$

From (4.4), we have

$$S_1 = S_3 + S_4 + S_5. \qquad \ldots (4.23)$$

On the lines of the esimation of S_4 in the proof of the Theorem 1 and using (4.15), we obtain

$$S_4 = O\left(\sqrt{q} \int_0^{\pi/q} |\varphi_x(t)|\, dt \right) + O\left(\frac{1}{\sqrt{q}} \int_{\pi/q}^{\pi} \frac{|\varphi_x(t)|}{t}\, dt \right)$$

$$= O\,(q^{-\alpha+1/p}). \qquad \ldots (4.24)$$

Similarly,

$$S_5 = O(q^{-\alpha+1/p}). \qquad \ldots (4.25)$$

Applying Lemma 2 and (4.20) and using the fact that $m = m(p)$ is an integer valued function of p, we get

$$S_3 = \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a} \right) \sin\,(m+1/2)t\, dt + O(m^{-(\alpha+1/p)})$$

$$= I_1 + O\,(m^{-(\alpha+1/p)}).$$

As in the Theorem 1, we write

$$I_1 = \left(\int_0^{\pi/m} + \int_{\pi/m}^{\pi/m\,\beta} + \int_{\pi/m\,\beta}^{\pi} \right) \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a} \right) \sin\,(m+1/2)t\, dt$$

$$= I_{1.1} + I_{1.2} + I_{1.3}\,. \qquad \ldots (4.26)$$

Now using (4.15) we shall modify the estimates of $I_{1.1}$, $I_{1.2}$, $I_{1.3}$.

We have

$$I_{1.1} = O\left(\int_0^{\pi/m} \frac{|\varphi_x(t)|}{t} \exp\left(-\frac{mt^2}{4a} \right) \sin m + 1/2)t\, dt \right)$$

$$= O\left(\int_0^{\pi/m} \frac{t^{\alpha-1/p}}{t} (m+1/2)t\, dt \right)$$

$$= O\,(m^{-\alpha+1/p}). \qquad \ldots (4.27)$$

Again,

$$I_{1.3} = O\left(\int_{\pi/m\,\beta}^{\pi} t^{\alpha - 1/p - 1}\exp\left(-\frac{mt^2}{4a}\right)dt\right)$$

$$= O\,(m^{-(\alpha - 1/p)\,\beta}). \qquad \dots (4.28)$$

Lastly,

$$I_{1.2} = O\left(\int_{\pi/m}^{\pi/m\,\beta} \frac{|\varphi_x(t)|}{t}\exp\left(-\frac{mt^2}{4a}\right)dt\right)$$

$$= O\left(\int_{\pi/m}^{\pi/m\,\beta} t^{\alpha - 1/p - 1}\,dt\right)$$

$$= O\,(m - (\alpha - 1/p)\beta) + O\,(m^{-\alpha + 1/p})$$

$$= O\,(m^{-(\alpha - 1/p)\,\beta}). \qquad \dots (4.29)$$

Hence the Theorem 3 follows from (4.21) to (4.29).

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MHD FLUCTUATING HELE-SHAW FLOW OF A VISCOUS FLUID PAST A CIRCULAR CYLINDER

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ABSTRACT

In the present problem an attempt has been made to study the effects of boundary fluctuations and the transverse magnetic field on the Hele-Shaw flow of incompressible viscous fluid past a circular cylinder $x^2 + y^2 = r^2, -d \leq z \leq d$.

1. INTRODUCTION. In 1897, Prof. Hele-Shaw established that the Hele-Shaw cell is an arrangement of two parallel plates very close together, the space between the plates being occupied partly by fluid and partly by obstacles in the form of cylinders with generators normal to the plates. Several researchers have paid their attention towards the Hele-Shaw flows. Lamb (1932), Thompson (1968), Lee & Fung (1969) and Buckmaster (1970) have discussed such flows. In these investigations, the pressure gradient was assumed to be a constant and the flow in the cell to be steady. Swaminathan (1975) studied the unsteady Hele-Shaw flow of viscous incompressible fluid under time dependent pressure gradient. Lal & Singh (1981) have extended the work of Swaminathan to discuss the unsteady Hele-Shaw flow through porous media under time varying pressure gradient.

Reigels (1989) has discussed *MHD* Hele-Shaw flow of Rivlin-Ericksen fluid. Sengupta and Roy (1991) have also investigated *MHD* flow of n-immiscible viscoelastic fluid under gravity between two parallel plates. Recently, Ghosh & Sengupta (1995) have studied *MHD* Hele-Shaw flow of Rivlin-Ericksen fluid.

In this paper, we have studied the effects of boundary fluctuations under the influence of transverse magnetic field on the Hele-Shaw flow under constant pressure. The expressions for extact velocity distributions have been found and graphically represented under the influence of magnetic field of different intensities.

2. GOVERNING EQUATIONS OF MOTION. Let the viscous incompressible fluid flows past a vertical cylinder $x^2 + y^2 = r^2$, $-d \le z \le d$; placed between two parallel plates $z = \pm d$.

The equations of motion for a viscous incompressible fluid in the unsteady Hele-Shaw cell under transverse magnetic field are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_2}{\partial z^2} - \frac{\sigma}{\rho} B_0^2 u \qquad \text{... (2.1)}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2} - \frac{\sigma}{\rho} B_0^2 v \qquad \text{... (2.2)}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{... (2.3)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{... (2.4)}$$

where ρ the density, p the pressure of the field, σ the electrical conductivity, B_0 the strength of magnetic field. The boundary conditions are

$$u = u_0 e^{int},\ v = u_0 e^{int} \text{ at } z = \pm d \qquad \text{... (2.5)}$$

Normal velocity at the surface of the cylinder is zero. ... (2.6)

U is the finite and $V = 0$ as $|x|, |y| \to \infty$ ... (2.7)

3. MATHEMATICAL ANALYSIS. We introduce the following non-dimensional quantities

$$y' = \frac{y}{d},\ x' = \frac{x}{d},\ r' = \frac{r}{d},\ u' = \frac{u}{u_0},\ t' = \frac{\mu t}{\rho d^2},\ \nu' = \frac{\nu}{u_0},\ p' = \frac{pd}{\mu u_0},$$

$$d' = \frac{n\rho d^2}{\mu} \text{ and } M = B_0 d \sqrt{\sigma/\mu},$$

where M is the Hartmann number.

The equations (2.1) to (2.4) under the constant pressure (after dropping dashes) reduce to :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} - M^2 u \qquad \text{... (3.0)}$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial z^2} - M^2 v \qquad \text{... (3.1)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{... (3.2)}$$

The boundary conditions, accordingly, are

$$u = v = e^{int} \text{ at } z = \pm 1 \qquad \dots (3.3)$$

The conditions for ϕ are

$$\frac{\partial \phi}{\partial r} = 0 \text{ at the surface of the cylinder where } x^2 + y^2 = R^2 \qquad \dots (3.4)$$

$$\frac{\partial \phi}{\partial x} = 1 \text{ and } \frac{\partial \phi}{\partial y} = 0 \text{ as } |x| \to \infty,\ |y| \to \infty \qquad \dots (3.5)$$

4. SOLUTION. Let

$$u = U(z) \frac{\partial \phi}{\partial x} e^{int}, -1 \le z \le 1 \qquad \dots (4.1)$$

$$v = U(z) \frac{\partial \phi}{\partial y} e^{int}, -1 \le z \le 1 \qquad \dots (4.2)$$

$$u = v = U(z)\, e^{int} \text{ at } z = \pm 1 \qquad \dots (4.3)$$

The conditions (3.3) suggests that

$$U(z) = 1 \text{ at } z = \pm 1 \qquad \dots (4.4)$$

Now substituting from (4.1) to (4.3) in equations (3.0) to (3.2), we get

$$\frac{\partial^2 u}{\partial z^2} - (M^2 + in)\, U = 0 \qquad \dots (4.5)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \dots (4.6)$$

The solution of equation (4.5) under the conditions (4.4) can easily be obtained as

$$U = \frac{\cosh (\sqrt{M^2 + in})z}{\cosh (\sqrt{M^2 + in})} \qquad \dots (4.7)$$

Also the solution of (4.6) under the conditions (3.4) and (3.5) can be obtained as

$$\phi = \left(R + \frac{r^2}{R} \right) \cos \theta \qquad \dots (4.8)$$

Substituting from equations (4.7) and (4.8) in equations (4.1) and (4.2), the exact velocity distributions are obtained as

$$u = \frac{\cosh (\sqrt{M^2 + in})z}{\cosh (\sqrt{M^2 + in})} \left\{ 1 - \frac{r^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right\} e^{int} \qquad \dots (4.9)$$

and

$$v = \frac{\cosh (\sqrt{(M^2 + in})z}{\cosh (\sqrt{M^2 + in})} \left\{ - \frac{2r^2 xy}{(x^2 + y^2)^2} \right\} e^{int} \qquad \dots (4.10)$$

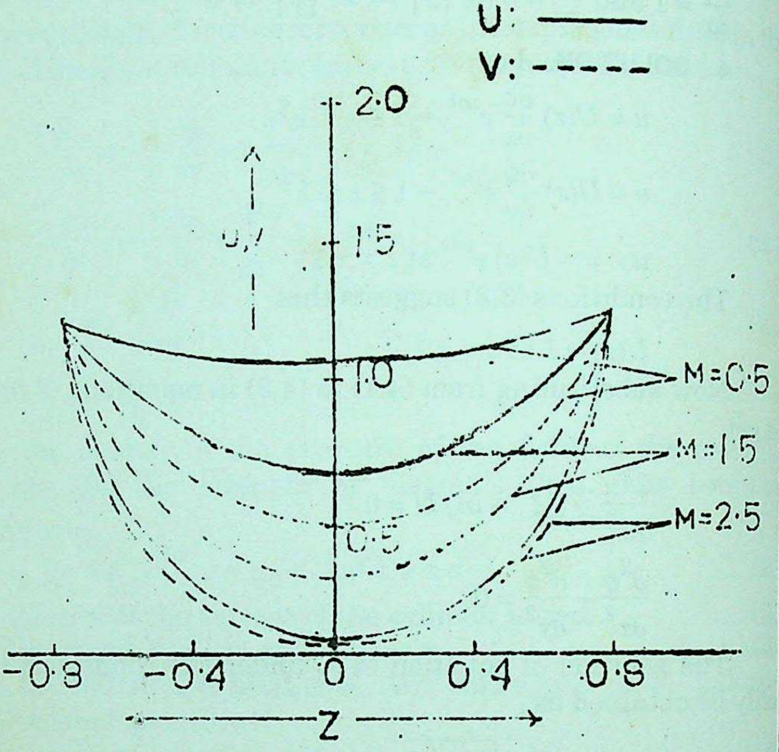

FIG.(I) VELOCITIES PLOTTED AGAINST Z AT n=1, m=0, X=1 AND Y=±1

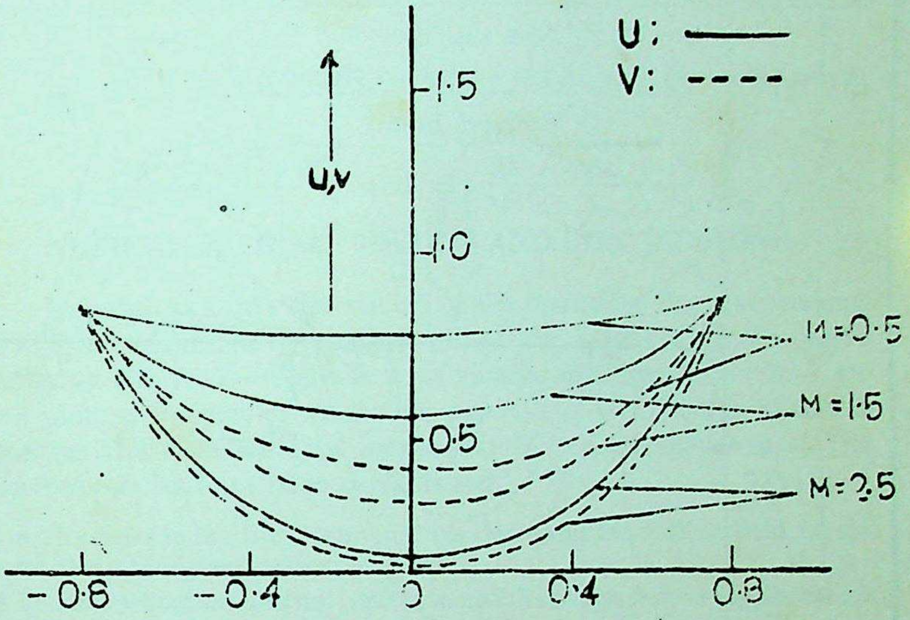

FIG.II VELOCITIES PLOTTED AGAINST Z AT nt = π/4, n=1, X=1, AND Y=-1

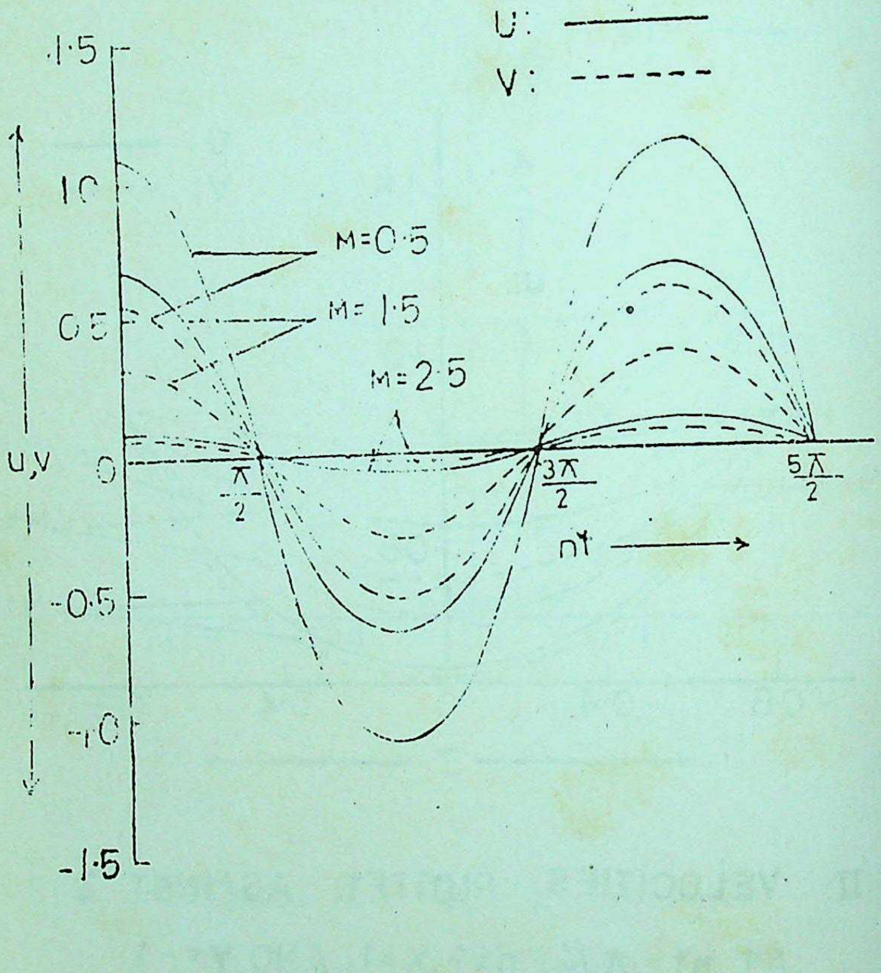

FIG.III VELOCITIES PLOTTED AGAINST nt AT Z=0, n=1, X=1 AND Y=−1

Equating the real parts, we have

$$u = (M_r \cos nt - M_i \sin at)\left\{1 - \frac{r^2(x^2 - y^2)}{(x^2 + y^2)^2}\right\} \qquad \dots (4.11)$$

and

$$v = (M_r \cos nt - M_i \sin nt)\left\{-\frac{2r^2(x^2 - y^2)}{(x^2 + y^2)^2}\right\} \qquad \dots (4.12)$$

where

$$M_r = \frac{\cos A \cos Az \cosh B \cosh Bz + \sin A \sin Az \sinh B \sinh Bz}{\cos^2 A + \sinh^2 B}$$

$$M_i = \frac{\sin A \sinh B \cos Az \cosh Bz - \sin Az \sinh Bz \cos A \cosh B}{\cos^2 A + \sinh^2 B}$$

$$A = -\frac{M^2 + \sqrt{(M^4 + n^2)}}{2} \text{ and } B = \frac{M^2 + \sqrt{(M^4 + n^2)}}{2}.$$

NUMERICAL DISCUSSIONS AND CONCLUSIONS :

In order to study the effects of the boundary fluctuations and the effects of Hartmann number (M) on the distribution of velocity profiles on the 'Hele - Shaw flow' of viscous incompressible fluid, we have plotted graphs of velocities against z, we observe that the velocities decreases with the increase in M or the increase in *nt*. The following results have been established.

[i] An Increase in the Hartmann number decreases the velocity field, i.e. the magnetic field decelerates the flow.

[ii] Boundary fluctuations are held responsible for the flow in the absence of the pressure gradient.

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This volume of

JÑĀNĀBHA

is being dedicated to honour

Professor J.N. Kapur

on his 75th Birthday

Professor Jagat Narain Kapur

Born : September 7, 1923

Felicitation on his 75th Birthday : September 7, 1998

Jñānābha, Vol. 29, 1999
(Dedicated to Professor J.N. Kapur on his Platinum Jubilee Celebrations)

PROFESSOR J.N. KAPUR AN AMAZING PERSONALITY

By

R.C. Singh Chandel
Secretary, Vijñāna Parishad of India
D.V. Postgraduate College, Orai-285001, U.P., India

There are millions of students who appear in university examinations. Out of these, there are a few thousands who get first division in all examinations. Out of these, there are a few hundreds who got first positions in all their examinations. Out of these, there may be a few who beat earlier university records in both their first and second degree examinations. Out of these there is only one who beat these earlier records by wide margins of 4% and 15% marks. This man is Professor Kapur.

There are tens of thousands of persons who get Ph.D. degrees. A number of their theses are based on ten or more published papers. There was one thesis of more than 900 pages submitted for Ph. D. degree and this was based on 30 published papers. This was submitted by Professor Kapur.

There are tens of thousand of mathematicians who publish research papers. Out of these, a few hundred publish more than 100 research papers. Professor Kapur is one of them, having published more than 600 research papers.

There are research workers who work in one area in their life. Some work in two or three areas. Very few work in ten or more different areas. Professor Kapur is one of them, having published in 38 different areas.

There are many persons who write general scholarly articles on educational matters. There are a few who publish more than 500 such articles. Professor Kapur is one of them with about 1000 articles.

There are many authors who write dozens of non-fiction books. Professor Kapur is one of them, having written more than 160 books, more than 100 of which are innovative in nature and unique of their kind in the world.

There are a large number of authors whose books are published locally. There is a smaller number of authors whose books are pubished by all-India publishers. There are only a few whose books are published by international publishers. Professor Kapur is one of them. Four of his books have already been buplished in USA and Canada.

There are persons who have written books for primary level or secondary level or senior secondary level or undergraduate level or postgraduate level or research level or general books or books on education and ethical and moral values. Professor Kapur has written books at all these levels.

There are tens of thousands of scientists in the country. Out of these about a thousand are Fellows of one or the other of the three National Science Academies. There are about 300 who are Fellows of all the three Academies. Professor Kapur is one of them and is quite senior among them.

There are many mathematicians who have been president of Indian Mathematical Society or Calcutta Mathematical Society or Indian Society of Theoretical and Applied Mechanics or Indian Society of Agricultural Statistics Mathematics Section of Indian Science Congress or Physical Sciences Section of National Academy of Science or of Associatins of Mathematics Teachers of India or of Vijñāna Parisad of India or of Mathematical Association of India or of Indian Society of Industrial and Applied Mathematics or of Indian Society of Information, Theory and Its Applications. Only Professor Kapur has been the President of all of them.

He has been Vice-President of Operation Research Society of India, Indian Society for History of Mathematics and Society of Scientific Values. This also reflects his wide interest in Pure and Applied Mathematics, Statistics, Theoretical and Applied Mechanics, Operations Research, Mathematics Education, History of Mathematics and Scientific Values.

There are many Indian Mathematicians who have been visiting professors abroad and out of them, some have been visiting professors in developed countries. He has been a visiting professor not only in departments of Pure and Applied Mathematics, but also in departments of Management Science, Industrial Engineering, Schools of Business and System Engineering. At present he is honorary professor in a school of Computer and Systems Sciences at Jawahar lal Nehru University and also in B.R. Ambedkar University, Agra and Netja Ji Subhash Institute of Technology, New Delhi. This also shows the wide range of his interests.

There are many persons who worked as editors of journals or on editorial boards of journals. Professor Kapur has worked as editor of four journals and has been on the editorial board of two dozen other journals.

Many pesons have directed summer schools organized by the UGC or NCERT. Some have directed a number of them. Professor Kapur has directed about 30 of them. What is more important is that he is one of those few perosns who have organized the summer schools on their own initiative.

There have been thousands of Vice-Chancellors of Indian Universities, but only a few who got this positions before they were even 47 years and fewer still who were requested to accept this position. In fact his Chancellor got leave of absence of him from IIT Kanpur for a second term without his even asking for it. He was a Vice-Chancellor who stood firmly for 100% fairness and impartiality in all admissions, appointments and examinations. It was he who conducted the CPMT examination inUP for four years without a single complaint. There were plenty of complaints before and after his time, but none during his time. In fact in an extraordinary resolution, the UP government congratulated him on his outstanding work which looked like a miracle at that time.

Again many eminent persons like their birthdays to be celebrated. However, when his students collected some funds for celebrating his 50th birthday, he prohibited them from doing so and these funds became the nucleus for a Mathematical Sciences Trust Society to work for the development of Mathematical Science in India.

When the UP Government wanted to give honorarium for the work he did for CPMT, he refused to accept it and at his instance the UP government donated the amount to MSTS. He has given the royality of all of his books to the Trust. The trust has published about 120 books so far and gives prizes worth bout Rs. 100,000 per year to Mathematical Olympiad winners and to all those who stand first in mathemaics in Board and Univesity examinations all over the country.

Many persons can claim to have done something for mathemaics education and research by using government money. He does not use government money. He believes in giving and not in taking. He believes that the Country will develop when the people depend on their own resources rather than on government resources for serving the country.

Many persons in India have personal libraries. Some mathematicians and Scientists in India have also their personal libraries, but few have a library of the size of Professor Kapur's Library which consists of more than 5000 books, 1000 journals and 300 Ph.D. Theses.

Professor Kapur has won many awards including GP Chatterji award of ISCA, Education Minister's Gold Medal of N.A.Sc., Yogiji Maharaj Centenary Award and best paper awards in many intenational conferences. He has won many distinguished service awards. He has won honors including Visiting Professorship, Fellowship of Academies, Presidentships of societies, Vice-chancellorship etc. but he has never asked for them. He has never hinted to anybody that he should be considered for any one of these.

He believes in joy of work. He enjoys reading, writing, lecturing, serving. He neither expects appreciation nor reward for his work. If there is anything he expects from his students, freinds and admirers, it is only that they may give their best to mathematics, science, country and mankind.

Professor Kapur has many firsts to his credit. He was the first to organise the summer schools in India, of course with the cooperation of others. He was the first head of the mathematics department of IIT Kanpur and he built this department to international standards in a short time. He is the founder-President of Vijñāna Parishad of India, Indian Society of Industrial and Applied Mathematics, Indian Society of Information Theory and its Application, Mathematical Sciences Trust Society, C.V. Kapur Education Foundation and Rotary Club of Meerut East, He is also the founder of Mathematial Association of India and founder editor of its publications. He was also the first convernor of Mathematical Education Forum of ISCA. He was the firt author to write books on Mathematical Modelling, Mathematical Models in Biology and Medicine, Maximum Entropy Models, Entropy Optimization Principles, Biographies of Indian Mathematicians, Mathematical Dramas and Olympiad Problems. He was the first Indian mathematician to write his autobiogrphies. In many cases his books were the first of their kind in the world.

Books and papers of most scientists are cited in journals of one subject. Professor Kapur's books and papers have been cited in papers published in 150 journals dealing with about fifty different subjects including Mathematics, Physics, Chemistry, Biology, all branches of Engineering, Environment, Remote Sensing Medicine, Microscopy, Pattern Recognition, Computers and Computer Vision, Sociology, Physiology, Philosophy, Mechanics, Education etc. His papers and books have been cited in 450 papers. One of his papers has been cited more than 150 times.

Six proceedings of national conferences and six issues of journals have been dedicated to him.

PROFESSOR J.N. KAPUR AT A GLANCE

Name	Jagat Narain Kapur
Date of Birth	7-9-1923
Place of Birth	Delhi
Affiliations	Honorary Professor School of Computer and Systems Science,

Jawahar Lal Nehru University,New Delhi- 110067
and
Honorary Director
Mathematical Science Trust Society, New Delhi

Address	C-766, New Friends Colony, New Delhi–110065 India (011) 6832290.
Education	B.A. (Hons.) Mathematics, 1942, Delhi University, India **First Class First** M.A. Mathematics, 1944, Delhi University, India **First Class First** Ph. D. Mathematics, 1957, Delhi University, India Certificate Course in Statistics (ICAR) **First Class First** in first batch. **Broke Previous Delhi University Records** in both B.A. (Hons.) and M.A. examinations by margins of 4 percent and 15 percent marks by securing 91 and 97 percent marks respectively. **Won Three Gold Medals and Six Scholorships. Seventh Position in the first earliest examination held in 1946.**

Positions Held

1944-59	Senior Lecturer, Hindu College (Graduate School) Delhi University, India 1944-1959.
1959-61	Reader (Associate Professor) Institute of Postgraduate Studies, Delhi University, India. 1959-1961
1961-86	Professor, Indian Institute of Technology, Kanpur, India 1961-1986.
1971- 74	Vice Chancellor, Meerut University, Meerut
1986-88	Visiting Professor, Mathematics Departement, IIT Delhi and Delhi Universtiy
1987-89	Senior Scientist, Indian National Science Academy
1988-90	Adjunct Professor, Waterloo University Canada
1988 onwards	Hony. Professor, School of Computer and Systems Science Jawahar Lal Nehru University, New Delhi.
2000 onwards	Hony. Professor B.R. Ambedkar Uiversity, Agra and Netaji Subhas Institute of Technology, New Delhi

Visiting Professor

University.	*Country*	*Year*	*Department*
Arkansas	USA	1969	Mathematics
Carnegie-Mellon	USA	1969-70	Mathematics

Siena	Italy	1970	Mathematics
Manitoba	Canada	1980-81	Business and Acturial Mathematiccs
Waterloo	Canada	1981-82	Applied Mathematics
Waterloo	Canada	1982	Sytstems Engineering
Manitoba	Canada	1983	Actuarial and Management Sciences
Manitoba	Canada	1984	Industrial Engineering
New South Wales	Australia	1984	Applied Mathematics
Flinders	Australia	1984	Mathematical Sciences
Carleton	Canada	1985	Business School
Waterloo	Canada	1985	Systems Design Engineering
Waterloo	Canada	1986-88	Systems Design Engineering
I.I.T. Delhi	India	1986-88	Mathematics
Waterloo	Canada	1987	Systems Design Engineering
Delhi	India	1987-88	Mathematics
Waterloo	Canada	1988-89	Systems Design Engineering
Waterloo	Canada	1990-92	Systems Design Engineering
Carleton	Canada	1997	School of Business
Manitoba	Canada	1999	Managements Science

Fellowships in Professional Organisations, Year of Elections

Indian Academy of Science	*F.A.Sc.*	1965
National Academy of Sciences	*F.N.A.Sc.*	1965
Institute of Mathematics and its Applications, UK	*F.I.M.A.*	1966
Indian National Science Academy	*F.N.A.*	1969
Vijñāna Parishad of India **(First Honorary Fellow)**	*F.V.P.I.*	1993

Presidentship of Professional Organisations

Indian Science Congress Association (Mathmatics Section)	1968
Bharat Ganita Parishad	1969
Indian Mathematical Society	1971, 1972
Calcutta Mathematical Society	1975
Association of Mathematics Teachers of India	1977-90
Indian Society of Theroretical and Applied Mechanics	1978
Mathematical Association of India	1979*
Indian Society of Agricultural Statistics	1984
National Academy of Sciences (Physical Sciences Sect.)	1984
Mathematical Sciences Trust Society	1978*

Vijñāna Parishad of India	1985*
Indian Society of Industrial and Applied Mathematics	1993*
Indian National Commission on History of Sciences (Modern Period)	1996
Indian Congress on Mathematics Education	1993
C.V. Kapur Education Foundation	1994*
Society for Scientific Values (Vice-President)	1993-1998

Chief Editor

The Mathematics Seminar	1963-68
Bulletin, Mathematical Association of India	1979*
Tha Mathematics Student (IMS)	1968-70
The Mathematical Education (UGC)	1984*
National Council of Educational Reserch and Training Mathematics Text Books	1962-65
National Council of Educational Reserch and Training Upper Primary Mathematics Text Books	1987-89
Indira Gandhi National Open Universtiy Text Books on Mathematics	1987-89
SSV Newsletter	2000

Membership (Past or Current)

- of Editorial Boards of Twenty Indian and Foreign Journals
- of 25 Professional Societies in the world
- International Commission of Mathematics Instruction (India's Representative for 15 years)
- Executive Committee and Council, Indian Science Congress Association
- Council, of National Academy of Sciences and Indian National Academy.
- Organizing Committee of Fourth International Conference on Mathematical Modelling.
- Organizing Committee of First and Second International Congress of Teaching of Mathematical Modelling.
- International Conference on Transportation (Chairman, Programme Commitee)
- Central Board of Secondary Education (Member, Chairman, Mathematics Committee)
- National Council of Teacher Education.
- University Grants Commission Mathematics Panel
- Innovations Committee of National Council of Educational Research and Training, India
- Review Committee of Regional Colleges of Education (Chairman)

- Council of Indian Society of History of Mathematics (Vice-President)
- Council of Operations Research Society of India (Vice-President).
- Science Education Forum of Indian Sceince Congress (Convenor).
- Parmar Institute of Mathematical Sciences Shimla, India (Honorary Director).
- Indian Mathematical Society (Academic Secretary)
- National Committee on Mathematical Sciences (DST) (Chairman)
- National Committee on Mathematical Education and Research (DST)
- Governing bodies of Nine U.P. Engineering Colleges.
- Mathematics Section Committee and Council of INSA
- Indian National Commission on History of Science and Chairman of its Modern Period Committee.
- Chairman INSA Committee for Cooperation with International Commission on History and Philosophy and Science.
- Chairman UGC Committee for DSA Programmes in Mathematics.
- Member Selection Committee of Vice-Chancellorship of Agra, Meerut and Jiwaji Universities.
- Member UGC Advisory Commitees for Mathematics of Panjab, Roorkee and Jadavpur Universities.
- Chairman CSIR Post Doctoral Fellowship Committee.
- Chair Professor ACME
- Member UGC Committee of Three Vice-Chancellors for visiting 9 Universities of Germany to Discuss Collaboration between Indian and German Universities.
- Invited to Visit 11 Australian Universities at the Invitation of Australian Vice-Chancellors Committee.
- Resource Person for two UNESCO conference in Bangladesh and Pakistan.
- Invited foreign lecturer at NCTM golden jubiliee conference Washington.

Publications :

Research Papers	:	600	About 6000	Pages
General Articles	:	1000	About 12,000	Pages
Books	:	183	About 25,000	Pages

Administrative Experience

Vice-Chacellor, Meerut Universtiy, India (1971-74). Head of Postgraduate Mathematics Department. (32 years), Acting Director, Indian Institute of Technology, Kanpur, on different occasions for approximately 250 days, Director of Thirty Summer Schools of Mathematics Organizer of a dozen national and International conferences.

Awards

Gold Medal for best research National Academy 1980

National Lecturer, University Grants Commission 1982-83

G.P. Chatterjee Award, Indian Science Gongress Association 1988

Distinguished Service Award, Mathematical Association of India 1983

Platinum Jubilee Lectureship, Indian Science Congress Association 1988

Best paper Award, Administrative Science Association, Canada 1986

Felicitation by Yogiji Maharaj Trust, Gandhigram 1992

Distinguished Service Award, Vijñāna Parishad of Inida 1996

Distinguished Service Award, Indian Society of Information Theory 1998

Distinguished Service Award, Mathematika Sciences Society 2000

Life Long Achievement Award, Indian Society of Industrial and Applied Mathematics 2000

Selected for Inclusion in more than a Dozen Biographies of American *Biographical Institute, Marquis* who is who, Distinguished Men of India and Many Others.

Hindu education trust award for being the best academission produced by Hindu College Delhi in its first hundred years 2000

Distinguished Plaque of Industrial Engineering Department, Manitoba University, Canada for outstanding contributions to Industrial Engineering.

Universities Visited

USA (39), Canada (12), Italy (2), UK (12), West Germany (9), Australia (12), Netherlands (3), Singapore (1), Iran (2), Thailand (1), Bangladesh (1). Pakistan (3).

Mathematics Education Projects Visited

USA (14), UK (8), Netherlands (2), Italy (3), Australia (8).

Seminars/Lectures

Given in India (about 1000), Given in other countries (about 200).

Inaugural/Valedictory

At about 100 conferences

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INVOLUTIONS AND B-INJECTORS OF *Fi22*

By

M.I. Alali* , Ch. Hering and A. Neuman

Mu'tah University, Tubingen University

(Received : November 24, 1998)

ABSTRACT

The aim of this paper is to find a simple approach to study centralizers of 3-transpositions in $F_{i22} = \widetilde{M}\,(22)$ and using these results to determine the B-Injectors of F_{i22}

1. Introduction. A finite group G is said to be of type F_{i22} if G possesses an involution d such that $H = G_G\,(d)$ is quasimple with $H/<d> \cong U_6(2)$ and d is not weakly closed in H with respect to G. For more information one is referred to [2]. The uniqueness of groups of this type is proved in [2]. Here we give a very simple characterization to the centralizer of involutions (3-transpositions) in F_{i22} , and among other results transvections in $GU_6\,(2)$ are proved to have 3-transpositions. Most of these results are in [2], [3], [7], but here we use very simple notions of group theory and linear algebra to exhibit these results in a more readable form. The concept of N-injectors is due to B. Fischer who proved that if G is solvable, then N-injectors exist and any two of them are conjugate [8]. N-injectors exist and any two of them are conjugate [8]. N-injectors of finite solvable groups, symmetric groups Sn, alternating groups An and classical groups where studied in [11], [7], [6], [5] and [12].

2. Preliminaries and Notations. Let F denote the group F_{i22} . The group $F_{i22} = <D>$ where D is a class of involutions (3-transpositions) with the property, $t_1, t_2 \in D$ implies that $o(t_1 t_2) = 1, 2,$ or 3 where $o(t_1 t_2)$ denotes the order of $t_1 t_2$. So if $t_1 \neq t_2$, then $< t_1, t_2 >$ is a group of order 4 i.e. $<t_1, t_2> = 2^2$ or $<d_1, d_2> = S_3$ the symmetric group of degree 3. There are 3 classes of involutions j in F_{i22} with representatives $(i) : j = d \in D$ such that $C_F\,(d) = 2\,U_6\,(2)$, (ii) : $j = d_1 d_2 = d_2 d_1$ where d_1, d_2 are uniquely determined by j and if $g \in C_G\,(j)$, one obtains $d_1^g = d_1$, $d_2^g = d_2$ or $d_1^g = d_2$, $d_2^g = d_1$, so $C_F(d_1) \cap C_F(d_2)$ is a normal subgroup of $C(j)$ of order 2^2. $\underline{C_F(d_1) = 2U_6,(2)}$, this implies that $C_F\,(d_1) / <d_1> = U_6\,(2)$ and since $d_2 \in$

---

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$C_F(d_1)$ one obtains $d_2<d_1>$ is a transvection in $U_6(2)$. This can easily be seen as follows:

Let V be a 6-dimentional vector space over the field $GF(4)$, and let $(|)$ be a unitary form on V, this means a function $(|)$:

$(|) : V \times V \to GF(4)$

$(x, y) \to (x|y)$ such that the following conditions hold

1. $(x_1+x_2|y_1+y_2) = (x_1|y_1) + (x_1|y_2) + (x_2|y_1) + (x_2,y_2)$
2. $(\lambda x|y) = \lambda(x|y)$
3. $(x|\lambda y) = \bar{\lambda}(x|y)$ where $\bar{\lambda}$ is the complex conjugate of λ in $GF(4)$.
4. $(x,y) = \overline{(y,x)}$, se the general unitary group $GU_6(2)$ is defined by $GU_6(2) = \{g \in GL(V):(x^g|y^g) = (x,y) \ \ \forall\, x,y \in V\}$ and the special unitary group $SU_6(2) = \{g \in GU_6(2): det\ (g) = 1\}$, the center of $GU_6(2) = Z(GU_6(2)) = \{\lambda I : \lambda\bar{\lambda} = 1\} = \{\lambda I : \lambda^3 = 1\} \leq SU_6(2)$, so $U_6(2) = SU_6(2)/Z(G(U_6(2))$.

Transvections in the unitary group is defined as follows. Let $v \in V$ with $(v|v) = 0$. Define a transvection $t_v : V \to V$ by $t_v(x) = x - (x|v)v$, sometimes $t_v(x)$ is written as x^{t_v}. This transvection preserves the unitary form for

$$(x^{t_v}|y^{t_v}) = (x-(x|v)\,v|y-(y|v)v = (x,y) - \overline{(y|v)}\,(x|v) - (x|v)\,(v|y) + (x|v)$$
$$\overline{(y|v)}\,(v|v) = (x,y) - 2(v|y)(x/v) + (x|v)\,\overline{(y|v)}.0 = (x|y).$$

It is very clear that transvections have 3-transpositions, this means that if t_v, t_w are two transvections then $o\,(t_v, t_w) = 1,2,$ or 3 and $t_v^2 = 1$ and $t_v\,t_w = t_w\,t_v$ iff $(v/w) = 0$.

Lemma1. $C\,(d_1d_2)$ is of the form 2-group. $U_4(2).2$

Proof.

$C_F(d_1) \cap C_F(d_2) = C_{C_F(d_1)}(d_2)$, So

$(C_F(d_1) \cap C_F(d_2))/<d_1> = C_{U_6\,(2)}(d_2<d_1>/<d_1>) = C_{U_6\,(2)}(\bar{d})$

where $\bar{d} = d_2<d_1>/<d_1>$ is a transvection in $U_6(2)$

So $C_{U_6\,(2)}(\bar{d}_2) = H.U_4(2)$, H is some elementrary abelian 2-group and hence, $\{C(d_1) \cap C(d_2)\}.2 = C\,(d_1d_2)$. Since $d_1 \in C_{C(d_1)}(d_2)$ one has, $C(d_1) \cap C(d_2) = 2.\ C_{C(d_1)}(d_2)/<d_1> = C_{U_6\,(2)} = 2^{1+8}: U_4(2)$ a stabilizer of transvection. Therefore

$C(d_1d_2) = 2.2^{1+8}: U_4(2).2$

3. The third class of involutions in F_{i22} is of the form

$j = j_3 = d_1 d_2 d_3$ where $d_i d_j = d_j d_i$.

Lemma 2. $C_G(d_1 d_2 d_3)$ $2 -$ group A_6.

Proof. There exist exactly 3 other transpositions d'_1, d'_2, d'_3 and $j = d'_1 d'_2 d'_3$ and $<d_1, d_2, d_3, d'_1, d'_2, d'_3> = 2^5$ for, j can be writtin in exactly two ways as products of 3 commuting transpositions, it follows that $g \in C(j)$ implies that :

(a) $\{d_1, d_2, d_3\}^g = \{d_1, d_2, d_3\}$, $\{d'_1, d'_2, d'_3\}^g = \{d'_1, d'_2, d'_3\}$

or (b) $\{d_1, d_2, d_3\}^g = \{d'_1, d'_2, d'_3\}$.

In any case g normalizes $<d_1, d_2, d_3, d'_1, d'_2\, d'_3>$. So

$C_G(j) \le N_G(<d_1, d_2, d_3, d'_1, d'_2\, d'_3>)$ = stablizer of a hexad in $2^{11}. M_{22} = (2^{11}. M_{22})_{hexad} = 2 - group.\ A_6$.

Now we come to the following conclusion :

Let $C_G(j_1) = 2\ U_6 = H_1$, $C_G(j_2) = 2$- group. $U_4(2).2 = H_2$ and $C_G(j_3) = H_3$. The generalized Fitting group $F^*(G)$ is defined by $F^*(G) = F(G)\, E(G)$ where $F(G)$ is the Fitting group of G and $E(G)$ is a subgroup of $G.A.$ group L is called quasi simple iff $L' = L$ where L' is the derived group of L and $L'/Z(L)$ is simple. So one has :

$F^*(H_2) = O_2(H_2)$ and $F^*(H_3) = O_2(H_3)$ because $F^*(H) = F(H)$.

This can be prooved as follows:

Lemma 3. Let $H = M.A_6$ where M is a 2-group, then $F^*(H) = F(H)$.

Proof. Let $F^*(H) \ne F(H)$ this means that $E(H) \ne \{1\}$ so there exists a quasi simple subnormal subgroup L such that $[F(H), L] = \{1\}$ (use theorem1) and the fact that $L \le E(H)$. It follows that $L \le C_H(F(H)$. Since $M < H$, $M \le O_2(H) = F(H)$ then $ML \le H$ and $ML/M \cong ML/M \le H/M \cong A_6$. In particular $5 \mid L \mid$ as $5 \mid IA_6 \mid$, take $g \in L$ such that order $(g) = 5$, since $[M, L] = 1$. This implies that $[M, g] = 1$ and one obtains $M \le C_H(g) \le C_G(g)$. So $|M| \le |C(g)|$ which is a contradiciton because H does not contain an element g of order 5 with $|M| \le |C(g)|$. Thus $F^*(H) = F(H)$.

3. Definitions. Let G be a finite group. Define

$d_2(G) = max\ \{\mid x \mid : x \le G;\ x$ is nilpotent of class $\le 2\}$

$\sigma_2(G) = \{x \le G;\ x$ is nilpotent of class ≤ 2 and $\mid x \mid = d_2(G)\}$

B-Injector of $G = \{H \le G : H$ is a maximal nilpotent in G and contains a subgroup $x \in \sigma_2(G)\}$.

For these defintions one is referred to [2]. Now we define

$m_k(G)$, $Om_k(G)$ as follows :

$m_k(G) = max\ \{|C_G\ (x)| : x \in G,\ o(x) = p_1.p_2 \dots p_k,\ p_i \neq p_j$ if $i \neq j$ and let $|C_G(g)| = p_1^{a_1} . p_2^{a_1} \dots p_k^{a_k} .m$ where $p_i \mid m$, $i = 1, \dots, k$ then define $Om_k\ (G) = max\ \{|C_G\ (x)| : o(x) = p_1.p_2, \dots p_k,\ 2 < p_1 < p_2 < \dots < p_k\}$. So we get the following critirion : If $H \leq G$ such that H is nilpotent and H has at least k prime divisors different from 2 then $|H| \leq O\ m_k\ (G) \leq m_k(G)$. The proof goes as in [1].

Theorem 2. [1] If $F^*(G) = O_p\ (G)\ Z(G)$ then B-injectors of G are of the form $PZ(G)$ where P is a p-Sylow subgroup, where $O_p(G)$ is the unique maximal normal p-group G and $Z(G)$ is the centre of G.

Corollary . G is N-constrained iff $E(G) = \{1\}$.

Theorem 2. Let B be a B-injector in F_{i22}, it follows that if $2 \mid |B|$ then B is a 2-Sylow subgroup.

Proof. If $2 \mid |B|$ then $2 \mid |Z(B)|$, so there exists an involution z such that $B \leq C(Z)$ and z is conjugate to j_1 or j_2 or j_3. If z is conjugate to j_2 or z is conjugate to j_3 then there exists a subgroup H such that $B \leq H \leq G$ and $F^*(G) = O_2\ (H)$. So the claim follows from theorem 1.

If z is conjugate to j_1, then with out loss of generality, $B \leq C(j_1) = 2U_6\ (2)$. So the claim follows from results on classical groups [12]. Assume that B is a B-injector and $2 \mid |B|$ then B is a p-group for: Since $|B| \geq 2^{11}$ and $m_3\ (F_{i22}) = 30 < 2^{11}$ it follows that $|B|$ has at most 2 prime divisors. If B has exactly two prime divisors this implies that $|B| \geq O\ m_2\ (F_{i22}) = 21$ which is a contradiction. Therefore B is a p- group, and since $|B| \geq 2^{11}$, this means that B is a 3-Sylow subgroup. Concequently the B-injectors are either 2-Sylow subgroups or 3-Sylow subgroups. Assume that the B-injectors are 3-Sylow sub groups, this implies that there exists a 3-subgroup of order $\geq 2^{11}$ and class ≤ 2. Since the 3-Sylow subgroups of F_{i22} are isomorphic to the 3-Sylow subgroups of $O(7,3)$, it follows that there exists a 3-subgroup $Y \leq O\ (7,3)$ of order $\geq 2^{11}$ and class ≤ 2, which is impossible. Therefore the B-injectors of F_{i22} are 2-Sylow subgroups and this completes the proof of the theorem.

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REAL INVERSION AND REPRESENTATION THEOREMS FOR A GENERALISED LAPLACE TRANSFORM

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ABSTRACT

The object of this paper is to give real inversion and representation theorems for the generalised Laplace transform

$$f(s) = \int_0^\infty (st)^\lambda e^{-st} L_k^\alpha(st)\ \phi(t)\, dt \qquad \ldots (1.1)$$

where $L_k^\alpha(z)$ is a Laguerre polynomial. For $\lambda = k = 0$ the integral (1.1) reduces to the well known Laplace integral

$$f(s) = \int_0^\infty e^{-st}\ \phi(t)\, dt.$$

2. Singular Integral.

Lemma 2.1. If

(i) $a < a + \xi < b$

(ii) $\gamma(t)\ \varepsilon\ c^2, \quad \gamma'(a) = 0, \quad \gamma''(a) < 0;\ a \le t \le a + \xi$

$\gamma(t)$ is non increasing in $a \le t \le b$

then

$$\int_a^b e^{k\gamma(t)}\, t^{\lambda+n}\, dt \sim e^{k\gamma(a)}\, a^{\lambda+n} \left[\frac{-\pi}{2k\ \gamma''(a)}\right]^{1/2} \quad ; k \to \infty .$$

Lemma 2.2

If in addition to the conditions (i) and (ii) of lemma 2.1 we have

(iii) $\phi(t)\ \varepsilon\ L,\ a \le t \le b\ ; \qquad \phi(a) \ne 0$

(iv) $\alpha(t) = \int_a^t [\phi(x) - \phi(a)]\, dx = o(t - a)\ ;\ t \to a+.$

then

$$\int_a^b \phi(t)\, t^{\lambda+n}\, e^{k\gamma(t)}\, dt = \phi(a)\, a^{\lambda+n}\ e^{k\gamma(a)} \left[\frac{-\pi}{2k\ \gamma''(a)}\right]^{1/2} \quad ; k \to \infty.$$

Lemma 2.3

In lmma 2.2. if we replace the condition (iii) and (iv) by

$$\phi(t) \in L,\ a \le t \le b\ ; \qquad \phi(b) \ne 0$$
$$\alpha(t) = \int_t^b [\phi(x) - \phi(b)]\, dx = o\,(b - t)\ ; \quad t \to b-.$$

then

$$\int_a^b \phi(t)\, t^{\lambda+n}\, e^{k\gamma(t)}\, dt \sim \phi(b)\, b^{\lambda+n}\ e^{k\gamma(b)} \left[\frac{-\pi}{2k\ \gamma''(b)}\right]^{1/2} \quad ; k \to \infty.$$

All the results given above have been proved as in Tivari [3].

Theorem 2.1

If (i) $\phi(t) \in L,\ (0 < x \le t \le R\ )$ for fixed x and large R.

(ii) $\int_x^\infty e^{-ct}\,\phi(t)\, dt$ converges for a fixed positive c,

(iii) $\int_x^\infty [\,\phi(y) - \phi(x)]\, dy = o\,(t-x)\ ;\ t \to x+$

then

$$\underset{k\to\infty}{Lim.} \frac{k^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} \int_x^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-(k/x)t}\,\phi(t)\, dt = \frac{\phi(x)}{2},$$

λ and k being real.

Proof. Choose any positive number δ and let

$$\alpha(y) = \int_{x+\delta}^y e^{-ct}\,\phi(t)\, dt\ ,\ \ y \ge x + \delta \qquad \ldots (2.1.1)$$

Then the condition (ii) implies the existence of a constant M such that

$$|\,\alpha(y)\,| \le M, \quad x \le y < \delta\,.$$

Set

$$I_k = A\int_x^{x+\delta} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-(k/x)t}\,\phi(t)\, dt + A\int_{x+\delta}^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-(k/x)t}\,\phi(t)\, dt$$

$$= I'_k + I''_k \ \text{ where } A = \frac{k^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)}\ .$$

Substituting for $\phi(t)$ from (2.1.1) and integrating I_k'' by parts we get

$$I_k'' = -\frac{A}{x^{\lambda+n+k+1}} \int_{x+\delta}^\infty \alpha(t)\, d\,[t^{\lambda+k+n}\, e^{-(k-cx)\,t/x}]\,.$$

The integrated part vanishes for $n > cx$ and $\alpha(x+\delta) = 0$.

Let us consider the function

$$X(t) = t^{\lambda+k+n}\, e^{-(k-cx)\,t/x}.$$

For sufficiently large $k > k_0$, $X(t)$ is maximum at

$$t=(\lambda+k+n)\frac{x}{k-cx}=x\left[1+\frac{\lambda+n}{k}\right]\left[1-\frac{cx}{x}\right]^{-1} .$$

Thus $X(t)$ is decreasing in $x+\delta \le t < \infty$ when $k > k_0$.

Hence

$$I_k'' \le \frac{A}{x} M\,(1+\delta/x)^{\lambda+k+n}\, e^{-(k-cx)\,(1+\delta/x)} = J_k \text{ (say)}$$

Now

$$J_{k+1}/J_k = (x+\delta)/x \ \ e^{-1-\delta/x} .$$

Therefore $\lim\limits_{k\to\infty} I_k'' = \lim\limits_{k\to\infty} J_k = 0$, since $(1+\delta/x)\, e^{-1-\delta/x} < 1$.

Evaluating I'_k by substituting $\gamma(t) = \log t - t/x$ in lemma 2.2 and using the Sterling's formula the theorem is proved.

Theorem 2.2 If,

(i) $\phi(t)\ \varepsilon\ L,\ (0 < \in \le t < x)$ for a fixed x and small positiv $\in$.

(ii) $\int_{0+}^{x} t^r \phi(t)\, dt$ converges for a fixed constant r.

(iii) $\int_{x}^{y} [\,\phi(t) - \phi(x)]\, dt = o\,(x-y)\ ;\ (y \to x-)$

then

$$\lim_{k\to\infty} \frac{k^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} \int_{0+}^{x} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-(k/x)t}\, \phi(t)\, dt = \frac{\phi(x)}{2}.$$

Proof. Let us choose a positive δ less than x and set

$$\alpha(y) = \int_{y}^{x-\delta} t^r\, \phi(t)\, dt$$

then there exists a constant M such that $|\,\alpha(t)\,| < M ;\ \ 0 < y \le x-\delta$.

Set

$$I_k = A\int_{0}^{x-\delta} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-(k/x)t}\, \phi(t)\, dt + A\int_{x-\delta}^{x} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k}}\, e^{-(k/x)t}\, \phi(t)\, dt$$

$$= I'_k + I''_k \text{ where } A = \frac{k^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} .$$

Integrating I'_k by parts we get

$$I_k = -\frac{A}{x^{\lambda+n+k+1}} \int_{0+}^{x-\delta} \alpha(t)\, d\,[t^{\lambda+k+n-r}\, e^{(-k/x)t}]$$

The integrated part vanishes for $\lambda+k+n-r > 0$ and $\alpha(x-\delta) = 0$.

Now consider the function $H(t) = t^{\lambda+k+n-r}\, e^{(-k/x)t}$.

$H(t)$ is maximum at $t = x\ [\,1+(\lambda+n-r)/k] > x-\delta\ ;$ for $k > k_0$.

Thus $H(t)$ is non decreasing in $0 \le t \le x-\delta$.

Hence

$$|I_k'| \le A/x \; M(1-\delta/x)^{\lambda+k+n-r} e^{-k(1-\delta/x)}.$$

Therefore as in theorem 2.1 we have $\lim_{k\to\infty} I_k' = o(1)$

Using lemma 2.2 and taking $\gamma(t) \ne \log t - t/x$ we have $I_k'' = [\phi(x)]/2$.

Hence the theorem.

Theorem 2.3 If,

(i) $\phi(t) \,\varepsilon\, L, \; (1/R \le t \le R)$ for every $R > 1$.

(ii) $\int_1^\infty e^{-ct} \phi(t)\, dt$ converges for a fixed $c > 0$.

(iii) $\int_{0+}^1 t^r \phi(t)\, dt$ converges for a fixed r,

(iv) $\int_x^y [\phi(t) - \phi(x)]\, dt = o(|y-x|)\,;\; y \to x$,

then,

$$\lim_{k\to\infty} \frac{k^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} \int_x^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-(k/x)t} \phi(t)\, dt = \phi(x).$$

The theorem can be proved by combining theorems 2.1 and 2.2. The conclusions hold good if the conditions (iv) in the theorem is replaced by $\phi(x) = 1/2\,[\phi(x+) + \phi(x-)]$.

3. The Inversion Operator.

If $f(s)$ has derivatives and integrals of all orders, an operator $M_{n,x}[f(s)]$ is defined for any positive x and any positive integer n by

$$M_{n,x}[f(s)] = (-1)^n \frac{s^{\lambda+n+1}}{\Gamma(\lambda+n+k+1)} D^n [\, s^{-\alpha-k} D^{-n} \{s^k D^n (s^{\alpha-\lambda} f(s)\}] \,;\; s = n/x$$

where

$$D \equiv \frac{d}{ds} \text{ and } D^{-1}(s^\mu) = \int_0^s s^\mu\, ds \quad \text{if } Re(\mu+1) > 0$$

$$= \int_s^\infty s^\mu\, ds \quad \text{if } Re(\mu+1) < 0.$$

4. Inversion Theorems

Lemma 4.1

If $F(s) = (st)^\lambda e^{-st} L_k^\alpha(st)$, then

$$M_{n,x}[F(s)] = \frac{s^{\lambda+n+1}}{\Gamma(\lambda+n+k+1)} [\, t^{\lambda-n} e^{-st} L_k^\alpha(st)\,].$$

Since, (Slater [2]), $L_k^{\alpha}(z) = \dfrac{(-1)^k}{k\,!}\, U(-k\,;\,1+\alpha\,;\,z)$,

we have $F(s) = \dfrac{(-1)^k}{k\,!}\, U(-k\,;\,1+\alpha\,;\,st)\,(st)^{\lambda}\, e^{-st}$.

Using results 2.1.32, 2.1.31, 2.1.30 of Slater [2] and taking $A = \dfrac{(-1)^k}{k!}$, we get

$M_{n,x}\,[F(s)] = (-1)^n\, \mathrm{A}\, t^{\lambda+n}\; e^{-st}\; U(-k\,;\,1+\alpha\,;\,st)$.

Transforming to Laguerre polynomials we get the result.

Theorem 4.1

If $\phi(t)\ \varepsilon\ L$ in $0 \le t \le R$ for every positive R and is such that the integral (1.1) converges for some $s = (s_0$ *say*) then

$$\lim_{n\to\infty} M_{n,x}\,[F(s)] = \phi(x)$$

for all positive x in the Lebesgue set for $\phi\,(t)$.

Proof. Since the integral (1.1) converges for $s > s_0$, we can evaluate the derivatives of $f(s)$ by differentiation under the sign of integration. Hence with the help of lemma 4.1 we have

$$M_{n,x}\,[F(s)] \sim \frac{n^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} \int_0^{\infty} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-(n/x)t}\ \phi\,(t)\,dt.$$

Since $L_k^{\alpha}(x) \sim z^k\,;\, z \to \infty$. Hence by theorem 2.3 $\lim\limits_{n\to\infty} M_{n,x}\,[F(s)] \sim \phi(x)$.

Now to verify the conditions of theorem 2.3 we see that $\phi(t)\ \varepsilon\ L$ in $(0 \le t \le R)$ for every positive R. Hence r of theorem 2.3 may be taken zero. Also c may be taken greater than s_0 (assumed real) because $\int_0^{\infty} e^{-ct}\,\phi(t)\,dt$ converges. The Lebesgue set for $\phi(t)$ is the set of numbers x_0 so that $\int_{x_0}^{x} |\,\phi(t) - \phi(x_0)\,|\,dx = o\,(\,|\,x - x_0\,|\,)\,;\, x \to x_0$. For such set of numbers hypothesis (iv) of theorem 2.3 is satisfied. Hence the Theorem.

5. Representation Theorem.

In this section we establish necessary and sufficient conditions that a given function must satisfy in order that it may be expressed as a generalised Laplace transform with determining function belonging to certain familiar class. We shall take s, λ, α, k to be real. We first give certain results which will be used in the later computations.

Lemma 5.1 If n is a positive integer and t, x are positive then

$$\frac{\partial^n}{\partial x^n}\,[\,x^{n-1}\, e^{-(n/x)\,t}\, L_k^{\alpha}(\tfrac{n}{x}\,t)] = n^n\, t^n\, x^{-n-1}\, e^{-(n/x)\,t}\, U\,(-k\,;\,1+\alpha+n\,;\,\tfrac{n}{x}\,t\,)$$

Proof. We have

$$x^{n-1}\, e^{-(n/x)\,t}\, L_k^{\alpha}(\frac{n}{x}\,t) = \frac{(-1)^k}{k\,!}\; x^{n-1}\, e^{-(n/x)\,t}\, U\,(-k\,;\,1+\alpha\,;\,\frac{n}{x}\,t\,)$$

and $(x/t)^{n-1}\, e^{-(n/x)t}\, U\,(-k\,;\,1+\alpha\,;\,\frac{n}{x}\,t\,)$ is a homogeneous function of order zero. Hence using Euler's theorem repeatedly n times and using Slater ([2],p.16) we have

$$\frac{\partial^n}{\partial x^n}\,[\,\frac{x^{n-1}}{t^n}\; e^{-(n/x)\,t}\, U\,(-k\,;\,1+\alpha\,;\frac{n}{x}\,t\,)] = n^n\, t^n\, x^{-n-1}\, e^{-(n/x)\,t}$$

$$U\,(-k\,;\,1+\alpha+n;\,\frac{n}{x}\,t\,)$$

Hence the lemma.

Lemma 5.2 If n is a positive integer and x, t are positive then

$$D^{-n}\,[x^{-\alpha-k-n-1}\, e^{-(n/x)t}\, U\,(-k\,;\,1+\alpha\,;\frac{n}{x}\,t\,)]$$

$$= x^{-\alpha-k-1}\, e^{-(n/x)t}\, U\,(-k-n\,;1+\alpha+n;\,\frac{n}{x}\,t\,)\,,$$

where D^{-1} denotes the integration as defined in section 3.

Proof Let $D' = d/dt$. Then by Euler's theorem

$$\frac{\partial}{\partial x}\,[\,\frac{t^{\,\alpha+n+k}}{x^{\,\alpha+n+k+1}}\, e^{-(n/x)\,t}\, U\,(-k\,;\,1+\alpha\,;\,\frac{n}{x}\,t\,)]$$

$$= -\frac{\partial}{\partial x}\,[\,\frac{t^{\alpha+n+k-1}}{x^{\,\alpha+n+k}}\; e^{-(n/x)t}\;\; U\,(-k\,;\,1+\alpha+n\,;\,\frac{n}{x}\,t\,)$$

Therefore

$$D^{-1}\,[\,\frac{t^{\alpha+n+k}}{x^{\,\alpha+n+k+1}}\, e^{-(n/x)\,t}\, U\,(-k\,;\,1+\alpha+n\,;\,\frac{n}{x}\,t\,)]$$

$$= (-1)\,D^{-1}\left[\frac{t^{\alpha+n+k-1}}{x^{\,\alpha+n+k}}\;\; U\,(-k\,;\,1+\alpha+n;\,\frac{n}{x}\,t\,)\, e^{-(n/x)\,t}\right].$$

Repeating the process and using Slater([2], p. 16) we get the result.

Lemma 5.3 If n is a positive integer and x,t are positive then

$$D^{-n}\,[x^{-1-\alpha}\, e^{-(n/x)t}\, U\,(-k+n\,;\,1+\alpha+n\,;\,\frac{n}{x}\,t\,)]$$

$$= x^{-n-\alpha-1}\, e^{-(n/x)t}\, U\,(-k\,;\,1+\alpha\,;\,\frac{n}{x}\,t\,).$$

Proof. As in lemma 5.1.

Lemma 5.4 If $g\,(s\,,y) = y^{n-1}\, e^{-(n/y)s}\, L_k^{\alpha}\,(\frac{n}{x}\,s)$ then

$$\begin{array}{lll} \dfrac{\partial^r}{\partial x^r}[\,g\,(s,y)\,] & = o\,(y^{n-k-r-1}\,e^{-(n/y)\,s})\,; & y \to 0 \\ & = o\,(y^{n-r-1}) \quad ; & y \to \infty \\ & & (r = 1,\,2,\,3,\,\ldots) \end{array}$$

Proof. We have

$$g\,(s,y) = \frac{(-1)^k}{k\,!}\;y^{n-1}\,e^{-(n/y)s}\,U\,(-k\,;\,1+\alpha\,;\,\frac{n}{y}\,s\,)\,.$$

Now

$$\begin{aligned} \frac{\partial^2}{\partial y^2}[\,g\,(s,y)] &= \frac{(-1)^k}{k\,!}\,s^n\,\frac{\partial^2}{\partial s^2}\,[\,\frac{y^{n-3}}{s^{n-2}}\;\;e^{-(n/y)s}\,U\,(-k\,;\,1+\alpha\,;\,\frac{n}{y}\,s\,) \\ &= \frac{(-1)^k}{k\,!}\;y^{n-3}\,e^{-(n/y)s}\,[(2-n)\,(1-n)\;U\,(-k\,;\,1+\alpha\,;\,\frac{n}{y}\,s\,) - 2(2-n)\,\frac{n}{y}\,s \\ &\qquad U\,(-k\,;\,2+\alpha\,;\,\frac{n}{y}\,s\,) + \frac{n^2}{y^2}\,s^2\;U\,(-k\,;\,3+\alpha\,;\,\frac{n}{y}\,s\,)]\,; \end{aligned}$$

using Slater ([2], p. 17).
Now using the estimates of $U\,(a;b;x)$, Slater ([2] p. 60) we have

$$\begin{array}{lll} \dfrac{\partial^2}{\partial y^2}[\,g\,(s,y)\,] & = o\,(y^{n-3-k}\,e^{-(n/y)\,s})\,; & y \to 0 \\ & = o\,(y^{n-3}) \quad ; & y \to \infty \end{array}$$

Now using the theorems of Hardy and Littlewood (Widder [4]; pp. 193), we have

$$\begin{array}{lll} \dfrac{\partial^r}{\partial y^r}[\,g\,(s,y)\,] & = o\,(y^{n-k-r-1}\,e^{-(n/y)\,s})\,; & y \to 0 \\ & = o\,(y^{n-r-1}) \quad ; & y \to \infty \end{array}$$

Hence the lemma.

Theorem 5.1

If $f(y)$ has derivatives and integrals of all orders in $(0 \le y < \infty)$ and if for each non negative integer n and some positive c.

(i) $f^{(n)}\,(y) = o\,(y^{-n})\,;$ $\qquad y \to \infty$

(ii) $f^{(n)}\,(y) = o\,(y^{c/y})\,;$ $\qquad y \to 0+$

then

$$\lim_{n\to\infty}\int_0^\infty (sx)^\lambda\,e^{-sx}\,L_k^\alpha(sx)\,M_{n,x}\,(f(s)\;dx \sim f(s).$$

Proof. Let s be any number greater than zero. Choose n so large that for real s, $ns > 0$. Let us set

$$I_n = \int_0^\infty (sx)^\lambda\,e^{-sx}\,L_k^\alpha(sx)\,M_{n,x}\,[f(s)]\;dx$$

$$= (-1)^n \frac{1}{\Gamma(\lambda+n+k+1)} \int_0^\infty (sx)^\lambda e^{-sx} L_k^\alpha(sx)(n/x)^{\alpha+n+1} D^n [s^{-\alpha-k} D^{-n} \{s^k D^n (s^{\alpha-\gamma} f(s))\}] dx \ .$$

Putting $n/x = y$ and $A = n^{\lambda+1}/\Gamma(\lambda+n+k+1)$

$$I_n = (-1)^n A \int_0^\infty G(s, y) D^n [y^{-\alpha-k} D^{-n} \{y^k D^n (y^{\alpha-\lambda} f(y))\}] dy \, ,$$

where $G(s,y) = s^\lambda g(s,y)$ as defined in lemma 5.4.

Integrating I_n by parts repeatedly we have

$$I_n = A \int_0^\infty y^{\alpha-\lambda} D^n [y^k D^{-n} \{y^{-\alpha-k} D^n (G(s,y))\}] f(y)\, dy \, .$$

The integrated part vanishes after each operation by virtue of lemma 5.4. Now making use of lemmas 5.1; 5.2; 5.3 we get

$$I_n = A n^n \int_0^\infty \frac{s^{\lambda+n}}{y^{\lambda+n+1}} e^{-(n/y)s} L_k^\alpha\left(\frac{n}{y} s\right) f(y)\, dy$$

Taking $n \to \infty$ we get

$$\lim_{n\to\infty} I_n \sim \lim_{n\to\infty} \frac{n^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} \int_0^\infty \frac{s^{\lambda+n+k}}{y^{\lambda+n+k+1}} e^{-(n/y)s} f(y)\, dy$$

$$\sim \lim_{n\to\infty} \frac{n^{\lambda+n+k+1}}{\Gamma(\lambda+n+k+1)} \int_0^\infty (st)^{\lambda+n+k} e^{-nts} [t^{-1} f(t^{-1})]\, dt.$$

Now applying theorem 2.3 to this integral, taking the function $\phi(t)$ of the theorem equal to $t^{-1} f(t^{-1})$ and number r of the theorem 2.3 unity and the number c of the theorem 2.3 greater than the number c here, we get

$$\lim_{n\to\infty} I_n = f(s).$$

When $\lambda = k = 0$, we get a known theorem due to Widder ([4] p. 303) for the Laplace integral.

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COMMON FIXED POINT THEOREMS

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In a paper Fisher [1], Sharma and Bajaj [2] obtained some fixed point theorems in complete metric space. In the present paper we obtain some fixed point theorems in complete metric space for mapping satisfying rational inequalities.

Theorem 1: Let S and T be two self mapping of a complete metric space (X, d) satisfying :

$$\text{(I)}\qquad d(Sx, Ty) \leq \beta \frac{[d(s, Sx)]^2}{d(x, Sx) + d(x, Ty)}, \text{ where } d(x, Sx) + d(x, Ty) \neq 0$$

For all x, y in X and $0 < \beta < 1$. Then S and T have a unique common fixed point.

Proof : Let x_0 be any arbitrary in X. We define a sequence $\{X_n\}$ in X such that

$$x_n = Sx_{n-1} \text{ and } x_{n+1} = Tx_n \text{ for } n = 1,2,3 \ldots . \text{ Then,}$$

$$d(x_n, x_{n+1}) = d(Sx_{n-1}, Tx_n) \leq \beta \frac{[d(x_{n-1}, Sx_{n-1})]^2}{d(x_{n-1}, Sx_{n-1}) + d(x_{n-1}, Tx_n)} \text{ from (I)}$$

$$\leq \beta \frac{[d(x_{n-1}, x_n)]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})},$$

$$\leq \beta \frac{[d(x_{n-1}, x_n)]^2}{d(x_n, x_{n+1})},$$

i.e

$$d(x_n, x_{n+1}) \leq \sqrt{\beta}\, d(x_{n-1}, x_n).$$

Similarly $d(x_{n-1}, x_n) \leq \sqrt{\beta}\, d(x_{n-2}, x_{n-1})$ and so on.

Hence $d(x_n, x_{n+1}) \leq (\sqrt{\beta})^n d(x_0, x_1) \to 0$ as $n \to \infty$, since $0 < \beta < 1$.

This proves that $\{x_n\}$ is a Cauchy sequence in X which is complete so converges (*say z*) in X. Now,

$$\begin{aligned} d(z, Tz) &\leq d(z, x_n) + d(x_n, Tz) \\ &\leq d(z, x_n) + d(Sx_{n-1}, Tz) \\ &\leq d(z, x_n) + \beta \frac{[d(x_{n-1}, Sx_{n-1})]^2}{d(x_{n-1}, Sx_{n-1}) + d(x_{n-1}, Tz)} \text{ from (I)} \\ &\leq d(z, x_n) + \beta \frac{[d(x_{n-1}, x_n)]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, Tz)} \to 0 \text{ as } n \to \infty \end{aligned}$$

implies that $Tz = z$ i.e. z is the fixed point of T. Similarly it can be shown that z is the fixed point of S. Consequently z is the common fixed point of S and T. Now for uniquness of z let z' be another common fixed point of S and T, then

$$d(z, z') = d(Sz, Tz') \leq \beta \frac{[d(z, Sz)]^2}{d(z, Sz) + d(z, Tz')} \leq 0, \text{ which is the}$$

contradiction so $z = z'$. This completes the proof.

Our second result is

Theorem (2) . Let S and T be two self mappings of a complete metric space (X,d) satisfying :

$$\text{(II)} \quad d(Sx, Ty) \leq \beta \frac{[d(x, Sx) + d(y, Ty)]^2}{d(x, Sx) + d(x, Ty)},$$

where $d(x, Sx) + d(x, Ty) \neq 0$ for all x, y in X and $0<\beta<1/4$. Then S and T have a unique common fixed point.

Proof . Let x_0 be any arbitrary point in X. We define a sequence $\{X_n\}$ in X such that $x_n = Sx_{n-1}$, and $x_{n+1} = Tx_n$, $n=1,2,3,\ldots$ Then

$$\begin{aligned} d(x_n, x_{n+1}) = d(Sx_{n-1}, Tx_n) &\leq \beta \frac{[d(x_{n-1}, Sx_{n-1}) + d(x_n, Tx_n)]^2}{d(x_{n-1}, Sx_{n-1}) + d(x_{n-1}, Tx_n)} \text{ From (II)} \\ &\leq \beta \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})}, \\ &\leq \beta \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]^2}{d(x_n, x_{n+1})}, \end{aligned}$$

i.e. $$d(x_n, x_{n+1}) \leq \frac{\sqrt{\beta}}{1 - \sqrt{\beta}} d(x_{n-1}, x_n).$$

$$\text{or,} \quad d(x_n, x_{n+1}) \le \lambda\, d(x_{n-1}, x_n), \text{ where } \lambda = \frac{\sqrt{\beta}}{1-\sqrt{\beta}} \text{ and } 0<\lambda<1,$$

Since $0<\beta<1/4$. Similarly $d(x_{n-1}, x_n) \le \lambda\, d(x_{n-2}, x_{n-1})$ and so on.
Hence $\quad d(x_n, x_{n+1}) \le \lambda^n\, d(x_0, x_1) \to 0$ as $n \to \infty$, since $0<\lambda<1$. This proves that $\{x_n\}$ is a Cauchy sequence is X which is complete so converges (say z) in x.

$$\begin{aligned}
\text{Now } d(z, Tz) &\le d(z, x_n) + d(x_n, Tz) \\
&\le d(z, x_n) + d(Sx_{n-1}, Tz) \\
&\le d(z, x_n) + \beta\,\frac{[d(x_{n-1}, Sx_{n-1}) + d(z,Tz)]^2}{d(x_{n-1}, Sx_{n-1}) + d(x_{n-1},Tz)}\,. \\
&\le d(z, X_n) + \beta\,\frac{[d(x_{n-1}, x_n) + d(z,Tz)]^2}{d(x_{n-1}, x_n) + d(x_{n-1},Tz)}\,.
\end{aligned}$$

Letting $n \to \infty$ we have $(1-\beta)\, d(z, Tz) \le 0$ which is the contradiction and so $Tz = z$ i.e. z is the fixed point of T. Similarly it can be shown that z is the fixed point of S. Consequently z is the common fixed point of S and T. Now for uniqueness of z let z' be another common fixed point of S and T, then

$$d(z, z') = d(Sz, Tz') \le \beta\,\frac{[d(z, Sz) + d(z',Tz')]^2}{d(z, Sz) + d(z,Tz')} \le 0 \text{ which}$$

is the contradiction so $z = z'$. Thin completes the proof.

Next we prove analogous results for a sequence of mappings. Our third result is

Theorem 3. Let $\{T_k\}$ be a squence of self mappings of a complete metric space (X, d) satisfying :

$$\text{(III)} \quad d(T_i x, T_j y) \le \beta\,\frac{[d(x, T_j x)]^2}{d(x, T_i x) + d(x,T_j y)}\ ,$$

where $d(x,T_i x) + d(x,T_j y) \neq 0$

for all x,y in X and $0<\beta<1$. Then $\{T_k\}$ have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that $x_n = T_i x_{n-1}$, and $x_{n+1} = T_j x_n$, $n=1,2,3,\ldots$ Then

$$d(x_n, x_{n+1}) = d(T_i x_{n-1}, T_j x_n) \le \beta\,\frac{[d(x_{n-1}, T_j x_{n-1})]^2}{d(x_{n-1}, T_i x_{n-1}) + d(x_{n-1}, T_j x_n)} \text{ from (III)}$$

$$\leq \beta \frac{[d(x_{n-1}, x_n)]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})},$$

$$\leq \beta \frac{[d(x_{n-1}, x_n)]^2}{d(x_n, x_{n+1})},$$

i.e $\quad d(x_n, x_{n+1}) \leq \sqrt{\beta}\, d(x_{n-1}, x_n).$

Similarly $\quad d(x_{n-1}, x_n) \leq \sqrt{\beta}\, d(x_{n-2}, x_{n-1})$ and so on.

Hence $\quad d(x_n, x_{n+1}) \leq (\sqrt{\beta})^n d(X_0, X_1) \to 0$ as $n \to \infty$, since $0 < \beta < 1$.

This proves that $\{X_n\}$ is a Cauchy sequence is X which is complete so converges (say z) in X. Now,

$$\begin{aligned} d(z, T_j z) &\leq d(z, x_n) + d(x_n, T_j z) \\ &\leq d(z, x_n) + d(T_i x_{n-1}, T_j z) \\ &\leq d(z, x_n) + \beta \frac{[d(x_{n-1}, T_i x_{n-1})]^2}{d(x_{n-1}, T_i x_{n-1}) + d(x_{n-1}, T_j z)} \text{ from (III)} \\ &\leq d(z, x_n) + \beta \frac{[d(x_{n-1}, x_n)]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, T_j z)} \to 0 \text{ as } n \to \infty \end{aligned}$$

implies that $T_j z = z$ i.e. z is the fixed point of T_j. Simlarly it can be shown that z is the fixed point of T_i. Consequently z is the common fixed point of $\{T_k\}$. Now for uniqueness of z let z' be another common fixed point of $\{T_k\}$, then

$$d(z, z') = d(T_i z, T_j z') \leq \beta \frac{[d(z, T_i z)]^2}{d(z, T_j z) + d(z, T_j z')} \leq 0, \text{ which is the}$$

contradiction so $z = z'$. This completes the proof.

Our last result is,

Theorem 4 : Let $\{T_k\}$ be a sequence of self mappings of a complete metric space (X,d) satisfying :

$$\text{(IV)} \quad d(T_i x, T_j y) \leq \beta \frac{[d(x, T_i x) + d(y, T_j y)]^2}{d(x, T_i x) + d(x, T_j y)},$$

where $d(x, t_i x) + d(x, t_j y) \neq 0$ for all x, y in X and $0 < \beta < 1/4$, . Then $\{T_k\}$ have a unique common fixed point.

Proof . Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X .

such that $x_n = T_i x_{n-1}$ and $x_{n+1} = T_j x_n$ $n = 1, 2, 3, \ldots$. Then,

$$d(x_n, x_{n+1}) = d(T_i x_{n-1}, T_j x_n) \le \beta \frac{[d(x_{n-1}, T_i x_{n-1}) + d(x_n, T_j x_n)]^2}{d(x_{n-1}, T_i x_{n-1}) + d(x_{n-1}, T_j x_n)} \text{ from (IV)}$$

$$\le \beta \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})},$$

$$\le \beta \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]^2}{d(x_n, x_{n+1})},$$

i.e. $$d(x_n, x_{n+1}) \le \frac{\sqrt{\beta}}{1-\sqrt{\beta}}\, d(x_{n-1}, x_n).$$

or, $$d(x_n, x_{n+1}) \le \lambda\, d(x_{n-1}, x_n), \text{ where } \lambda = \frac{\sqrt{\beta}}{1-\sqrt{\beta}} \text{ and } 0 < \lambda < 1,$$

since $0 < \beta < 1/4$. Similarly $d(x_{n-1}, x_n) \le \lambda\, d(x_{n-2}, x_{n-1})$ and so on.

Hence $d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1) \to 0$ as $n \to \infty$, since $0 < \lambda < 1$. This proves that $\{x_n\}$ is a Cauchy sequence in X which is complete so converges (say z) in X.

Now
$$\begin{aligned} d(z, T_j z) &\le d(z, x_n) + d(x_n, T_j z) \\ &\le d(z, x_n) + d(T_i x_{n-1}, T_j z) \\ &\le d(z, x_n) + \beta \frac{[d(x_{n-1}, T_i x_{n-1}) + d(z, T_j z)]^2}{d(x_{n-1}, T_i x_{n-1}) + d(x_{n-1}, T_j z)}, \text{ from (IV)} \\ &\le d(z, x_n) + \beta \frac{[d(x_{n-1}, x_n) + d(z, T_j z)]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, T_j z)}, \end{aligned}$$

letting $n \to \infty$, we have $(1-\beta) d(z, Tz) \le 0$, which is the contradiction and so $T_j z = z$ i.e. z is the fixed point of T_j .Similarly it can be shown that z is the fixed point of T_i. Consequently z is the common fixed point of $\{T_k\}$. Now for uniqueness of z let z' be another common fixed point of $\{T_k\}$ then,

$$d(z, z') = d(T_i z, T_j z') \le \beta \frac{[d(z, T_i z) + d(z', T_j z')]^2}{d(z, T_i z) + d(z, T_j z')} \le 0 \text{ which}$$

is the contradiction so $z = z'$. This completes the proof.

Corollary 1 . If we put $S = T$ in (I) and for all k, $T_k = T$ in (III) then T has unique fixed point.

Corollary 2 . If we put $S = T$ in (II) and for all k, $T_k = T$ in (IV) then T has a unique fixed point.

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SOME SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE AND MISSING COEFFICIENTS

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ABSTRACT

In the present paper we have obtained coefficient estimates, distortion theorem, redius of convexity and closure theorems for the class $H_p^+(A,B,\alpha,\beta,\gamma)$. Various results obtained in the present paper are shown to be sharp.

1. Introduction and Definitions. Let H denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \qquad \text{... (1.1)}$$

which are analytic in $U^* = \{z : 0 < |z| < 1 \text{ and } z \in C\}$, having simple pole at the origin and residue 1 there. Also H^* is the class of all function in H which are univalent in U^*.

Definition 1. Let $f(z)$ defined by (1.1) with $a_i = 0$ $(i = 1, 2, \dots, p-1, p \in N = \{1,2, \dots,\})$ that is

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1}, \qquad p \in N \qquad \text{... (1.2)}$$

which are analytic in U^*. We denote by H_p, the class of all such functions.

Definition 2. A function $f \in H_p$ is said to be in the class H_p^* if it is univalent in U^*.

Definition 3. A function $f(z)$, analytic and univalent in U^*, is said to be in the class $H_p^*(s)$ if it is given by (1.1) with

$$a_i = 0 \ (i = 1, 2, \dots, p-1, p \in N),$$

$$a_{P+i-1} \geq 0 \ (i \in N)$$

that is, by

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1}, \qquad \text{... (1.3)}$$

$$(a_{n+p-1} \geq 0, \ p \in N).$$

Hence we have relationship

$$H_p^*(s) \subset H_p^* \subset H^* \subset H \text{ and } H_p \subseteq H \quad (p \in N). \qquad \dots (1.4)$$

Definition 4. A function $f \in H^*$ is said to be meromorphically starlike of order a, if $Re\,[-zf'(z)/f(z)] \quad > \alpha,\ (0 \le a < 1,\ z \in U^*)$.

We denote by $H^*(\alpha)$, the class of all functions which are meromorphically starlike of order α.

Definition 5. A funciton $f \in H^*$ is said to be meroomorphically convex of order α, if $Re\,\{-[1 + zf''(z)/f'(z)]\} > \alpha,\ (0 \le a < 1,\ z \in U^*)$.

$H_k(\alpha)$ is the class of all functions which are meromorphically convex of order α.

Motivated especially by works of Mogra [6], Aouf [1]. We aim at presenting here a systematic study of basic properties [coefficient estimates, distortion theorem, radii of starlikeness, closure under convex linear combinations] of some subclasses of the class H_p^*.

Definition 6. A function $f \in H_p$ is said to be in the class $H_p\,(A,B,\alpha,\beta,\gamma)$ if it satisfies

$$\left| \frac{zf'(z)/f(z)+1}{(B-A)\,\gamma\,[zf'(z)/f(z)+\alpha] - B\,[zf'(z)/f(z)+1]} \right| < \beta,\ z \in U^* \qquad \dots (1.5)$$

where $0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1,$

$$B/(B-A) < \gamma \le \begin{cases} B/(B-A)\,\alpha & \alpha \ne 0 \\ 1 & \alpha = 0. \end{cases}$$

Let us write

$$H_p^+\ (A,B,\alpha,\beta,\gamma) = H_p^+ \cap H_p\ (A,B,\alpha,\beta,\gamma). \qquad \dots (1.6)$$

where H_p^+ denotes the class of functions $f(z)$ defined by (1.3) which are analytic in U^*.

2. Main Results Coefficient Estimates.

Theorem 1. Let $f \in H_p$. If

$$\sum_{n=1}^{\infty} \{(n+p) + \beta\,[(B-A)\,\gamma\,(n+p-1+\alpha) - B\,(n+p)]\}\,|a_{n+p-1}| \le (B-A)\,\gamma\,\beta\,(1-\alpha) \qquad \dots (2.1)$$

then, $f \in H_p\ (A,B,\alpha,\beta,\gamma)$, where α, β, γ, A and B are constrained as in Definition 6.

Proof. Let us assume that (2.1) holds for all admissible values of α, β, γ, A

and B. Making use of definition 6, we consider the expression

$$H(f, f') = |zf'(z) + f(z)| - \beta\, |(B-A)\,\gamma\,(zf'(z) + \alpha\, f(z)) - B\,(zf'(z) + f(z)|,$$
$$z \in U^* \qquad \ldots (2.2)$$

Replacing $zf'(z)$ and $f(z)$ by their series expansion, we have for $0 < |z| = r < 1$, from (2.2) that

$$H(f,f') = \Big|\sum_{n=1}^{\infty} (n+p)\, a_{n+p-1}\, z^{n+p-1}\Big| - \beta\, \Big|(B-A)\,\gamma\,(1-\alpha) - [\,(B-A)\,\gamma\,(n+p-1+\alpha)$$
$$-B\,(n+p)\,] \sum_{n=1}^{\infty} a_{n+p-1}\, z^{n+p-1}\Big|$$

$$\leq \sum_{n=1}^{\infty} (n+p)\, |a_{n+p-1}|\, r^{\,n+p-1} - (B-A)\,\gamma\,\beta\,(1-\alpha) + \beta\,[\,(B-A)\,\gamma$$
$$(n+p-1+\alpha) - B\,(n+p)\,] \sum_{n=1}^{\infty} |a_{n+p-1}|\, r^{n+p-1}$$

$$= \sum_{n=1}^{\infty} \{\,(n+p) + \beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha) - B\,(n+p)\,]\}\, |a_{n+p-1}|\, r^{n+p-1}$$
$$- (B-A)\,\gamma\,\beta\,(1-\alpha) \qquad \ldots (2.3)$$

The above inequality in (2.3) holds for all r $(0 < r < 1)$. Hence, letting $r \to 1^-$ in (2.3) we get

$$H(f, f') \leq \sum_{n=1}^{\infty} \{\,(n+p) + \beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha) - B\,(n+p)\,]\}\, |a_{n+p-1}| -$$
$$(B-A)\,\gamma\,\beta\,(1-\alpha) \leq 0\,, \ (p \in N). \qquad \ldots (2.4)$$

by the hypothesis, the assertion of Theorem 1, follows immediately from definition 6.

Theorem 2. Let the function $f(z)$ defined by (1.3) be analytic in U^*, that is, $f \in H_p^+$. Then $f \in H_p^+\ (A,B,\alpha,\beta,\gamma)$ ... (2.5)
if and only if

$$\sum_{n=1}^{\infty} \{\,(n+p) + \beta[\,(B-A)\,\gamma\,(n+p-1+\alpha) - B\,(n+p)\,]\}\, a_{n+p-1} \leq B-A)\,\gamma\,\beta\,(1-\alpha)$$
$$(p \in N) \qquad \ldots (2.6)$$

provided α, β, γ, A and B are constrained as in definition 6.

Proof. If $f \in H_p^+$, we have

$$|a_{n+p-1}| = a_{n+p-1}\,,\ (a_{n+p-1} \geq 0\,,\ p \in N,\ n \in N)$$

and in view of Theorem 1, condition (2.6) is sufficient for the assertion

(2.5) of Theorem 2.

To show that condition (2.6) is necessary, let us assume that (2.5) holds true. Then, in view of (1.6) and as $f \in H_p^+$, we find from Definition 6

$$\frac{zf'(z)/f(z)+1}{(B-A)\gamma\,[zf'(z)/f(z)+\alpha]-B\,[zf'(z)/f(z)+1]}\,.$$

$$=\left|\frac{\sum\limits_{n=1}^{\infty}(n+p)\,a_{n+p-1}\,z^{n+p-1}}{(B-A)\gamma(1-\alpha)-\sum\limits_{n=1}^{\infty}[\,(B-A)\gamma(n+p-1+\alpha)-B\,(n+p)]\}\,a_{n+p-1}z^{n+p-1}}\right|<\beta. \quad \text{.. (2.7)}$$

As $Re\,(z) \le |\,z\,|$ for all z, it follows from (2.7) that

$$Re\left|\frac{\sum\limits_{n=1}^{\infty}(n+p)\,a_{n+p-1}\,z^{n+p-1}}{(B-A)\gamma(1-\alpha)-\sum\limits_{n=1}^{\infty}[\,(B-A)\gamma\,(n+p-1-\alpha)-B\,(n+p)\,]\}\,a_{n+p-1}z^{n+p-1}}\right|<\beta. \quad (2.8)$$

Now choose the values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing denominator in (2.8) and letting $z\to 1$ through real values we obtain

$$\sum_{n=1}^{\infty}\{\,(n+p)+\beta[\,(B-A)\;\gamma\,(n+p-1+\alpha)\;-B\,(\,n+p)\,]\}\,a_{n+p-1}\le\;B-A)\,\gamma\,\beta\,(1-\alpha)$$

Hence the result (2.6) follows.

Corollary. If $f \in H_p^+\;(A,B,\alpha,\beta,\gamma)$ then

$$a_{n+p-1}\le\frac{(B-A)\,\gamma\,\beta\,(1-\alpha)}{\{\,(n+p)+\beta\,[\,(B-A)\,\gamma\,(n+p-1-\alpha)-B\,(\,n+p)\,]\}}\qquad n\in N,$$

with equality for functions of the form

$$f_{n+p-1}(z)=z^{-1}+\frac{(B-A)\,\gamma\,\beta\,(1-\alpha)\;z^{n+p-1}}{\{\,(n+p)+\beta\,[\,(B-A)\,\gamma\,(n+p-1-\alpha)-B\,(\,n+p)\,]\}}\qquad n\in N,\ldots\;(2.9)$$

3. Distortion Theorem and Radius of convexity :

Theorem 3. If $f\in H_p^+\;(A,B,\alpha,\beta,\gamma)$, then for $0<|\,z\,|=r<1$,

$$r^{-1}-\frac{(B-A)\,\gamma\,\beta\,(1-\alpha)\,r^{p}}{\{\,(1+p)+\beta\,[\,(B-A)\,\gamma\,(p+\alpha)-B\,(\,n+p)\,]\}}\le|\,f\,(z)\,|$$

$$\le\;r^{-1}+\frac{(B-A)\,\gamma\,\beta\,(1-\alpha)\,r^{p}}{\{\,(1+p)+\beta\,[\,(B-A)\,\gamma\,(p+\alpha)-B\,(\,n+p)\,]\}}\,.\qquad\ldots\;(3.1)$$

The result (3.1) is sharp, the extremal function being

$$f_p(z) = z^{-1} + \frac{(B-A)\,\gamma\,\beta\,(1-\alpha)\,z^p}{\{\,(1+p) + \beta\,[\,(B-A)\,\gamma\,(p+\alpha) - B\,(\,n+p)\,]\}} \text{ at } z = \pm\, r. \qquad \ldots (3.2)$$

Proof. Since $f \in H_p^+\ (A,B,\alpha,\beta,\gamma)$,. In view of Theorem 2, we have

$$\sum_{n=1}^{\infty} a_{n+p-1} \le \frac{(B-A)\,\gamma\,\beta\,(1-\alpha)}{\{\,(1+p) + \beta\,[\,(B-A)\,\gamma\,(p+\alpha) - B\,(\,n+p)\,]\}}.$$

Then for $0 < |\,z\,| = r < 1$, we have

$$\begin{aligned} |\,f(z)\,| &\le r^{-1} + \sum_{n=1}^{\infty} a_{n+p-1}\, r^{n+p-1}, \\ &\le r^{-1} + r^{\,p} \sum_{n=1}^{\infty} a_{n+p-1} \\ &\le r^{-1} + r^{\,p}\, \frac{(B-A)\,\gamma\,\beta\,(1-\alpha)}{\{\,(1+p) + \beta\,[\,(B-A)\,\gamma\,(p+\alpha) - B\,(\,n+p)\,]\}}. \end{aligned}$$

and

$$\begin{aligned} |\,f(z)\,| &\ge r^{-1} - \sum_{n=1}^{\infty} a_{n+p-1}\, r^{n+p-1}, \\ &\ge r^{-1} - r^{\,p} \sum_{n=1}^{\infty} a_{n+p-1} \\ &\ge r^{-1} - r^{\,p}\, \frac{(B-A)\,\gamma\,\beta\,(1-\alpha)}{\{\,(1+p) + \beta\,[\,(B-A)\,\gamma\,(p+\alpha) - B\,(\,n+p)\,]\}}. \end{aligned}$$

It can be easily seen that the function $f_p\,(z)$ defined by (3.2) is extremal for the theorem.

Theorem 4. If $f \in H_p^+\ (A,B,\alpha,\beta,\gamma)$, then $f\,(z)$ is meromorphically convex in $|\,z\,| < r = r_p\ (A,B,\alpha,\beta,\gamma)$,where

$$r_p\,(A,B,\alpha,\beta,\gamma) = \inf \left[\frac{\{(n+p) + \beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha) - B\,(\,n+p)\,]\}}{(B-A)\,\gamma\,\beta\,(1-\alpha)\,(n+p-1)(n+p+1)}\right]^{1/(n+p)}, n \in N.$$

The bounds for $|\,z\,|$ is sharp for each n, with extremal function being of the form (2.9).

Proof. It is sufficient to show that

$$\left|\, 2 + \frac{zf''(z)}{f'(z)} \,\right| \le 1, \text{ for } |\,z\,| < r_p\,(A,B,\alpha,\beta,\gamma)\,,$$

where $r_p\ (A,B,\alpha,\beta,\gamma)$ is as stated in the statement of the theorem.

We have

$$\left|\frac{2f'(z)+zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{n=1}^{\infty}(n+p-1)(n+p)\,a_{n+p-1}\,z^{n+p-2}}{-1/z^2+\sum\limits_{n=1}^{\infty}(n+p-1)\,a_{n+p-1}\,z^{n+p-2}}\right|$$

$$\leq \frac{\sum\limits_{n=1}^{\infty}(n+p-1)(n+p)\,a_{n+p-1}\,|z|^{n+p}}{1-\sum\limits_{n=1}^{\infty}(n+p-1)\,a_{n+p-1}\,|z|^{n+p}} \leq 1$$

provided that

$$\sum_{n=1}^{\infty}(n+p-1)\quad(n+p+1)\,|z|^{n+p}\,a_{n+p-1} \leq 1. \qquad \ldots(3.4)$$

By Theorem 2 we have

$$\sum_{n=1}^{\infty}\frac{\{(n+p)+\beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}}{(B-A)\,\gamma\,\beta\,(1-\alpha)}\,a_{n+p-1} \leq 1. \quad \ldots(3.5)$$

In view of (3.5), it follows that (3.4) is true if

$$(n+p-1)\quad(n+p+1)\,|z|^{n+p} \leq \frac{\{(n+p)+\beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}}{(B-A)\,\gamma\,\beta\,(1-\alpha)}$$

$$n \in N$$

or

$$|z| \leq \left[\frac{\{(n+p)+\beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}}{(B-A)\,\gamma\,\beta\,(1-\alpha)\,(n+p-1)(n+p+1)}\right]^{1/n+p}, n \in N \qquad \ldots(3.6)$$

setting $|z| = r_p\,(A,B,\alpha,\beta,\gamma)$ in (3.6), the result follows. Sharpness can be verified easily.

4. Convex linear Combinations.

In this section we shall prove that the class $H_p^+\,(A,B,\alpha,\beta,\gamma)$, is closed under convex linear combinations.

Theorem 5. Let

$$\psi_0(z) = 1/z\;(z \in U^*),$$

and

$$\psi_{p+n-1}\,(z) = z^{-1}+\frac{(B-A)\,\gamma\,\beta\,(1-\alpha)\;z^{\,n+p-1}}{\{(n+p)+\beta\,[\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}}\,(p,n \in N,\; z \in U^*).$$

Then the function $\psi(z)$ is in $H_p^+\;(A,B,\alpha,\beta,\gamma)$ if and only if it can be expressed in the form

$$\psi(z) = \sum_{n=0}^{\infty} \lambda_n \psi_n(z)$$

$$(\lambda_n \geq 0, \lambda_j = 0 \ (j=1, \ldots, p-1, \text{ if } p \geq 2), \quad \sum_{n=0}^{\infty} \lambda_n = 1.$$

Proof. Let $\psi(z) = \sum_{n=0}^{\infty} \lambda_n \psi_n(z)$

$$\psi(z) = z^{-1} + \sum_{n=1}^{\infty} \lambda_{n+p-1} \frac{(B-A)\gamma\beta(1-\alpha)\, z^{n+p-1}}{\{(n+p)+\beta[(B-A)\gamma(n+p-1+\alpha)-B(n+p)]\}}.$$

Since

$$\sum_{n=1}^{\infty} \frac{\{(n+p)+\beta[(B-A)\gamma(n+p-1+\alpha)-B(n+p)]\}}{(B-A)\gamma\beta(1-\alpha)}$$

$$\frac{(B-A)\gamma\beta(1-\alpha)\,\lambda_{n+p-1}}{\{(n+p)+\beta[(B-A)\gamma(n+p-1+\alpha)-B(n+p)]\}} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1.$$

Hence $\psi(z) \in H_p^+ (A,B,\alpha,\beta,\gamma)$.

Conversely assume that

$$\psi(z) \in H_p^+ (A,B,\alpha,\beta,\gamma).$$

Since

$$a_{n+p-1} \leq \frac{(B-A)\gamma\beta(1-\alpha)}{\{(n+p)+\beta[(B-A)\gamma(n+p-1+\alpha)-B(n+p)]\}}, \ n \in N, \ p \in N.$$

Setting

$$\lambda_{n+p-1} = a_{n+p-1} \frac{\{(n+p)+\beta[(B-A)\gamma(n+p-1+\alpha)-B(n+p)]\}}{(B-A)\gamma\beta(1-\alpha)} \qquad (p,n \in N).$$

$\lambda_j = 0 \ (j=1, \ldots, p-1, \text{ if } p \geq 2)$,

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n \geq 0 .$$

Hence, it follows that

$$\psi(z) = \sum_{n=0}^{\infty} \lambda_n \psi_n(z)$$

Theorem 6. The class $H_p^+ (A,B,\alpha,\beta,\gamma)$ is closed under convex linear combinations.

Proof. Suppose $f_k(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n+p-1,\,k}\, z^{n+p-1} \ (k = 1,2)$ are in $H_p^+ (A,B,\alpha,\beta,\gamma)$.

Setting

$$f(z) = (1-t)f_1(z) + t\, f_2(z), \qquad (0 \le t \le 1).$$

Then

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} [(1-t)\, a_{n+p-1,1} + t\, a_{n+p-1,2}]\, z^{n+p-1}$$

In view of Theorem 2,

$$\sum_{n=1}^{\infty} \{(n+p)+ \beta\, [\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}\, \{(1-t)\, a_{n+p-1,1} + t\, a_{n+p-1,2}\}$$

$$= (1-t) \sum_{n=1}^{\infty} \{(n+p)+ \beta\, [\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}\, \{\, a_{n+p-1,1}$$

$$+\, t \sum_{n=1}^{\infty} \{(n+p)+ \beta\, [\,(B-A)\,\gamma\,(n+p-1+\alpha)-B\,(\,n+p)\,]\}\, a_{n+p-1,2}\}$$

$$\le (1-t)\,(B-A)\,\gamma\,\beta\,(1-\alpha) + t\,(B-A)\,\gamma\,\beta\,(1-\alpha).$$

$$= (B-A)\,\gamma\,\beta\,(1-\alpha)\,,$$

which shows that $f(z) \in H_p^{+}\ (A,B,\alpha,\beta,\gamma)$

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CONTINUITY CONDITION FOR MEASURABLE MULTI-VALUED MAPPING

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1. Introduction. The concept of continuity of a funciton has been the concern of many authors since the time of Newton who defined the concept of limit of real valued function to determine the continuity of a function by showing that if the limit of function exists at a given point and if that limit of the function is equal to the value of function at that point then function is continuous. This concept of continuity of function was generalised from real valued to complex valued function as well as function defined on metric spaces and topological spaces by the use of open sphere and open sets etc. In this generalisation the ontinuity of function is investigated in three main directions viz.

(1) How the continuity of function is affective by changing the natue of a function without being changing the nature of a function without being changing the nature of spaces?

(2) How the continuity of funciton is affected by changing the nature of space under consideration, without changing the nature of function?

(3) How the continuity of function is affected when the nature of function as well as space under consideration, both are subject to change?

Since we are investigating the concept of continuity related to multi-valued maping so we will confine above three aspects. Many authors like Plis, Aumann, Jacob, Himmelberg, have tried to investigate such relationship in the multi-valued mapping under specific condition on a metric space. In the work of all these author's the principle involved has been generalisation of continuity of open sets in metric spaces in terms of measure i.e. any multi-valued function to be continuous, we must show that for any $\varepsilon > 0$ there exist an open set E_ε contain in the given metric space such that the measure $\mu(E_\varepsilon) < \varepsilon$, where μ is Randon measure, when the restriction of measurable mapping to E_ε is continuous. Plis has tried to obtain the continuity condition for a measurable multi-valued mapping by assuming the metric space to be a compact space, while Casting has tried to obtain a similar condition for the continuity of measurable multi-valued mapping when space is seperable instead of compact. Jacob further generalised the condition of both Plis and Casting. His study was moltivated due to the considerable direction porvided by the work of Casting, Kurtawaski, Plis and Michal.

Our study will also confine to such similar situation for multi-valued mapping when it has compact value.

2. Preliminary. Through out the paper (T,e) will denote compact metric space and (X, d) be a seperable metric space. A multifunction $\Omega :\rightarrow X$ is a subset of $T \times X$ with domain equal to T. If $B \in X$, then $\Omega^{-1}(B) = \{t \in T \mid \Omega\ (t) \cap B \neq \phi\}$.

Definition 2.1. A multifunction $\Omega : \mathrm{T} \rightarrow X$ is (weakly) measurable if $\Omega^{-1}(B) = \{t \in T \mid \Omega\ (t) \cap B \neq \phi\}$ is measurable for every closed (respectively, open) subset B of X.

Definition 2.2. If $\Omega : T \rightarrow X$ is a multifunction, then φ_Ω is the real valued function defined on $T \times X$ by

$$\varphi_\Omega(t, x) = d\ [X\ ,\ \Omega\,(t)],\ \text{for}\ (t,x) \in T \times X.$$

Definition 2.3. If X be a seperable metrizable space, then X is said to be *Polish*, if X is seperable and metrizable by a complete metric. Again, if X is metrizable and continuous image of Polish space then X is called *Souslin Space.*

Definition 2.4. For the determination of continuity condition in multi-valued mapping we need following

(a) If $t_0 \in T$, and $S_\varepsilon\,(t_0) = \{t \in T \mid \rho\,(t,\ t_0) < \in\}$ then $\xi(t_0) = \{S_\varepsilon\,(t_0), \in > 0\}$ is said to be *neighborhood filter base* at t_0.

(b) The *grill* of $\xi(t_0)$ is defined and denoted as

$$\xi''(t_0) = \{\ S''\,(t_0) \subset T \mid S''(t_0) \cap S_\varepsilon\,(t_0) \neq 0,\ \text{for}\ \varepsilon > 0\ \}$$

(c) If range of Ω is non empty closed subset of X then the *Pseudo limit superior* of Ω as $t \rightarrow t_0$ [abbreviated : *p–limsup*$_{t\rightarrow t_0}$ $\Omega\,(t)$] is defined to be

$$\cap_{S_\varepsilon(t_0) \in \xi(t_0)}\ CL\ [\cup_{t \in S_\varepsilon(t_0)} \Omega\,(t)\,]$$

where CL denotes the closure,

and the *Pseudo limit inferior* of Ω as $t \to t_0$ (abbreviated : *p-lim* $inf_{t\to t_0}$) is defined by

$\cap_{S''(t_0)\in\xi(t_0)} CL\,[\cup_{t\in S''(t_0)}\Omega(t)]$

(d) Ω is *Pseudo upper semicontinuous* at $t_0 \to T$ (abbreviated : *p–usc* at $t_0 \in T$) if *p–lim sup* $_{t\to t_0}\Omega(t)\subset\Omega(t_0)$; Ω is *Pseudo lower semi continuous* at $t_0 \in T$ [abbreviated : *p–lsc* at $t_0\in T$] if *p-limit*$_{t\to t_0}\Omega(t) \supset \Omega(t_0)$; Ω is Pseudo continuous [abbreviated : *p-continuous* at $t_0 \in T$] if Ω is *p-usc* and *p–lsc* at $t_0 \in T$.

(e) Ω is upper semi continuous at $t_0\in T$ [abbreviated *usc* at $t_0\in T$] if for each open G containing (t_0) there is an $S_\varepsilon(t_0)\in\xi(t_0)$ such that

$t\in S_\varepsilon(t_0)\Rightarrow\Omega(t)\subset G$.

Ω is *Lower semi continous* at $t_0\in T$ [abbreviated *lsc* at $t_0\in T$] if for every open G meeting $\Omega(t_0)$ there is an

$S_\varepsilon(t_0)\in\xi(t_0) \Rightarrow\Omega(t)\subset G\neq 0.$

(f) If collection of non-empty closed subset of X have finite topology i.e. X has an open base then Ω is continous if any only if Ω is at $t_0\in T$ and *lsc* at $t_0\in T$. The following result is due to Himmerlberg, Jacob, Van Vleck [5].

Definition 2.5. If T be locally compact and σ– compact Hausdorff space, and X be seperable metric space, and $\Omega : T\to X$ a multifunction. Then among the following statement we have relations (i)⇒(ii)⇒(iii).

(i) Ω is weakly measurable :

(ii) $P\,\xi_\Omega(t,x)$ is measurable function of t for each $x\in X$;

(iii) For all $\epsilon>0$ there exists a closed subset T_ε of T such that $\mu(T-T_\varepsilon)<\epsilon$ and $\xi_\Omega / T_\varepsilon\times X$ is continuous.

The following results are due to Himmelberg [3]:

2.6. Let x be seperable metric space and $\Omega : T\to X$ be multifunction with compact values. Then Ω is mesurable iff. Ω is weakly measurable.

2.7. Let X be seperable metric space and let $\Omega : T\to X$ be a multifunction then

(i) Ω is weakly measurable ,

(ii) $t\to d\,[X,\Omega(t)]$ is measurable function of t for each X,

(iii) $Gr(\bar F)$ is $A\times B$ – measurable [$\bar F$ is defined by $\bar F(t)=\overline{F(t)}$]. Then (i)$\Leftrightarrow$ (ii)$\Leftrightarrow$ (iii),

where *Gr* denotes graph.

2.8. Let T be complete, X be Souslin, and $\Omega : T \to X$ a multifunction such that $Gr\ (F)$ is $A \times B$ – measureable then Ω is B– measurable.

3. Main Result. Let T be a compact metric space and (X,d) be seperable metric space, $\Omega : T \to X$ be a multifunctioning with compact value and μ be Randon measure, then the following statements are equivalent :

(i) Ω is measurable,

(ii) Ω is weakly measurable ,

(iii) $\xi_\Omega(t,x)$ is measurable function of t for each $x \in X$,

(iv) For all $\in > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and $\xi_\Omega / T_\varepsilon \times X$ is continuous

(v) For all $\in > 0$ there exists a Souslin space T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and graph $Gr\ (\Omega / T_\varepsilon)$of Ω / T_ε is Souslin sub–space.

(vi) There exist a Souslin space T' of T such that $\mu(T - T') = 0$ and graph $Gr\ (\Omega / T')$ is Souslin.

(vii) For all $\in > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and Ω / T_ε is *p-usc*.

(viii) For $\in > 0$ there exists a closed subspace T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and $Gr\ (\Omega / T_\varepsilon)$ is closed subst of $T \times X$.

Proof . Our scheme to prove the resutl is

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i).

and (iv) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (v).

First Part .

(i) $\Rightarrow$ (ii)

Since the Himmelberg [3] in his paper proved that if T be any arbitrary measurbale space and X be any metric space such that the multi–valued functions $\Omega : T \to X$ has compact value then mesurability and weak measurability of multi–valued function are equivalent. Hence (i) $\Rightarrow$(ii) .

(ii) $\Rightarrow$ (iii)

It follows directly by the virtue of Lemma 3 due to Himmelberg–Jacob–Vanveleck [5].

(iii) $\Rightarrow$ (iv)

By the same said Lemma.

(iv) ⇒ (v)

To prove it, since for any $\in > 0$ and $\xi_\Omega | T_\varepsilon \times X$ is continuous, we claim $Gr\ (\Omega / T_\varepsilon)$ is closed. To have it we know $\xi_\Omega : T \times X \rightarrow X$ defined by $\xi_\Omega (t,x) = d\,[x, \Omega (t)]$ and by given (iv) there exists a closed subset T_ε of T such that $\mu\ (T - T_\varepsilon) < \in$ and $\xi_\Omega | T_\varepsilon \times X$ is continous and therefore if (t_n, x_n) be any arbitrary convergent sequence in $G_r\ (\Omega / T_\varepsilon)$ s.t. $(t_n, x_n) \rightarrow (t, x)$ then by the continuity of $\xi_\Omega / T \times X$, we have,

$$d\,[x, \Omega (t)] = \xi_\Omega\,(t,x) = Lim\ d\,[x_n, \Omega (t_n)] = 0$$

because for each $t_n \in T_\varepsilon$, $t_n \in \Omega (t_n)$ and therefore $d\,[x, \Omega (t)] = 0$ but t_ε being closed subset of compact metric space T so it is compact and every compact metric space is complete so (t_n) in T is connvergent to t, so $t \in T_\varepsilon$. Further Ω has compact value therefore $\Omega (t)$ is compact subset of X, which is Hausdorff and every compact subset of Hausdorff is closed hence $\Omega (t)$ is closed, and

$d\,[x, \Omega (t)] = 0$, so $x \in \Omega (t) \Rightarrow (x, t) \in Gr\ (\Omega / T_\varepsilon)$ hence $Gr\ (\Omega / T_\varepsilon)$ of Ω / T_ε is closed. Again we know every compact space is seperable therefore T_ε being the closed subset of compact metric space is compact, so it must be seperable. Moreover every compact space is locally compact. So T_ε is locally compact seperable and every locally compact seperable metric space is an open subset of compact metric space, it follows that every space is polish, hence it is Souslin

thus T_ε and $G_r\,(\Omega / T_\varepsilon)$ are Souslin space because they are closed subspace of suslin space T and $T \times X$ respectively. Moreover we know that every closed subspace of Souslin space is Souslin, hence the result.

(v) ⇒ (vi)

Let us select $T_{1/n}$ for each $n = 1,2,...$ by taking n so large that $1/n < \in$ and let

$$T' = \cup\ \{ T_{1/n} / n = 1,2\ \ldots \}$$

We claim

(a) $\mu(T - T') = 0$ and

(b) Graph $Gr\ (\Omega / T')$ is Souslin.

To prove (a), we know from given (v) that $(T - T_\varepsilon) < \in$.

So we have $\mu(T - T') < \lim\limits_{n \rightarrow \infty} 1/n \Rightarrow \mu(T - T') = 0$, but we know that $\mu(T-T') > 0$, hence $\mu(T-T') = 0$.

To prove (b), we have $Gr\,(\Omega / T') = \cup_n Gr\,(\Omega / T_{1/n})$.

and again by (v) we have $Gr\,(\Omega / T_{1/n})$ of $(\Omega / T_{1/n})$ is a Souslin space of $T \times X$ and countable union of Souslin subspace is Souslin. Hence $Gr\,(\Omega / T')$ is Souslin.

Thus the result.

(vi) ⇒ (i)

Now to prove it, it is sufficient to show that inverse image of any closed subset of X under Ω is closed in T. For this, let B be any arbitrary closed subset in X. Then we claim $\Omega^{-1}(B)$ is closed. To have this claim let T_1, $T_2, T_3, \ldots, T_n$ be any arbitary covering of T where each T_i is a closed subspace, since each closed subspace of compact space T is compact therefore each T_i is compact subspace and has a compact value, therefore $\Omega^{-1}(B)$ is the projecion of T of $Gr\,(\Omega) \cap (T \times B)$ but,

$$Gr\,(\Omega) \cap (T \times B) = \cup \{Gr(\Omega) \cap (T \times B) \mid n = 1,2 \ldots\}$$

Since T being compact hence every convering is reducible to finite sub–covering therefore,

$$\cup \{Gr(\Omega) \cap (T_n \times B) \mid n = 1,2 \ldots\} = \cup_{i=1}^{k} \{Gr(\Omega) \cap (T_i \times B)\},$$

but for each i, $Gr\,(\Omega) \cap (T_i \times B)$, can be expressed as the union of sets given by

$$Gr\,(\Omega / T') \quad \cap \quad \{(T_i \cap T') \times B)\} \qquad \ldots (1)$$

$$\text{and } Gr\,(\Omega) \quad \cap \quad \{(T_i - T') \times B)\} \qquad \ldots (2)$$

Now (1) is Souslin and is projection on the Souslin by our given hypothesis and therefore measurable and (2) is projection on a set of measure zero and we know that countable union of measerable is measurable hence every finite union of measurable set is measurable as every finite set is countable. Thus $\Omega^{-1}(B)$ is measurable being countable union of measurable sets. Therefore Ω is measurable and hence the result.

Second Part

(iv) ⇒ (vii)

Now to prove this we must show that if (t_n) and (x_n) are sequences in T_ε and X respectively such that $x_n \in \Omega\,(t)$ for all n and such that $t_n \to t$ and $x_n \to x$ then we claim $x \in \Omega\,(t)$. Now to prove this condition, since by (iv) we have that $\xi_\Omega / (T_\varepsilon \times X)$ is continuous and therefore we have by definition of ξ_Ω

$$d\,[x, \Omega(t)] = \lim_{n\to\infty} d\,[x_n, \Omega(t_n)] = 0$$

and $t \in T$ because T_ε is closed and since $\Omega(t)$ is compact subset of Hausdorff space X so it must be closed therefore,

$$d\,[x, \Omega(t)] = 0 \Rightarrow x \in \Omega(t)$$

and thus the conditon.

(vii) ⇒ (viii)

Since by given (vii) we have for any fixed $\epsilon > 0$ If we select T_ε satisfying the condition (vii) then

we claim $Gr\,(\Omega / T_\varepsilon)$ is closed subset of $T \times X$. To prove the claim it is sufficient to show that $Gr\,(\Omega / T_\varepsilon)$ contains all of its limit points. To have this condition, let $(t,x,)$ be any arbitrary limitpoint of $Gr\,(\Omega / T_\varepsilon)$ then we know by the property of limit point of set in metric space, that if x is any limit point of subset A of metric space X then there exists a sequence (x_n) in A such that $x_n \to x$ therefore if (t, x) is limit point of $Gr\,(\Omega / T_\varepsilon)$. Then there exsits a sequence (t_n, x_n) in $Gr\,(\Omega / T_\varepsilon)$ such that $(t_n, x_n) \to (t,x)$. Since Ω / T_ε is *p-u.s.c.* therefore $\Rightarrow x \in \Omega / T_\varepsilon$ which implies that $(t,x) \in Gr\,(\Omega / T_\varepsilon)$. Hence $Gr\,(\Omega / T_\varepsilon)$ is closed subset of $T \times X$.

(viii) ⇒ (v)

The phenomena is trivial.

The results in Section -2 due to Himmelberg, Jacob and Van Vleck [5] becomes particular case of our main result because every compact value of metric space is closed and complete.

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ON PARA-KAHLER MANIFOLD WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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ABSTRACT

Sawaki and Sekigawa [3] and Defever and others [1] have studied semi-symmetric and Ricci-symmetric Para-Kahler manifolds. We have, in this paper, studied Para Kahler manifold with constant holomorphic sectional curvature and have proved that this manifold is Semi-Symmetric as well as Ricci semi-symmetric.

1. Preliminaries. Let (M,G,F) be a connected, $n = 2m$–*diamensional* $(m>2)$ manifold of class C^{∞} with complex structure tensor F satisfying.

(1.1) (a) $\overline{\overline{X}} + X = 0$

where

(1.1) (b) $\overline{X} \overset{def}{=} FX$;

and a non-necessarily definite metric tensor G satisfying :

(1.2) $G(\overline{X}, \overline{Y}) = G(X, Y)$

Then (M,G,F) is called an almost Hermite manifold. The almost Hermite manifold (M,G,F) is said to be a para Kahler manifold [3] if its Riemannion Curvature tensor $'K$ satisfies the Kahler identity.

(1.3) $'K(X, Y, Z, U) = {}'K(X, Y, \overline{Z}, \overline{U})$.

Remark (1.1) In the above and in what follows the letters X, Y, Z, U ... etc. denote the C^{∞} vector fields in the Lie-algebra $L(M)$ of the manifold M.

We know that every Kahler manifold satisfies (1.3) but converse is not true. In views of the following identities for Riemannion curvature tensor $'K$:

(1.4) $'K(X, Y, Z, U) = -{}'K(Y,X,Z,U) = +{}'K(Z,U,X,Y) = -{}'K(X, Y, Z,U)$
$= -{}'K(X,Y,U,Z)$

and the equation (1.1) the equation (1.3) gives the following results :

(1.5) (a) $'K(X,Y,Z,U) = {}'K(\overline{X},\overline{Y},Z,U) = {}'K(X,Y,\overline{Z},\overline{U}) = {}'K(\overline{X},\overline{Y},\overline{Z},\overline{U})$,

and

(1.5) (b) $\quad 'K(X,Y,\overline{Z},U) = +'K(X, Y, Z, \overline{U}) = 0$.

2. Para-Kahler Manifolds with Constant Holomorphic Sectional Curvature. In an almost Hermite manifolds the holomorphic sectional curvature with regard to any vactor field X is given by [2]

(2.1) $\quad k(X)\, G(X,X)\, G(X,X) + 'K(X, \overline{X}, X, \overline{X}) = 0$

Also, we know that the condition that the Almost Hermite manifold be of constant holomorphic sectional curvature is [2]

$$\begin{aligned}(2.2)\quad & 3'K(X, Y, Z, U) + 3'K(\overline{X}, \overline{Y}, Z, U) + 3'K(X, Y, \overline{Z}, \overline{U}) + 'K(\overline{X}, \overline{Y}, ZU) \\ & + 'K(X, U, \overline{Z}, \overline{Y}) + 'K(Y, U, \overline{X}, \overline{Z}) + 'K(Y, Z, \overline{U}, \overline{X}) + 'K(Z, X, \overline{U}, \overline{Y}) + \\ & 'K(X, Z, \overline{U}, \overline{Y}) + 'K(Y, \overline{U}, Z, \overline{X}) + 'K(\overline{U}, X, Z, \overline{Y}) + 'K(Z, \overline{Y}, U, \overline{X}) \\ & = -4K\{2'F(X,Y)\, 'F(Z,U) + G(X,Z)G(Y,U) - G(Y,Z)\, G(X,U) + \\ & 'F(Y,Z)\, 'F(U,X) + 'F(X,Z)\, 'F(Y,U)\}.\end{aligned}$$

Now using the conditions (1.5) in (2.2), we get

$$\begin{aligned} & 6'K(X,Y,Z,U) + 'K(Z,Y,X,U) + 'K(X,Z,Y,U) + 'K(X,\overline{Z},Y,\overline{U}) + 'K(\overline{Z},Y,X,\overline{U}) \\ & = -2K\{2'F(X,Y)\, 'F(Z,U) + G(X,Z)\, G(Y,U) - G(Y,Z)\, G(X,U) + 'F(Y,Z) \\ & \quad 'F(U,X) + 'F(X,Z)\, 'F(Y,U)\}.\end{aligned}$$

Further using Binchies first identity and (1.5) in the above relation, we get

$$\begin{aligned}(2.3)\ (a)\quad & 4'K(X,Y,Z,U) = -k\{2'F(X,Y)\, 'F(Z,U) + G(X,Z)\, G(Y,U) - G(Y,Z) \\ & \qquad G(X,U) - 'F(Y,Z) + 'F(X,U) + 'F(X,Z)\, 'F(Y,U)\},\end{aligned}$$

$$\begin{aligned}(2.3)\ (b)\quad & 4K(X,Y,Z) = -K\{2'F(X,Y)\overline{Z} + G(X,Z)\, Y - G(Y,Z)\, X - F(Y,Z)\, \overline{X} \\ & \qquad + 'F(X,Z)\, \overline{Y}\}.\end{aligned}$$

Contracting (2.3) (b) with respect to X, we get

(2.4) (a) $\quad 4Ric(Y,Z) = k\,(n+2)\, G(Y,Z).$

Again Barring Y, Z in (2.3) (b) and then contracting with respect to X, we get

(2.4) (b) $\quad 4Ric(Y,Z) = k\,(n+2)\, G(Y,Z).$

Combining the above two results, we can write

(2.4) (c) $\quad Ric(Y,Z) = Ric(\overline{Y},\overline{Z}) = \dfrac{k\,(n+2)}{4}\, G(Y,Z).$

Thus, we have

Theorem (2.1) The condition that the para-Kahler manifold be of constant holomorphic sectional curvature is given by (2.3) (a) or (2.3) (b).

Corollary (2.1). In a para-Kahler manifold with constant holomorphic sectional curvature, we have $Ric(Y,Z) = Ric(\overline{Y},\overline{Z})$.

The proof is obvious from (2.4) (c).

Corollary (2.2). The Para Kahler manifold with constant holomorphic sectional curvature is an Einstein manifold.

The proof follows from from (2.4) (C).

Now, a Riemannion manifold is defined to be Semi-Symmetric and Ricci–Semi–Symmetric [1], if

(2.5) (a) $(K.{}'K)\ (Z,U,V,W;X,Y) = -{}'K\ (K(X,Y,Z),\ U,V,W)$
$-\ {}'K\ (Z,\ K\ (X,Y,U),\ V,W) -\ {}'K\ (Z,U,K\ (X,Y,V),W)$
$-\ {}'K\ (Z,U,V,K\ (X,Y,W) = 0$

and

(2.5) (b) $(K.RIc)\ (Z,U;X,Y) = -Ric\ (K(X,Y,Z),U) -Ric\ (Z,K(X,Y,U)) = 0.$

Here the Tensors $(K'K)$ and $(K.\ Ric)$ are tensors of type (0,6) and (0,4).

Another tensor Q of type (0,6) is defined as

(2.6) $Q\ (Ric,\ {}'K)\ (Z,U,V,W;X,Y) = -{}'K\ (Ric\ (Z,Y)\ X - Ric\ (Z,X)\ Y,\ U,V,W)$
$-{}'K\ (Z,Ric\ (U,Y)\ X - Ric\ (U,X)\ Y,V,W) -\ {}'K\ (Z,U,\ Ric\ (V,Y)\ X$
$-Ric\ (V,X)\ Y,\ W) -\ {}'K\ (Z,U,V,\ Ric\ (W,Y)X - Ric\ (W,X)\ Y)\ .$

Now, taking account of (2.3), we can write the equation (2.5) (a) as

$$4^2\ (K.{}'K)\ (Z,U,V,W;X,Y) = 4{}'K(4K(X,Y,Z),\ U,V,W) -4{}'K(Z,4K$$
$$(X,Y,U,)V,W) - 4{}'K(Z,U\ 4K\ (X,Y,V),W) - 4{}'K\ (Z,U,V,\ 4K\ (X,Y,W)).$$

Using (2.3) in this equation, we get

$$4^2\ (K.{}'K)\ (Z,U,V,W,\ X,Y) = -\ 4{}'K\ (-k\{2{}'F\ (X,Y)\overline{Z} + G\ (X,Z)Y - G\ (Y,Z)\ X$$
$$-{}'F\ (Y,Z)\ \overline{X} + {}'F(X,Z)\overline{Y}\ \},\ U,\ V,W) -\ 4{}'K\ (Z,\ -k\ \{2{}'F(X,Y)\overline{U} + G(X,U)Y -$$
$$G(Y,U)X - {}'F(Y,U)\ \overline{X} + {}'F(X,Y),\overline{Y}\},V,W) - 4{}'K(Z,U,-k\ \{2{}'F\ (X,Y)\ \overline{V} + G\ (X,V)$$
$$Y - G\ (Y,V)\ X - {}'F\ (Y,V)\ \overline{X} + {}'F\ (X,V)\ \overline{Y}\},\ W) -4{}'K\ (Z,\ U,V - k\ \{2{}'F(X,Y)\ \overline{W}$$
$$+ G(X,W)\ Y - G(Y,W)\overline{X} - {}'F(Y,W)\overline{X} + {}'F(X,W)\overline{Y}\})$$
$$= 4\ k\{2{}'F(X,Y)\ {}'K\ (\overline{Z},U,V,W) + G\ (X,Z)\ {}'K\ (Y,U,V,W) - G\ (Y,Z){}'K\ (X,U,V,W)\}$$
$$-{}'F\ (Y,Z)\ {}'K\ (\overline{X},\ U,V,W) + {}'F\ (X,Z)\ {}'K\ (Y,U,V,W) + 2{}'F\ (X,Y)\ {}'K\ (Z,\overline{U},V,W)$$
$$+ G(X,U)\ {}'K\ (Z,Y,V,W) - G\ (Y,U){}'K(Z,X,V,W) - {}'F\ (Y,U)\ {}'K\ (Z,\ \overline{X},\ V,W) +$$
$${}'F\ (X,U)\ {}'K\ (Z,\ \overline{Y},V,W) + 2{}'F\ (X,Y)\ {}'K\ (Z,U,\overline{V},W) + G\ (X,V)\ {}'K\ (Z,U,Y,W) -$$
$$G\ (Y,V)\ {}'K\ (Z,U,X,W) -\ {}'F\ (Y,V)\ {}'K(Z,U,\overline{X},W) +\ {}'F\ (X,V)\ {}'K\ (Z,U,\overline{Y},W) +$$
$$2{}'F\ (X,Y)\ {}'K\ (Z,\ U,\ V,\overline{W}) + G\ (X,W){}'K\ (Z,U,V,Y)4^2(K.Ric)\ (Z,U;\ X,Y)\ {}'F$$
$$(X,W)\ {}'K\ (Z,U,V,Y)].$$

Again, using (2.3) (a) in the above expression and taking account of the equation (1.1) and (1.2), we obtain

(2.7) $(K.{}'K)\ (Z,U,V,W;\ X,Y) = 0$

Thus, we have

Theorem (2.2) A Para-Kahler manifold with constant holomorphic sectional Curvature is semi-symmetic.

Again, using (2.3) (b) in (2.5) (b), we write

$$4^2 (K.\, Ric)(Z,U,X,Y) = -4\, Ric\,\{4K(X,Y,Z)]U\} - 4\, Ric\,(Z, 4K(X,Y,U)\}$$
$$= -4\, Ric\,(-k\,\{2{}'F(X,Y)\bar{Z} + G(X,Z)Y - G(Y,Z)X\} - F(Y,Z)\bar{X} + {}'F(X,Z)\bar{Y}\}, U) - 4\, Ric\,(Z, -k\,\{2{}'F(X,Y)\bar{U} + G(X,U)X - {}'F(Y,U)\bar{X} + {}'F(X,Y)Y\})$$

$$= 4k\,[2{}'F(X,Y)\, Ric\,(\bar{Z},U) + G(X,Z)\, Ric\,(Y,U) - G(Y,Z)\, Ric\,(X,U) - {}'F(Y,Z)\, Ric\,(\bar{X},U) + {}'F(X,Z)\, Ric\,(\bar{Y},U) + 2\,{}'F(X,Y)\, Ric\,(Z,\bar{U}) + G(X,U)\, Ric\,(Z,Y) - G(Y,U)\, Ric\,(Z,X) - {}'F(Y,U)\, Ric\,(Z,\bar{X}) + {}'F(X,U)\, Ric\,(Z,\bar{Y})]$$

Now, using (2.4) (c) in the above expression, we easily get

$$(2.8) \quad (K.Ric)(Z,U;X,Y) = 0\,.$$

Thus, we have

Theorem (2.3) A para-Kahler manifold with constant holomorphic sectional curvature is Rieci-semisymmetric.

Further, using (2.3) and (2.4) (c) in (2.6), we have

$$4^2 Q\,(Ric, {}'K)(Z,U,V,W;X,Y) = -4^2\,[Ric\,(Y,Z)]\,{}'K(X,U,V,W) - Ric\,(X,Z)\,{}'K(Y,U,V,W) + Ric\,(Y,U)\,{}'K(Z,X,V,W) - Ric\,(X,U)\,{}'K(Z,Y,V,W) + Ric\,(Y,V)\,{}'K(Z,U,X,W) - Ric\,(X,V)\,{}'K(Z,U,Y,W) + Ric\,(Y,W)\,{}'K(Z,U,V,X) - Ric\,(X,W)\,{}'K(Z,U,V,Y)]$$

$$= k^2 (n+2)\,[G(Y,Z)\,\{2{}'F(X,U)\,{}'F(V,W) + G(X,V)\,G(U,W) - G(U,V)\,G(X,W) - {}'F(U,V)\,{}'F(X,W) + {}'F(X,V)\,{}'F(U,W)\} - G(X,Z)\,\{2{}'F(Y,U)\,{}'F(V,W) + G(Y,V)\,G(U,W) - G(U,V)\,G(Y,W) - {}'F(U,V)\,{}'F(Y,W) + {}'F(Y,V)\,{}'F(U,W)\} + G(Y,U)\,\{2\,{}'F(Z,X)\,{}'F(V,W) + G(Z,V)\,G(X,W) - G(X,V)\,G(Z,W) - {}'F(X,V)\,{}'F(Z,W) + {}'F(Z,V)\,{}'F(X,W)\} - G(X,U)\,\{2{}'F(Z,Y)\,{}'F(V,W) + G(Z,V)\,G(Y,W) - G(Y,V)\,G(Z,W) - {}'F(Y,V)\,{}'F(Z,W) + {}'F(Z,V)\,{}'F(Y,W)\} + G(Y,V)\,\{2{}'F(Z,U)\,{}'F(X,W) + G(Z,X)\,G(U,W) - G(V,X)\,G(Z,W) - {}'F(U,X)\,{}'F(Z,W) + {}'F(Z,X)\,{}'F(U,W)\} - G(X,V)\,\{2{}'F(Z,U)\,{}'F(Y,W) + G(Z,Y)\,G(U,W) - G(U,Y)\,G(Z,W) - {}'F(U,Y)\,{}'F(Z,W) + {}'F(Z,Y)\,{}'F(U,W)\} - G(Y,W)\,\{2{}'F(Z,U)\,{}'F(V,X) + G(Z,V)\,G(U,X) - G(U,V)\,G(Z,X) - {}'F(U,V)\,{}'F(Z,X) + {}'F(Z,V)\,{}'F(U,X)\}] - G(X,W)\,\{2{}'F(Z,U)\,{}'F(V,Y) + G(Z,V)\,G(U,Y) - G(U,Y)\,G(Z,Y) - {}'F(U,V)\,{}'F(Z,Y) + {}'F(Z,V)\,{}'F(U,Y)\}]$$

or

$$4^2 Q\,(Ric\,{}'K)(Z,U,V,W;X,Y) = k^2 (n+2)\,[2{}'F(V,W)\,\{{}'F(X,U)\,G(Y,Z) - {}'F(Y,U)\,G(X,Z) + {}'F(Y,Z)\,G(X,U) - {}'F(X,Z)\,G(X,U)\}] + {}'F(U,V)\,\{{}'F(X,Z)\,G(Y,W) - {}'F(Y,Z)\,G(X,W) - {}'F(X,W)\,G(Y,Z) + {}'F(Y,W)\,G(X,Z)\} + {}'F(U,W)\,\{{}'F(X,V)\,F(Y,Z) - {}'F(Y,V)\,G(X,Z) + {}'F(Y,Z)\,G(X,V) - {}'F(X,Z)\,G(Y,V)\} + {}'F(Z,W)\,\{{}'F(Y,V)\,G(X,U) - {}'F(X,V)\,G(Y,U) + {}'F(X,U)\,G(Y,V) - {}'F(Y,U)\,G(X,V)\} + {}'F(Z,V)\,\{{}'F(X,W)\,G(Y,U) - {}'F(Y,W)\,G(X,U) + {}'F(Y,U)$$

$G(X,W) - {}'F(X,U)\, G,W)\} + 2'F(Z,U)\, \{'F(V,X)\, G(Y,W) - {}'F(V,Y)\, G(X,W) + {}'F(X,W)\, G(Y,V) - {}'F(Y,W)\, G(X,V)\}$

Interchanging (z,v) and (u, w) in the above equation, we get

$$Q(Ric, 'K)(Z,U,V,W ; X,Y) = Q(Ric, 'K)(V,W,Z,U; X,Y). \tag{2.9}$$

So, we have

Theorem (2.4). In a para-Kahler manifold with constant holomosphic sectional curvature, we have (2.9).

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UNSTEADY FREE CONVECTION MHD DUSTY FLOW BETWEEN TWO HEATED VERTICAL PLATES

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ABSTRACT

Free convective dusty flow of a viscous, incompressible, electrically conducting fluid between two vertical heated porous plates in the presence of magnetic field acting in the horizontal direction and perpendicular to the flow. Using Laplace transform technique exact solution for the liquid velocity, particle velocity and temperature distribution have been obtained. The effect of magnitude of Hartmann number M, Prandtl number P_r on liquid velocity and the tempersture has been studied with the help of graphs.

1. Introduction. The problem of unsteady free convection in incompressible fluids in porous medium is of importance in geophysics, ground water hydrology, oil recovery, soil sciences and astrophysics. It is an established fact that comets consist of a dusty 'Snowball' of a mixture of frozen gases which in the process of their journey changes from solid to gas and vica-versa. The physical properties of comets, mebeorites and interplantary dust strongly suggest the importance of porosity in astrophysical context. The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of earth's core where the earth's mantle, which consists of conducting fluid, behaves like a porous medium which can become convectively unstable as a result of different diffusion. Following Saffman [4] many investigators namely Raptis and Perdikis [2], Sharma et al [5], Ramamurthy [3], Gowrla and Katoch [1], Singh [6] have studied laminar flow of a dusty fluid taking various flow models.

2. Formulation of the Problem. Let us consider the fully developed unsteady flow of an incompressible slightly conducting dusty viscous fluid between two heated porous infinite parallel plates (distance $2h$ apart) under the influence of uniform magnetic field applied perpendicular to the flow region. We assume x-axis along the flow in the mid-way of the palte and y–axis perpendicular to it. Let u be the liquid velocity, υ the particle velocity in the direciton of x–axis and initially the plates and the liquid are at the same temperature T_0 and there is no flow. Besides our analysis is based on the facts that (i) the flow in the direction of x-axis and is driven by a constant pressure $\partial p/\partial x$ with negligible body forces, (ii) the dust particles are non-conducting, solid, spherical, equal in size, uniformly distributed and their number density N_0 is constant throughout the molecule and (iii) there is no external applied electric field and the induced field is neglected.

Under these assumptions continuity and momentum equations governing the velocity distribution are :

$$\frac{\partial u}{\partial t} = \beta g\,(T - T_0) + \upsilon\,\frac{\partial^2 u}{\partial y^2} + \frac{KN_0}{\rho}\,(\upsilon - u) - \frac{\upsilon}{k}\,u - \frac{\sigma}{\rho}\,\beta_0^2 u \qquad \text{... (2.1)}$$

$$m\,\frac{\partial \upsilon}{\partial t} = K\,(u - \upsilon) \qquad \text{... (2.2)}$$

$$\frac{\partial T}{\partial t} = \frac{K_1}{\rho c}\,\frac{\partial^2 T}{\partial y^2} \qquad \text{... (2.3)}$$

where ρ is the density of the liquid, υ is the kinematic viscosity, σ is the electrical conductivity, β is th coefficient of thermal expansion, K_1 is the thermal conductivity, c_p is specific heat of the liquid and B_0 is the magnetic field component, g is the acceleration due to gravity, K is the porosity of the porous medium, t is the time. At $t = 0$, the temperature T of each plate changes according to the law.

$$T = T_0 + (T_w - T_0)\,(1 - e^{-at}) \qquad \text{... (2.4)}$$

where a is the decay factor.

The initial and boundary conditions for the present problem are :

$$\begin{aligned} t = 0 &: u = 0,\ T = T_0,\ \forall\, y \in [-d, d] \\ t > 0 &: u = 0,\ T = T_0 + (T_w - T_0)\,(1 - e^{-at}) \text{ for } y = -d \\ &\quad u = 0,\ T = T_0 + (T_w - T_0)\,(1 - e^{-at}) \text{ for } y = d\ . \end{aligned} \qquad \text{... (2.5)}$$

Introducing the non-dimensional quantities :

$$u^* = \frac{\upsilon u}{\beta g\, d^2\,(T_w - T_0)},\ y^* = \frac{y}{d},\ T^* = \frac{T - T_0}{T_w - T_0},\ t^* = \frac{\upsilon t}{d^2},$$

$a^* = \frac{d^2 a}{\upsilon}$, $K^* = \frac{k}{d^2}$, $p_r = \frac{\mu c}{K_1}$ (Prandtl number), and

$M = B_0 d\sqrt{\frac{\sigma}{\mu}}$ (Hartmann number).

The equations (2.1), (2.2), (2.3) and (2.5) (ignoring stars) reduce to

$$\frac{\partial u}{\partial t} = T + \frac{\partial^2 u}{\partial y^2} + \frac{l}{w}(\upsilon - u) - K^{-1}u - M^2 u \qquad \text{... (2.6)}$$

$$w\frac{\partial \upsilon}{\partial t} = (u - \upsilon) \qquad \text{... (2.7)}$$

$$\frac{\partial^2 T}{\partial y^2} - p_r\frac{\partial T}{\partial t} = 0 \qquad \text{... (2.8)}$$

$$\begin{aligned} t = 0 &: u = 0,\ T = T_0\ \forall\ y \in [-1, 1] \\ t > 0 &: u = 0,\ T = 1 - e^{-at}\ \text{for } y = -1 \\ &\quad u = 0,\ T = 1 - e^{-at}\ \text{for } y = 1 \end{aligned} \qquad \text{... (2.9)}$$

3. Solution of the Problem. Taking Laplace transform of equation (2.6) to (2.9) and solving the differential equations under the corresponding transformed boundary conditions. Using Tranter (1951) and Calculus of Residues, we get

$$T = 1 - e^{-at}\frac{\cos\sqrt{p_r a}\,y}{\cos\sqrt{p_r a}} + \sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)\,a\,\pi\,e^{-p_1 t}}{p_1(p_1 - a)\,p_r}\cos\left(\frac{2m+1}{2}\right)\pi y \quad \text{... (3.1)}$$

$$\begin{aligned} u = {} & \frac{1}{b^2}\left[1 - \frac{\cosh by}{\cosh b}\right] + \frac{e^{-at}}{(b_1^2 - ap_r)}\left[\frac{\cosh b_1 y}{\cosh b_1} - \frac{\cosh\sqrt{p_r a}\,y}{\cosh\sqrt{p_r a}}\right] + \frac{a\,e^{p_2 t}}{p_2(p_2 - a)\,b_3} \\ & \left[\frac{\cosh\sqrt{p_r p_2}\,y}{\cosh\sqrt{p_r p_2}} - \frac{\cosh b_2 y}{\cosh b_2}\right] + \frac{a\,e^{p_3 t}}{p_3(p_3 - a)\,b_5}\left[\frac{\cosh\sqrt{p_r p_3}\,y}{\cosh\sqrt{p_r p_3}}\ \frac{\cosh b_4 y}{\cosh b_4}\right] \\ & - \sum_{m=0}^{\infty}\frac{a(-1)^m(2m+1)\,\pi\,(1 + wp_4)\,e^{p_4 t}\cos[(2m+1)/2]\,\pi y}{p_4(p_4 + a)(b_6^2 - p_r p_4)\,[(1 + wp_4)^2 + l]} \\ & - \sum_{m=0}^{\infty}\frac{a(-1)^m(2m+1)\,\pi\,(1 + wp_5)\,e^{p_5 t}\cos[(2m+1)/2]\,\pi y}{p_5(p_5 + a)(b_7^2 - p_r p_5)\,[(1 + wp_5)^2 + l]} \\ & - \sum_{m=0}^{\infty}\frac{a(-1)^m(2m+1)\,\pi\,e^{-p_1 t}\cos[(2m+1)/2]\,\pi y}{p_1(p_1 - a)(b_8^2 - p_1 p_r)\,p_r} \end{aligned} \qquad \text{... (3.2)}$$

where $p_1 = \frac{(2m+1)^2\,\pi^2}{4\,P_r}$, and

$$v = \frac{1}{b^2}\left[1 - \frac{\cosh by}{\cosh b}\right] + \frac{e^{-at}}{(1-wa)(b_1^2 - aP_r)}\left[\frac{\cosh b_1 y}{\cosh b_1} - \frac{\cosh\sqrt{p_r a}\,y}{\cosh\sqrt{p_r a}}.\right]$$

$$+ \frac{a\,e^{P_2 t}}{P_2(P_2+a)(1+wp_2)\,b_3}\left[\frac{\cosh\sqrt{p_r p_2}\,y}{\cosh\sqrt{p_r p_2}} - \frac{\cosh b_2 y}{\cosh b_2}\right] + \frac{a\,e^{p_3 t}}{p_3(p_3-a)(1-wp_3)b_5}$$

$$\left[\frac{\cosh\sqrt{p_r p_3}\,y}{\cosh\sqrt{p_r p_3}} - \frac{\cosh b_4 y}{\cosh b_4}\right] - \sum_{m=0}^{\infty} \frac{a(-1)^m(2m+1)\pi(1+wp_4)e^{p_4 t}\cos(m+\tfrac{1}{2})\pi y}{p_4(p_4+a)(b_6^2 - p_r p_4)\,[(1+wp_4)^2+l]}$$

$$- \sum_{m=0}^{\infty} \frac{a(-1)^m(2m+1)\,\pi(1+wp_5)\,e^{p_5 t}\cos(m+1/2)\pi y}{p_5(p_5+a)(b_7^2 - p_r p_5)\,[(1+wp_5)^2+l]}$$

$$+ \sum_{m=0}^{\infty} \frac{a(-1)^m(2m+1)\,\pi\,e^{-p_1 t}\cos(m+1/2)\pi y}{p_1(p_1-a)(b_2^2 - p_1 p_r)\,p_r} \qquad \text{... (3.3)}$$

where,

$b = (M^2 + K^{-1})^{1/2}$, $b_1 = (M^2 + K^{-1} - a - \dfrac{la}{1-wa})^{1/2}$,

$2\,w\,(1-p_r),\ p_2;\ 2\,w\,(1-p_r)\,p_3 = -L_1 \pm \sqrt{L_1^2 - 4w\,(1-p_r)\,(M^2 + K^{-1})}$,

$L_1 = 1 \ - p_r \ + w^2(M^2 + K^{-1})$, p_2 , p_3 are the roots of the equation, $w\,(1-P_r)\,s^2 + L_1 s + (M^2 + K^{-1}) \ = 0$,

$b_2 = (M^2 + K^{-1} + p_2 + \dfrac{lp_2}{1+wp_2})^{1/2}$, $b_3 = [1 - p_r + \dfrac{l}{(1+wp_2)^2}]$,

$b_4 = (M^2 + K^{-1} - p_3 - \dfrac{lp_3}{1+wp_3})^{1/2}$, $b_5 = [1 - p_r + \dfrac{l}{(1+wp_3)^2}]$

$2wp_4$, $2wp_5 = -a_1 \pm \sqrt{a_1^2 - 4wa_2}$

where, $a_1 = \left[1 + l + \left\{\dfrac{(2m+1)^2\pi^2}{4} + M^2 + K^{-1}\right\} w\right]$

and $\quad a_2 \ = \ \dfrac{(2m+1)^2\pi^2}{4} + M^2 + K^{-1}$

p_4, p_5 are the roots of the equation, $ws^2 + a_1 s + a_2 = 0$

$b_6 = (M^2 + K^{-1} + p_4 + \dfrac{lp_4}{1+wp_4})^{1/2}$, $b_7 = (M^2 + K^{-1} + p_5 + \dfrac{lp_5}{1+wp_5})^{1/2}$,

and $b_8 = (M^2 + K^{-1} - p_1 - \dfrac{lp_1}{1+wp_1})^{1/2}$,

Figure 1. Velocity profiles u against y for different values of magnetic feild
$[a = 1.0,\ t = 1.0,\ I = 0.2,\ w = 0.8,\ k = 0.4$ and $P_r = 1.0]$

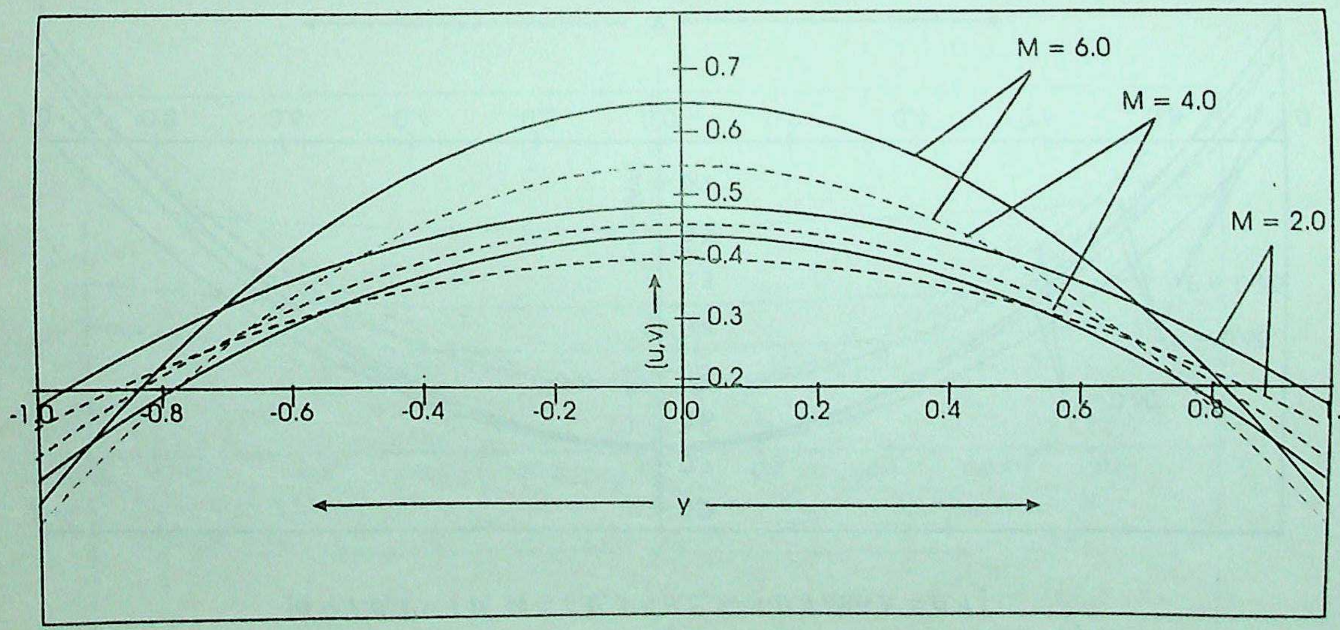


Figure 2. Velocity profiles u and v against y for different values or p_r
[$a = 1.0$, $t = 1.0$, $M = 6.0$, $I = 0.2$, $w = 0.8$ and $k = 0.4$]

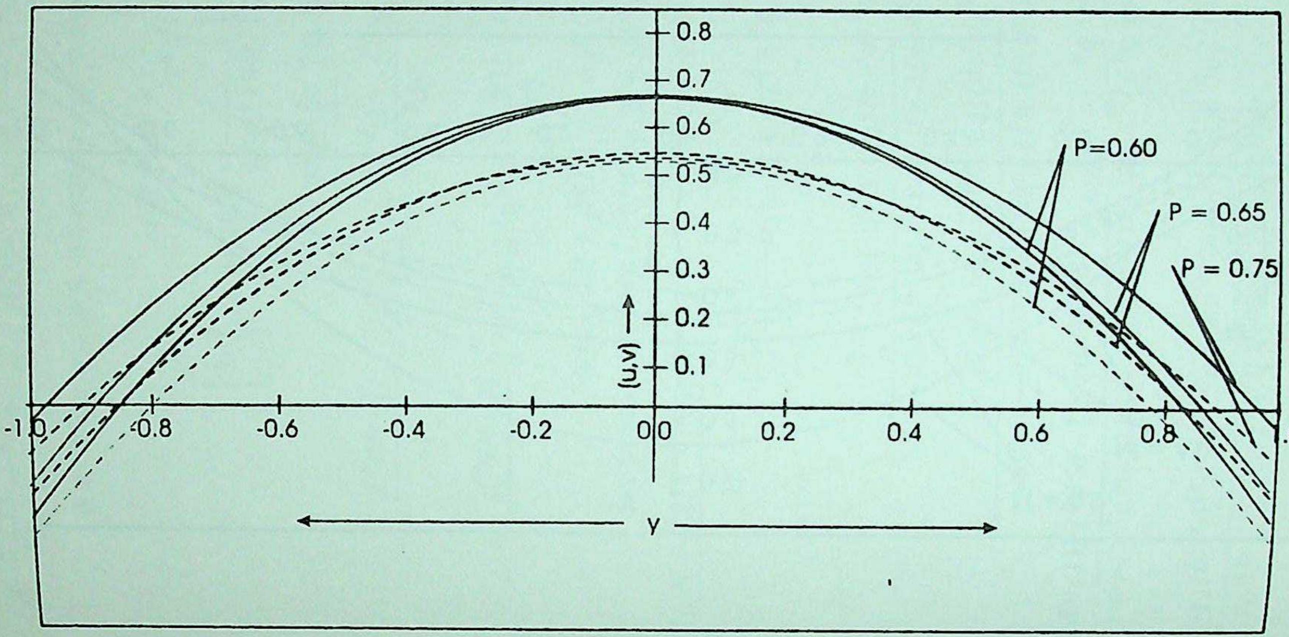


Figure 3. Temperature profiles T against y at u different time
$[a = 1.0, p_r = 0.75]$

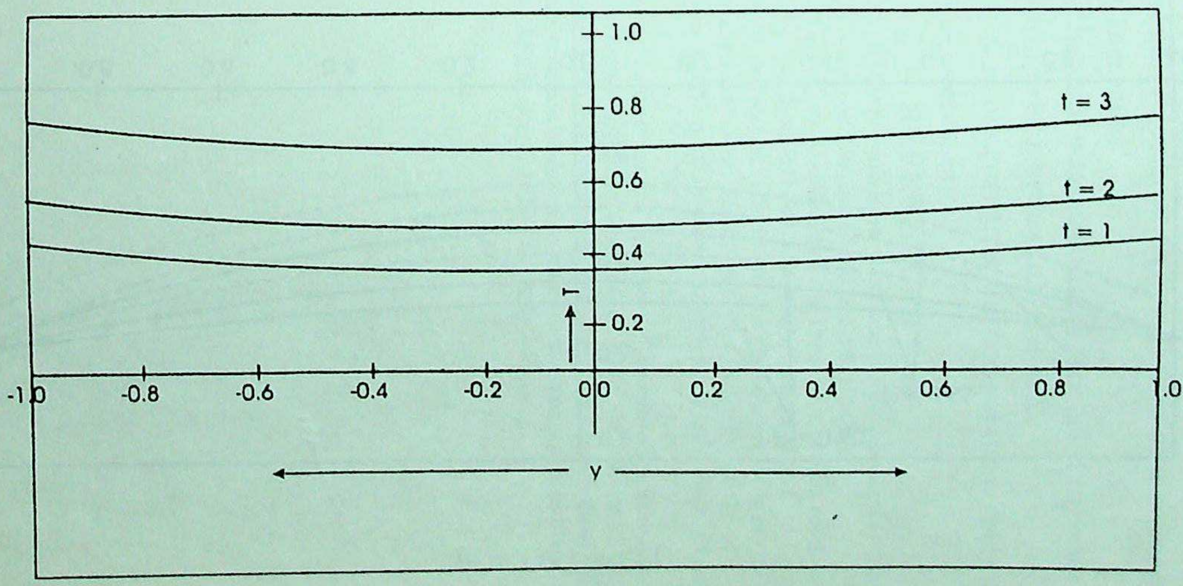
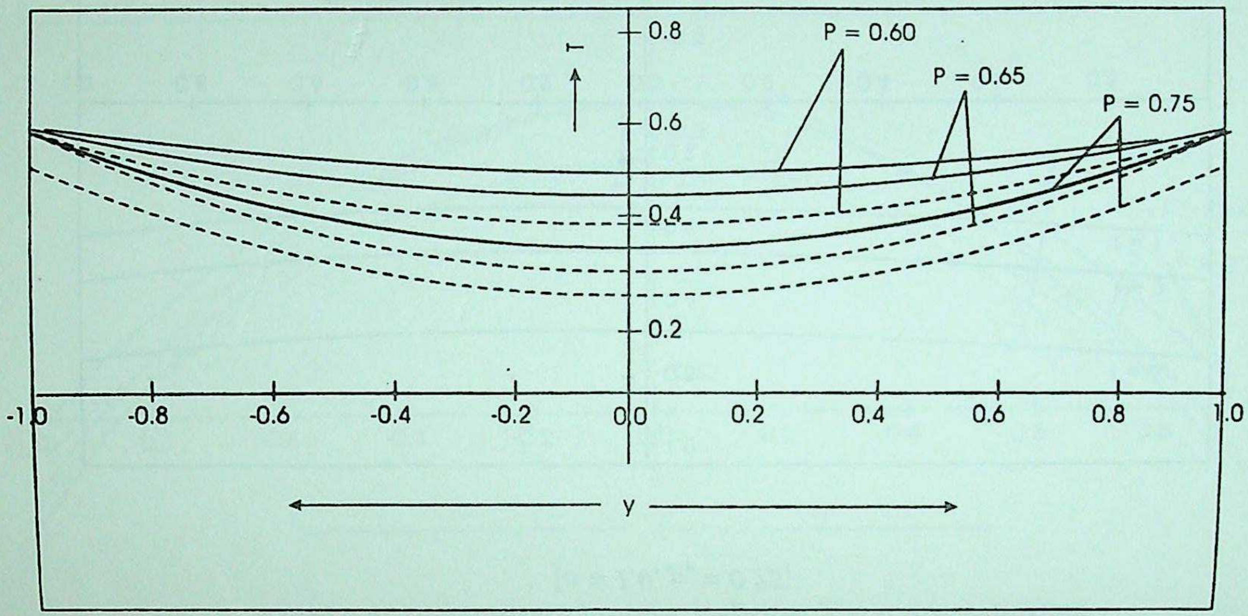


Figure 4. Temperature profiles T against y for various values of p_r

$[a = 1.0, t = 1.0]$



Discussion.

Figure -1 shows that on increasing the magnitude of Hartmann number the velocity of liquid and dust particles take the positive and negative values showing thereby backward and forward fluid flows.

Figure-2 shows that the velocity at any point increases as magnitude of Pradtl number P_r increases.

Figure-3 describes that the temperature at any point increases with the increase of magnitude of Hartmann number.

From the figure-4, it is obvious that the temperature decreases as the Prandtl number P_r increases.

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ON A FIXED POINT THEOREM

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ABSTRACT

The aim of the present paper is to obtain a generalization of a recent result due to Pant et al. Our theorem extends the scope of the study of common fixed point theorems from the class of compatible maps to the wider class of pointwise R-weakly commuting maps.

1. Introduction. Two selfmaps A and S of a metric space (X, d) are called compatible if $lim_n\ d\ (ASx_n, SAx_n) = 0$ whenever $\{X_n\}$ is a sequence such that $lim_n\ AX_n = lim_n\ Sx = t$ for some t in X. In [6], the first author obtained another generalization of weak commutativity by introducing the concept of R-weak commutativity. Two self maps A and S of a metric space (X,d) are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. It is obvious that A and S can fail to be pointwise R-weakly commuting only if there exists some x in X such that $Ax = Sx$ but $ASx \neq SAx$, that is, only if they possess a coincidence point at which they do not commute. Thus pointwise R-weakly commuting maps may equivalently be defined as coincidence preserving maps and we shall use these terms interchangeably. It follows easily from this, as is formally eastablished in [11], that pointwise R-weak commutativity is a necessary, hence minimal condition for the existence of common fixed points of contractive type maps. Moreover, compatible maps are necessarily coincidence preserving since compatible maps commute at coincidence points.

Recently, employing the notion of pointwise *R-weak* commutativity Pant et al [12], proved the following :

Theorem (Pant et al. [12]). *Let* $\{A_i\}\ i = 1,2,3,\ \dots$, S and T be selfmappings of a complete metric space (X,d) such that

(i) given u in X there exists y, υ in X such that $A_1u = Ty$ and $A_2y = S\upsilon$,

(ii) $d(A_1x, A_i y) < M_{1i}(x,y)$ for all x,y in X whenever $M_{1i}(x,y) > 0$ and $A_1x = A_iy$ whennever $A_1x = Sx$ and $A_i y = Ty$,

(iii) $d(A_1x, A_2y) \leq h M_{12}(x,y)$, $0 \leq h < 1$ whenever $y \in A_1X \cup A_2X$ or $Ty \in \overline{A_1X}$ and $x \in X$.

Let A_1 and S be compatible and A_2 and T be pointwise R-weakly Commuting. If A_1 or S be continuous then all the A_1,S and T have unique common fixed point.

In the present paper we obtain a generalization of the above theorem by replacing the Banach type contractive condition by a more general contravtive condition that employs a contractive gauge fucntion ϕ. The theorem employs the notion of pointwise R-weak commutativity as well as compatibility.

2. Results.

If $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a metric space (X,d), in the sequel for each $i > 1$ we shall denote

$M_{1i}(x,y) = \{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(A_1x, Tx) + d(A_iy, Sx)]/2\}$.

Theorem . Let$\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that

(i) given u in X there exists y, υ in X such that $A_1u = Ty$, $A_2y = S\upsilon$.

(ii) $d(A_1x, A_i y) < M_{1i}(x,y)$ whenever $M_{1i}(x,y) > 0$,

(iii) $d(A_1x, A_2y) < \phi M_{12}(x,y)$,

where $\phi : R_+ \to R_+$ is an upper semicontinuous (to be abbreviated as *u.s. continuous*) function such that $\phi(t) < t$ fro each $t > 0$. Let A_1 and S be compatible and A_2 and T be pointwise R-weakly Commuting. If A_1 or S be continuous then all the A_i, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of (i). If $A_1x_{2n} = A_2x_{2n+1}$ or $A_2x_{2n+1} = A_1x_{2n+2}$ for some value of n, it becomes easier to establish the existence of the fixed point. So let us assume that $A_1x_{2n} \neq A_2x_{2n+1}$ and $A_2x_{2n+1} \neq A_1x_{2n+2}$ for every value of n. Then, by virtue of (ii), we obtain

(2.1) $d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n})$,

and

(2.2) $d(y_{2n-1}, y_{2n}) \leq \phi(d(y_{2n-2}, y_{2n-1})) < d(y_{2n-2}, y_{2n-1})$.

We thus see that $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \geq 0$. Suppose $r > 0$. Then relation (1) on making $n \to \infty$ and in view of *u.s. continuity* of ϕ yields $r \leq \phi(r) < r$, a contradiction.

Hence $r = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. We show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\in > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon$. If not, then

$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < \varepsilon + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < 2\varepsilon]$ a contradiction. If m_i be the smallest integer such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon$, then

$$\varepsilon \leq d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$
$$< \varepsilon + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}).$$

That is, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon$ and

(2.3) $\lim_{n\to\infty} d(y_{n_i}, y_{m_i}) = \varepsilon$.

Without loss of generality we can assume that n_i is odd and m_i even. Now, by virtue of (2.1), we have

$$d(y_{n_i+1}, y_{m_i+1}) \leq \phi(\max\{d(y_{n_i}, y_{m_i}), [d(y_{n_i+1}, y_{m_i}) + d(y_{m_i+1}, y_{n_i})]/2\})$$

Now on letting $n_i \to \infty$ and in view of (2.3) and u.s. continuity of ϕ, the above relation yields $\varepsilon \leq \phi(\varepsilon) < \varepsilon$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $y_n \to z$. Also

$$y_{2n} = A_1 x_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = A_2 x_{2n+1} = Sx_{2n+2} \to z.$$

Suppose S is continuous. Then $SA_1 x_{2n} \to Sz$ and $SSX_{2n} \to Sz$. Compatibility of A_1 and S implies $\lim_{n\to\infty} d(A_1 x_{2n}, SA_1 x_{2n}) = 0$, that is, $A_1 Sx_{2n} \to Sz$. We claim that $z = Sz$. If $z \neq Sz$, the inequality

$$d(A_1 Sx_{2n}, A_2 x_{2n+1}) \leq \phi(\max\{d(SSx_{2n}, Tx_{2n+1}), d(A_1 Sx_{2n}, SSx_{2n}),$$
$$d(A_2 x_{2n+1}, Tx_{2n+1}), [d(A_1 x_{2n}, Tx_{2nz+1}) + d(A_2 x_{2n+1}, SSx_{2n})]/2\}),$$

on letting $n \to \infty$ leads to $d(z, Sz) \leq \phi(d(z, Sz)) < d(z, Sz)$, a contradiction. Therefore $z = Sz$. Similarly, If $z \neq A_1 z$ the inequality

$$d(A_1z, A_2x_{2n+1}) \le \phi\,(\max\{d(Sz, Tx_{2n+1}), d(A_1z, Sz), d(A_2x_{2n+1}, Tx_{2n+1}), [d(A_1z, Tx_{2n+1}) + d(A_2x_{2n+1}, Sz)]/2\}),$$

on making $n \to \infty$ yields, $d(z,A_1z) \le \phi\,(d(z,A_1z)) < d\,(z, A_1z)$, a contradiction. Thus $z = A_1z = Sz$. By virtue of (i), there exists a point w in X such that $z = A_1z = Tw$. We now show that $z = A_2w = Tw$. If $z \ne A_2w$ we get

$$d(A_1z, A_2w) \le (\max\{d(Sz, Tw), d\,(A_1z, Sz), d\,(A_2w, Tw), [d\,(A_1z,Tw) + d\,(A_2w, Sz)]/2\}),$$

i.e. $d\,(z, A_2w) \le \phi\ (d\,(z, A_2w)) < d\,(z, A_2w)$.
a contradiction. Hence $z = A_1z = Sz = Tw = A_2w$.

Pointwise R-weak commutativity of A_2 and T implies that $d\,(A_2Tw, TA_2w) \le R\,d\,(A_2w, Tw) = 0$ for some $R > 0$, we get $A_2Tw = TA_2w$, *i.e.* $A_2z = Tz$. If $z \ne A_2z$, the inequality

$$d(A_1z, A_2z) \le \phi\,(\max\{d(Sz, Tz), d\,(A_1z, Sz), d\,(A_2z, Tz), [d(A_1z, Tz) + d\,(A_2z,Sz)]/2\}).$$

leads to $d(z, A_2z) \le \phi\,(d(z, A_2z)) < d(z, A_2z)$ a contradiction. Hence z is a common fixed point of A_1, A_2, S and T. Now if $z \ne A_iz$ for some $i > 2$, (ii) yields

$$d(A_1z, A_iz) \le \max\{d(Sz, Tz), d\,(A_1z, Sz), [d\,(A_iz, Tz), [d(A_1z, Tz) + d(A_1z,Sz)]/2\}).$$

That is, $(d(z, A_iz)) < d(z, A_iz)$ a contradiction. We have thus shown that z is a common fixed point of all the A_i, S and T if S is assumed continuous. The proof is similar when T is assumed continuous.

Next, suppose that A_1 is continuous. Then $A_1A_1x_{2n} \to A_1z$ and $A_1Sx_{2n} \to A_1z$. Compatibility of A_1 and S implies $\lim_{n\to\infty} d(A_1Sx_{2n}, SA_1x_{2n}) = 0$, that is, $SA_1x_{2n} \to A_1z$. If $z \ne A_1z$ the inequality

$$d\,(A_1A_1x_{2n}, A_2x_{2n+1}) \le \phi\,(\max\{d(SA_1x_{2n}, Tx_{2n+1}), d\,(A_1A_1x_{2n}, SA_1x_{2n}), d\,(A_2x_{2n+1}, Tx_{2n+1}), [d\,(A_1A_1x_{2n+1}, Tx_{2n+1}) + d\,(A_2x_{2n+1}, SA_1x_{2n})]/2\}),$$

in view of u.s. continuity of ϕ will imply $d\,(z, A_1z) \le \phi\,(d(z, A_1z)) < d(z,A_1z)$. Hence $z = A_1z$. By virtue of (i), there exist points w, u in X such that $z = Tw = A_1z$ and

(2.4) $\quad A_2w = Su$.

We now show that $z = Tw = A_2w$. If $z \ne A_2w$, the inequality

$$d\,(A_1x_{2n}, A_2w) \le \phi\,(\max\{d(Sx_{2n}, Tw), d\,(A_1x_{2n}, Sx_{2n}), d\,(A_2w, Tw), [d\,(A_1x_{2n}, Tw) + d\,(A_2w, Sx_{2n})]/2\}).$$

on letting $n \to \infty$ yields $d(z, A_2w) \le \phi\,(d(z, A_2w)) < d\,(z, A_2w)$, a contradiction.

Hence $z = Tw = A_2 w$.

Pointwise R-weak commutativity of A_2 and T implies that $d(A_2Tw, TA_2w) \le Rd(A_2w, Tw)=0$ for some $R > 0$, that is, $A_2Tw = TA_2w$, or $A_2z = Tz$. Now, if $z \neq A_2 z$, the inequality

$$d(A_1 x_{2n}, A_2 z) \le \phi\,(max\,\{d(Sx_{2n}, Tz), d(A_1 x_{2n}, Sx_{2n}), d(A_2 z, Tz), [d(A_1 x_{2n}, Tz) + d(A_2 z, Sx_{2n})]/2\}).$$

will yield $d(z, A_2 z) \le \phi\,(d(z, A_2 z)) < d(z, A_2 z)$. Thus $z = A_1 z = A_2 z = Tz$. Since by (2.4) $A_2 w = Su$ for some $u \in X$, we show that $z = A_1 u = Su$. If not, the inequality

$$d(A_1 u, A_2 z) \le \phi\,(max\,\{d(Su, Tz), d(A_1 u, Su), d(A_2 z, Tz), [d(A_1 u, Tz) + d(A_2 z, Su)]/2\}),$$

will yield $d(A_1 u, z) \le \phi\,(d(A_1 u, z) < d(A_1 u, z)$, a contradiction. Hence $A_1 u = Su = z$. Since A_1 and S are compatible, we get $A_1 Su = SA_1 u$, that is $A_1 z = Sz$. Hence z is a common fixed point of A_1, A_2, S and T. Finally, if $z \neq A_i z$ for some $i > 2$, we get

$$d(A_1 z, A_i z) < max\,\{d(Sz, Tz), d(A_1 z, Sz), d(A_i z, Tz), [d(A_1 z, Tz) + d(A_i z, Sz)]/2\}),$$

that is, $d(z, A_i z) < d(z, A_i z)$, a contradiction. Hence z is a common fixed point of all the A_i, S and T when A_1 is continuous. The proof is similar when A_2 is assumed continuous.

Uniqueness of the common fixed point follows easily. This complets the proof of the theorem.

As a direct consequnce of the above theorem we get the following

Corollary. Let A_1, A_2, S and T be selfmapping of a complete metric space (X, d) such that

(i) $A_1X \subset TX, \qquad A_2X \subset SX$,

(ii) $d(A_1 x, A_2 y) \le \phi\,(max\,\{d(Sx, Ty), d(A_1x, Sx), d(A_2 y, Ty)\})$

$x, y \in X$.

where $\phi : R_+ \to R_+$ is an upper semicontinuous function such that $\phi(t) < t$ for each $T > 0$. Let $A_1 S$ be compatible and A_2, T be pointwise R-weakly communting. If A_1 or S be continuous than A_1, A_2, S and T have a unique common fixed point.

We now give an example of mappings satisfying the conditions of our theorem and having a unique common fixed point.

Example .

Let $X = [2,\infty)$ with usual metric d. Define mappings $A_i, S, T : X \to X$,

$i=1,2,3...$, by

$Sx = x + 5$ when $x > 2$, $Sx = 2$ when $x = 2$

$Tx = 4x$ when $x \geq 3$, $Tx = 2$ when $x < 3$,

$A_1x = 2$ for all x,

$A_2x = 2$ when $x = 2$, $A_2x = 2x$ when $x > 2$,

$A_ix = 2(3 + 1/i)$ when $x > 3 + 1/i$

$A_ix = 2$ when $x \leq 3 + 1/i$.

Then $\{A_i\}$, S and T satisfy all the conditions of our theorem and have a unique common fixed point $x = 2$.

In view of this example, a few observation will be in order:

1. **Theorem 5.1.** of Jachymski [4] assumes, for each $i > 2$, $A_iX \subset SX$ and A_i to be compatible with T. These conditions are neither required in our theorem nor are satisfied in the example. It is obvious that A_iX is not contained in SX when $i > 2$, let us consider a decreasing sequence $\{x_n\}$ in $X = [2,\infty)$ such that $x_n \to 3 + (1/i)$. Then $A_ix_n \to 2(3 + (1/i))$ and $Tx_n \to 4(3 + (1/i))$ But $A_iTx_n \to 2(3 + (1/i))$ while $TA_ix_n \to 8(3 + (1/i))$. Hence $\lim_n d(A_iTx_n, TA_ix_n) \neq 0$, that is, A_i and T are noncompatible.

2. The main theorem of Rhoades et al. [15] requires $A_iX \subset SX \cap TX$ and A_i to be compatible with both S and T for every value of $i > 2$. These conditions are not required in our theorem and are not satisfied in the above example. Moreover, their result requires each A_i, A_j to satisfy an (ε,δ) type contractive condition (the necessary correction in their (ε,δ) condition appeared in Jungck et al [7]). However, in the above example, A_1, A_1, $i > 2$, fail to satisfy (ε,δ) condition at $\varepsilon = 4 + (2/i)$.

3. Likewise, the present theorem is more general than Theorem 2 of pant, Joshi and Pande [11] in one respect. In theorem 2 of Pant et al. if we take $\{P_i\} = \{A_1, A_1, A_1, ...\}$ and $\{Q_i\} = \{A_2, A_3, A_4, ...\}$ then A_1, A_i are required to satisfy an $(\in, \delta)$ contractive condition for each $i > 2$. However, as discussed above, this condition is not satisfied in the above example at $\varepsilon = 4 + (2/i)$.

Remarks. In the above theroem if we let $\phi(t) = ht$, $0 \leq h < 1$, then we obtain the main theorem of Pant et al [12] as special case of the present theorem.

It is clear from the above discussion that our theorem applies to much wider class of mappings than covered by the above mentioned results. Moreover, since other results of this type can be obtained as special cases of the results in [4], [11], [16] mentioned above, they can also be obtained as special cases of our theorem. For a nice comarative study of various results one may refer to Rhoades et al. [15] or Jachymski [4]. Among the results which can be obtained as special cases of the present Theorem, we mention here due to Boyd and Wong [1], Das and Naik [2], Fisher [3], Joshi an Pant [5], Meir and Keeler [8], Pant [9,10], Pant et al [12], Park and Bae [13], Park and Rhoades [14], Rao and Rao [15] and Tivari and Singh [18].

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SOLUTION OF CERTAIN DUAL INTEGRAL EQUATIONS INVOLVING *I*-FUNCTION

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ABSTRACT

This paper presents a solution of the following dual integral equations:

$$\int_0^\infty I_1\,[ux]\; a\,(u)\, du = E\,(x)\;;\; 0 < x < 1,$$

$$\int_0^\infty I_2\,[ux]\; a\,(u)\, du = G\,(x)\;;\; x > 1.$$

Here, the functions I_1 and I_2 are special forms of *I*-functions introduced by Saxena [9] as generalization of hypergeometric function subsequent to Fox's *H*-functions. These equations, under prescribed mathematical conditions on treating with the fractional integral operator and using Mellin transform reduce to a single integral equation involving a summation kernel. The final solution is obtained in terms of finite sums of integrals involving *H*–functions.

1. Introduction and Definition. The dual integral equations which arise in mixed boundary value problems of mathematical physics can be put as :

$$\int_0^\infty u^\alpha D_k\,(ux)\, a(u)\, du = g\,(x)\;;\; 0 < x < 1 \tag{1}$$

$$\int_0^\infty u^\beta D_\upsilon\,(ux)\, a(u)\, du = h\,(x)\;;\; x > 1 \tag{2}$$

where $g(x)$ and $h(x)$ are given functions and $a(u)$ is to be determined. D_k and D_υ are kernels expressed generally in terms of hypergeometric functions and their special cases. These equations occur in several problems of physis and therefore have been widely studied. Various methods of solving the above and other types of dual integral equations have been given by number of authors such as Busbridge [1], Peter [3], Nobel [4] and Saxena [6,7].

An elegent generalization of the dual intergral equations (1) and (2) is given by Fox [5] involving H–funciton as kernels

Fox [5] has used the following generalized Erdèlyi Kober operators :

$$T\,[\gamma,\varepsilon;t]\; f(x) = \frac{t}{\Gamma\gamma}\, x^{-\gamma t-\varepsilon+t-1} \int_0^x (x^t-v^t)^{\gamma-1}\, v^{\varepsilon} f(v)\, dv \tag{3}$$

$$R\,[\gamma,\varepsilon;t]\; f(x) = \frac{t}{\Gamma\gamma}\, x^{\varepsilon} \int_x^{\infty} (v^t-x^t)^{\gamma-1}\, v^{-\varepsilon-t\gamma+t-1}\; f(v)\, dv \tag{4}$$

The operator T exists if $f(x) \in L_p\,(0,\infty)\; p > 1,\; \gamma > 0,\; \varepsilon > (1-p)/p$ and if $f(x)$ can be differentiated sufficient number of times, then the operator T exists for both negative and positive values of γ. Also, for R, $f(x) \in L_p\,(0,\infty)$ $p \geq 1$, and if $f(x)$ can be differentiated sufficient number of times. then R exists if $t > \varepsilon > -1/p$, while γ can take any negative or positive value.

Saxena [9] further generalized the dual integral equations by considering a summation kernel involving H–functions.

The Solution of these equations led to the following definition of I–function [9]:

$$I\,[Z] = I^{m,n}_{p,\,q,\,r} \left[Z \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,\,n},\; (a_{ji},\,\alpha_{ji})_{n+1,\,p_i} \\ (b_j,\beta_j)_{1,\,m},\; (b_{ji},\,\beta_{ji})_{m+1,q_i} \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\, t(s)\; Z^s ds \tag{5}$$

where

$$t\,(s) = \frac{\prod\limits_{j=1}^{m} \Gamma(b_j-\beta_j s) \prod\limits_{j=1}^{n} \Gamma(1-a_j+\alpha_j s)}{\sum\limits_{j=1}^{r} \left\{ \prod\limits_{j=m+1}^{q_i} \Gamma(1-b_{ji}+\beta_{ji} s) \prod\limits_{j=n+1}^{p_i} \Gamma(a_{ji}-\alpha_{ji} s) \right\}} \tag{6}$$

and $p_i\;(i = 1,2,\ldots,r)$, $q_i\;(i = 1,2,\ldots,r)$, m, n are integers satisfying $0 \leq n \leq p_i$, $1 \leq m \leq q_i\;(i = 1,2,\ldots r)$, r is finite $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive a_j, b_j, a_{ji}, b_{ji} and are complex such that $\alpha_j\,(b_h + v) \neq \beta_h\,(a_j - 1-k)$ for v, $k = 0, 1,2, \ldots$, $h = 1,2,..m; j = 1,2,...r$. L is a suitable contour of Barnes type which runs from $\sigma - i\infty$ to $\sigma + i\infty$ (σ is real), in the complex S-plane such that the points

$s = (a_j - 1 - k)/\alpha_j\,,\ j = 1,2,\ ...,\ n\ ;\ k = 0,1,2,\ ...$

$s = (b_j + \upsilon)/\beta_j\,,\ j = 1,2,\ ...,\ m\ ;\ \upsilon = 0,1,2,\ ...$

lie to the left hand side and right hand side of L respectively. The conditions under which (5) converges are as follows :

$$A_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_j ;\ (i = 1,2,\ ...,\ r) \qquad (7)$$

$$B_i = 1/2\,(p_i - q_i) + \sum_{j=1}^{q_i} b_j - \sum_{j=1}^{p_i} a_j ;\ (i = 1,2,\ ...,\ r) \qquad (8)$$

$$A_i > 0,\ |\arg z| < 1/2\,\pi A_i \text{ and } B_i \geq 0;\ (i = 1,2,\ ...,\ r) \qquad (9)$$

Formulation of the Problem

In the present paper we have endeavoured to solve a pair of dual integral equations involving I–functions. The results obtained are summarised in the form of following theorem :

Theorem. The solution of following dual integral equations

$$\int_0^\infty I_1(ux)\, a(u)\, du = E(x)\ ;\ 0 < x < 1 \qquad (10)$$

$$\int_0^\infty I_2(ux)\, a(u)\, du = G(x)\ ;\ x > 1 \qquad (11)$$

where the functions $E(x)$ and $G(x)$ are prescribed and $I_1(x)$ and $I_2(x)$ are abbreviated forms of the I–functions.

$$I^{m,n}_{p,\,q,\,r}\left[x \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{i,\,n},\ (a_{ji}, \alpha_{ji})_{n+1,\,p_i} \\ (b_j, \beta_j)_{i,\,m},\ (b_{ji}, \beta_{ji})_{m+1,\,q_i} \end{matrix}\right]$$

and

$$I^{m,n}_{p,\,q,\,r}\left[x \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,\,n},\ (a_{ji}, \alpha_{ji})_{n+1,\,p_i} \\ (b_j, \beta_j)_{1,\,m},\ (b_{ji}, \beta_{ji})_{m+1,\,q_i} \end{matrix}\right]$$

respectively, can be represented as

$$a(x) = \sum_{j=1}^{r} \int_0^\infty H_i\,[ux]\, P(u)\, du \qquad (12)$$

where $P(s) = M\,\{P(u)\}$, under the conditions :

$$-\min_{1 \leq i \leq m} Re\,(b_j/\beta_j) < Re\,(s) < \frac{1}{\alpha_j} - \underset{1 \leq j \leq n}{Max}\ Re\,(a_j/\alpha_j). \qquad (13)$$

$$|\arg x| < 1/2\,\pi D_i\ ;\ (i = 1,2,\ ...,\ r) \text{ and} \qquad (14)$$

$$D_i = \sum_{i=1}^{m} \beta_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=m+1}^{q_i} \beta_j - \sum_{j=1}^{n} \alpha_j ;\ (i = 1,2,\ ...\ r) \qquad (15)$$

Proof of theorem

The proof is based on the famous Parseval's theorem for Mellin transforms which is stated below for ready reference

If $M\ \{f(u)\} = F(\xi)$ and $M\ \{g(u)\} = G(\xi)$, then

$$\int_0^\infty g(u)\, f(u)\, du = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} G(\xi)\, F(1-\xi)\, d\xi,\ \sigma_0 = Re\,(\xi). \tag{16}$$

Thus if $f(ux)$ is considered to be a function of u with x as a parameter, where $x > 0$, then

$$M\,\{f(ux)\} = x^{-\xi}\, M\,\{f(u)\} \tag{17}$$

Therefore, from (16) and (17), we have

$$\int_0^\infty g(ux)\, f(u)\, du = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} x^{-\xi}\, G(\xi)\, F(1-\xi)\, d\xi. \tag{18}$$

Additional conditions, for the validity of (18) are that $F(\xi) \in Lp\ (\sigma_0 - i\infty,\ \sigma_0 + i\infty)$ and

$x^{1-\sigma_0}.\ g(x) \in Lp\ (0,\infty),\ p \geq 1.$

where Lp denotes the class of functions $g(x)$ such that

$$\int_0^\infty |g(x)|^p / x\, dx < \infty. \tag{19}$$

We consider the dual integral equations (10) and (11), which on applying Parseval's theorem reduce to the following forms :

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s) \right\}} x^{-s} A\,(1-s)\, ds$$

$$= E(x)\ ;\ 0 < x < 1\,. \tag{20}$$

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b'_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a'_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s) \right\}} x^{-s} A\,(1-s)\, ds$$

$$= G(x)\ ;\ x > 1 \tag{21}$$

In (20) we replace x by υ, multiply both sides by $(x^{1/\alpha_n} - \upsilon^{1/\alpha_n})^{(a'_n - a_n - 1)}.\ \upsilon^{(1-a'_n)/\alpha_n - 1}$ and integrate through the integral sign with respect to υ from 0 to x and using the well known integral

$$\int_0^x (x^{1/c} - \upsilon^{1/c})^{(d-c-1)} . \upsilon^{(d/e-s-1)} d\upsilon = \frac{c\Gamma(d-e)\Gamma(e-cs)}{\Gamma(d-cs)} x^{d/c-1/c-s} \tag{22}$$

provided, $d > e$ and $e/c > \sigma$, where $s = \sigma + it$, then the equation (20) takes the form

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n-1}\Gamma(1-a_j-\alpha_j s)\Gamma(1-a'n-\alpha_n s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right\}} x^{-s} A(1-s)\, ds$$

$$= \frac{x^{a_n/\alpha_n}}{\alpha_n\Gamma(a'_n-a_n)}\int_0^x (x^{1/\alpha_n} - \upsilon^{1/\alpha_n})^{(a'_n-a_n-1)} . \upsilon^{(1-a'_n)/\alpha_n-1} E(\upsilon)\, d\upsilon . \tag{23}$$

Now we use the definition (3) of the fractional integral operator T. For bervity we write

$$T[a'j-a_j, (1-a'j)/\alpha_j-1:1/\alpha_j]\, w(x) = T_j[w(x)]. \tag{24}$$

Evidently the right hand side of (23) can be written as $T_n[E(x)]$. In other words, we can transform $\Gamma(1-a_j-\alpha_j s)$ in (20) to $\Gamma(1-a'_j-\alpha_j s)$, if at the same time, we apply the operator T_j to the right hand side of (20). Consequently on step by step transformation for $j = n-1, n-2, \ldots, 3, 2, 1$, we obtain

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a'_n-\alpha_n s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right\}} x^{-s} A(1-s)\, ds$$

$$= \prod_{j=1}^{n}\{T_j[E(x)]\}\ ;\ 0 < x < 1. \tag{25}$$

Proceeding on the similar lines and using the second fractional integral operator R, along with the integral

$$\int_x^\infty (\upsilon^{1/c} - x^{1/c})^{(d-c-1)} . \upsilon^{(1/c-d/c-s-1)} d\upsilon = \frac{c\Gamma(d-e)\Gamma(e+cs)}{\Gamma(d+cs)} x^{-e/c-s}\ , \tag{26}$$

we can transform the fcator $\Gamma(b'_j+\beta_j s)$ of the equation (21) in to $\Gamma(b_j+\beta_j s)$: $(j = 1, 2, \ldots, m)$, respectively. This transforms the equation in the following form

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a'n-\alpha_n s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right\}}x^{-s}A(1-s)\,ds$$

$$=\prod_{j=1}^{n}\{R_j[G(x)]\};\ x>1, \tag{27}$$

where $R[b'_j-b_j, b_j/\beta_j : 1/\beta_j]\, w(x)$ is denoted by $R_j\,[w(x)]$. (28)

Now in (25), using the Perseval's theorem of Mellin transform, we get

$$\int_0^\infty I^{m,n}_{p,\,q,\,r}\left[ux\,\middle|\,\begin{matrix}(a'_j,\alpha_j)_{1,\,n},\,(a_{ji},\,\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,\,m},\,(b_{ji},\,\beta_{ji})_{m+1,q_i}\end{matrix}\right]a(u)\,du=P_1(x)\,;\ 0<x<1 \tag{29}$$

$$\text{where }\ P_1(x)=\prod_{j=1}^{n}\{T_j\,[E(x)]\}\ ;\ 0<x<1. \tag{30}$$

Similarly the equation (27) reduces to the following form

$$\int_0^\infty I^{m,n}_{p,\,q,\,r}\left[ux\,\middle|\,\begin{matrix}(a'_j,\alpha_j)_{1,\,n},\,(a_{ji},\,\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,\,m},\,(b_{ji},\,\beta_{ji})_{m+1,q_i}\end{matrix}\right]a(u)\,du=P_2(x)\,;\ x>1 \tag{31}$$

$$\text{where }\ P_2(x)=\prod_{j=1}^{n}\{R_j\,[G(x)]\}\ ;\ x>1\,. \tag{32}$$

In this way the dual integral equations (10) and (11) have been reduced to the equations (29) and (31) which have a common kernel. Adding the equations (29) and (31) we arrive at the following integral equation:

$$\int_0^\infty I^{m,n}_{p,\,q,\,r}\left[ux\,\middle|\,\begin{matrix}(a'_j,\alpha_j)_{1,\,n},\,(a_{ji},\,\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,\,m},\,(b_{ji},\,\beta_{ji})_{m+1,q_i}\end{matrix}\right]a(u)\,du=P(x):\ x>1 \tag{33}$$

$$\begin{aligned}\text{where }\ P(x)&=P_1(x)\text{ for }0<\ x<1\\&=P_2(x)\text{ for }\ x>1\end{aligned}$$

In this way our problem has been reduced to finding the solution of integral equation (33). To solve this equation we make use of the theorem[8] and assume as :

(i)

$$\begin{aligned}&\sum_{j=1}^{m}\beta_j-\sum_{j=n+1}^{p_i}\alpha_j=\sum_{j=m+1}^{q_i}\beta_j-\sum_{j=1}^{n}\alpha_j;\ (i=1,2,\ldots,r)\\&\sum_{j=1}^{m}b_j-\sum_{j=n+1}^{p_i}a_j=\sum_{j=m+1}^{q_i}b_j-\sum_{j=1}^{n}a'_j;\ (i=1,2,\ldots,r)\end{aligned} \tag{34}$$

(ii)
$$
\begin{aligned}
&Re\,[\,(1-a'_j)-\alpha_j/2\,] > \alpha_j/2D_i\,;\,(j=1,2,\ldots,n;\ i=1,2,\ldots,r)\\
&Re\,[\,(a_j-\alpha_j/2\,] > \alpha_j/2D_i\,;\,(j=n+1,\ldots,p_i;\ i=1,2,\ldots,r)\\
&Re\,[\,(b'_j-\beta_j/2\,] > \beta_j/2D_i\,;\,(j=1,\ldots,m;\ i=1,2,\ldots,r)\\
&Re\,[\,(1-b_j)-\beta_j/2\,] > \beta_j/2D_i\,;\,(j=m+1,\ldots,q_i;\ i=1,2,\ldots r)
\end{aligned}
\tag{35}
$$

where

$$
D_i = \sum_{j=1}^{m}\beta_j-\sum_{j=n+1}^{p_i}\alpha_j=\sum_{j=m+1}^{q_i}\beta_j-\sum_{j=1}^{n}\alpha_j;\ (i=1,2,\ldots,r) \tag{36}
$$

Under these conditions, we find that *I*–funciton involved in the left hand side of (33) is an un-symmetric Fourier kernel. Therefore, with the help of theorem[8], we find that its respective reciprocal kernel is

$$
I^{m,n}_{p,\,q,\,r}\left[x\,\middle|\,\begin{matrix}(a_{ji}+\alpha_{ji},\alpha_{ji})_{1,\,n},\,(a'_j-\alpha_j,\,\alpha_j)_{n+1,p_i}\\(b_{ji}-\beta_{ji},\beta_{ji})_{1,\,m},\,(b_j+\beta_j,\beta_j)_{m+1,q_i}\end{matrix}\right] \tag{37}
$$

Taking Mellin transform and making use of theorem[8] in (33), we arrive at the follwing solution

$$
a(x)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j-\beta_j-\beta_j s)\prod_{j=n+1}^{p_i}\Gamma(a'_j-\alpha_j+\alpha_j s)\right\}}{\prod_{j=1}^{m}\Gamma(b_{ji}-\beta_{ji}+\beta_{ji}s)\prod_{j=1}^{n}\Gamma(1-a_{ji}-\alpha_{ji}-\alpha_{ji}s)}
$$

$$
x^{-s}\,P\,(1-s)\,ds \tag{38}
$$

where $P(s)=M\,\{P(u)\}$ and $P\,(x)$ is defined in (33). Now using Parseval's Thorem in (38), we obtain

$$
a(x)=\sum_{i=1}^{r}\ \int_0^\infty H_i\,[ux]\,p(u)\,du. \tag{39}
$$

Particular Case

If we put parameter $r = 1$, then equation (10) and (11) reduce to dual integral equation involving *H*–function considered by Saxena[7], as given below:

$$
\int_0^\infty H^{m,n}_{p+n,\,q+m}\left[ux\,\middle|\,\begin{matrix}a_j,\ A_j\\ b_k,\ B_k\end{matrix}\right]a(u)\,du=g(x)\,;\ 0<x>1 \tag{40}
$$

$$
\int_0^\infty H^{m,n}_{p+n,\,q+m}\left[ux\,\middle|\,\begin{matrix}c_j,\ A_j\\ d_k,\ B_k\end{matrix}\right]a(u)\,du=h(x)\,;\ \ x>1 \tag{41}
$$

The Solution of this special case coincides with that of Saxena[7].

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(*Dedicated to Professor J.N. Kapur on his Platinum Jubilee Celebrations*)

SATURATED POROUS POISEUILLE FLOW OF FLUIDS WITH MICROSTRUCTURE BETWEEN TWO PARALLEL WALLS

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ABSTRACT

The governing equation for saturated porous flow of fluids with microstructure based on Brinkmann model between two walls have been derived. The equations are solved to study the Poiseuille flow for general A accounting for rotational surface friction of the fluid along the walls. Expressions for the velocity, micro-rotation, mass flow rate and the skin friction at the walls for general A has been obtained and in particular discussed for $A = 0$ and ∞ .

1. Introduction. The study of flow through porous media has become the basis for many scientific and engineering applications. This type of flow is of great importance to the petroleum engineer concerned with the movement of oil, gas and water through the reservoir of an oil or gas field, to the hydrologist in his study of the migration of underground water and to the chemical engineer in connection with the filtration processes. Beyond this, the study is widely applicable in soil mechanics, water purificaiton, ceramic engineering and powder metallurgy.

The first theoretical investigations of the fluid flow through porous medium based on Darcy's [4] empirical law were started by Dupuit [5]. However, there was no general theory involving defferential equations for fluid flow through porous media till Zhukovskii [16] published his paper.

After this with the rapid development of hydraulics and hydromechanics, several theoretical forms of Darcy's law have been developed and justified by numerous planned experiments to predict the flow in porous media of practical interest. One of the modifications of Darcy's law is due to Brinkmann [2].

The analysis of the mechanics of microstructured fluids normally begins with the development of the microdynamical equation for the state of a single microstructural element which includes the streching and orienting tendencies of the surrounding flow. Microdynamical equations have been derived by many authors [1,6,7,9] for many types of microstructure. Obtaining the solution of flow problems of a microstructured fluid is difficult, involving the effect of the flow on the local structure. The coupled effect of the local structure on the flow has recieved much attention due to its technological importance. Drag reduction in dilute polymer solutions, emulsification and the dynamics of filter properties in a flowing composite material are all examples of stretching and orienting effect which the flow has on microstructure. The intresting aspects of the dynamics of microstructural fluids can be seen in James and Saringer [8], who investigated solution for the flow of macromolecules through small orifices. Several authors [3,10,11,12,13,14] discussed the dynamical behaviour of stretchable, orientable microstructure suspended in two and three-dimensional fluid flow that are steady and time dependent. Vidyanidhi and Sithapathi [15] investigated the steady Poiseuille flow of microstructure fluid between two parallel plates.

2. Formualtion of the problem. Consider the steady laminar flow of an incomprssible fluid with microstructure permeating a saturated porous medium confined between two parallel walls under the action of a constant pressure gradient in the direction of the flow.

The governing equations, following Vidyanidhi and Sithapathi [15], for Brinkmann model [2], become

$$-\frac{\partial \bar{p}}{\partial x} + V'' + 2\,b\,\zeta' - \frac{V}{K} = 0 \tag{1}$$

$$\frac{a}{\alpha^2}\,\zeta'' - \frac{1}{2}\,V' - \left[1 + \frac{2}{K\alpha^2}\right]\zeta = 0 \tag{2}$$

in terms of the velocity $\bar{V}\,(V(y),0,0)$, the micro -rotaion $\bar{\zeta}\ (0,0,\zeta\ (y))$ and denoting K as the porous parameter or porosity.

The above equations (1) and (2) are to be solved subject to the boundary conditions,

(i) for no slip at the walls i.e., $V = 0$ at $z = \pm h$

(ii) for microsturcture spin as $\zeta' = A\zeta$ at $z = \pm h$. (3)

3. Method of Solution.

Substituting,

$$V = V_1 - \frac{2b}{\alpha^2}\zeta'_1 \text{ and } \zeta = \zeta_1 - \frac{1}{2}V'_1 \tag{4}$$

in equations (1) and (2), the uncoupled equations are given by,

$$-\frac{\partial \bar{p}}{\partial x} + aV_1'' - \frac{V_1}{K} = 0 \tag{5}$$

$$\zeta_1'' - \left[\alpha^2 + \frac{1}{K}\right]\zeta_1 = 0 \tag{6}$$

Using the non-dimensional quantities,

$$V^* = \frac{-V}{h^2 \dfrac{\partial \bar{p}}{\partial x}};\ \zeta^* = \frac{-\zeta}{h \dfrac{\partial \bar{p}}{\partial x}};\ y^* = \frac{y}{h};\ \alpha^* = \alpha h$$

$$A^* = Ah\ ;\ \beta^* = \frac{h}{\sqrt{aK}};\ Q^* = \frac{-Q}{h^3 \dfrac{\partial \bar{p}}{\partial x}};\ S^* = \frac{S}{\mu}, \tag{7}$$

we obtain from equations (1) and (2), V^* and ζ^* respectively, as

$$V^* = -\ [C_1 \cosh(\beta^* y^*) + C_2 \sinh(\beta^* y^*)] + \frac{1}{a\beta^{*2}} + \frac{2b}{\alpha^{*2}}$$

$$[\sqrt{\alpha^{*2} + a\beta^{*2}}]\ [\ C_3 \sinh(\sqrt{\alpha^{*2} + a\beta^{*2}}\ y^*) + C_4 \cosh(\sqrt{\alpha^{*2} + a\beta^{*2}}\ y^*)\ ] \tag{8}$$

$$\zeta^* = \frac{\beta^*}{2}[C_1 \sinh(\beta^* y^*) + C_2 \cosh(\beta^* y^*)] - [\ C_3 \cosh(\sqrt{\alpha^{*2} + a\beta^{*2}}\ y^*) +$$

$$C_4 \sinh(\sqrt{\alpha^{*2} + a\beta^{*2}}\ y^*)\ ], \tag{9}$$

where the cosntant appear in the Appendix.

Noting $Q^* = \int_{-1}^{+1} y^* dy$,

$$Q^* = 2\left[\frac{1}{a\beta^{*2}} - \frac{C_1 \sinh(\beta^*)}{\beta^*} + \frac{2b}{\alpha^{*2}} C_4 \sinh(\sqrt{\alpha^{*2} + a\beta^{*2}})\right] \ . \tag{10}$$

The shearing stress at walls is given by,

$$S^* = \frac{1}{a}[V^{*\prime} + 2b\zeta^*] \tag{11}$$

We obtain,

$$S^*|_{y^*=\pm 1} = \pm\{ -\beta^* \left[C_1 \sinh(\beta^*) + C_2 \cosh(\beta^*)\right] + \frac{2b}{\alpha^{*2}}\beta^{*2}\left[C_3 \cosh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right) + C_4 \sinh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)\right]\} . \tag{12}$$

3.1 When $A^* = 0$

$$V^* = \frac{1}{a\beta^{*2}}\left[1 - \frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\frac{\cosh(\beta^* y^*)}{\cosh(\beta^*)}\right] - \frac{b}{a\alpha^{*2}}\frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\frac{\cosh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\, y^*\right)}{\cosh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)} \tag{13}$$

$$\zeta^* = \frac{1}{2a\beta^*}\frac{\sinh(\beta^* y^*)}{\cosh(\beta^*)} + \frac{b\beta^*}{2a\,\alpha^{*2}}\frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\frac{\sinh(\beta^* y^*)}{\cosh(\beta^*)} - \frac{1}{2a}\frac{1}{\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)}\frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\frac{\sinh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\, y^*\right)}{\cosh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)}. \tag{14}$$

$$Q^* = 2\left[\frac{1}{a\beta^{*2}} + \frac{b}{a\,\alpha^{*2}\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)}\frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\tanh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right) - \frac{1}{a\beta^{*2}}\frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\frac{\tanh(\beta^*)}{\beta^*}.\right] \tag{15}$$

$$S^*|_{y^*=\pm 1} = \pm\frac{1}{a}\left[\frac{b\,\beta^{*2}}{\alpha^{*2}\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)}\frac{1}{\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\tanh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right) - \frac{1}{\beta^*\left[1-\frac{b\beta^{*2}}{\alpha^{*2}}\right]}\tanh(\beta^*).\right] \tag{16}$$

Special Cases

(i) When $\alpha^* = 0$

$$V^* = \frac{1}{a\beta^{*2}}\left[1 - \frac{\cosh\left(\sqrt{a\beta^*}\, y^*\right)}{\cosh\left(\sqrt{a\beta^*}\right)}\right], \tag{17}$$

$$\zeta^* = 0, \tag{18}$$

$$Q^* = \frac{2}{a\beta^{*2}}\left[1 - \frac{\tanh(\sqrt{a\beta^*})}{(\sqrt{a\beta^*})}\right], \tag{19}$$

$$S^*|_{y^*=\pm 1} = \pm \frac{1}{a}\,\frac{\tanh(\sqrt{a\beta^*})}{(\sqrt{a\beta^*})} \;. \tag{20}$$

(ii) When $\alpha^* \to \infty$

$$V^* = \frac{1}{a\beta^{*2}}\left[1 - \frac{\cosh(\beta^* y^*)}{\cosh(\beta^*)}\right], \tag{21}$$

$$\zeta^* = \frac{1}{2a\beta^{*2}}\,\frac{\sinh(\beta^* y^*)}{\cosh(\beta^*)} \;, \tag{22}$$

$$Q^* = \frac{2}{a\beta^{*2}}\left[1 - \frac{\tanh(\beta^*)}{(\beta^*)}\right], \tag{23}$$

$$S^*|_{y^*=\pm 1} = \pm \frac{1}{a}\,\frac{\tanh(\beta^*)}{(\beta^*)} \;. \tag{24}$$

3.2 when $A^* \to \infty$

$$V^* = \frac{1}{a\beta^{*2}}\left[1 - \frac{\cosh(\beta^* y^*)}{\cosh(\beta^*)}\right] - \frac{\Delta_1\Delta_2\cosh(\beta^* y^*)}{\Delta_3\Delta_4} + \frac{\Delta_9\Delta_8\cosh(\sqrt{\alpha^{*2}+a\beta^{*2}}\;y^*)}{\Delta_3\Delta_4}, \tag{25}$$

$$\zeta^* = \frac{1}{2a\beta^*}\,\frac{\sinh(\beta^* y^*)}{\cosh(\beta^*)} - \frac{\tanh(\beta^*)}{2a\,\beta^*}\,\frac{\Delta_8\sinh(\sqrt{\alpha^{*2}+a\beta^{*2}}\;y^*)}{\Delta_3\Delta_4} + \frac{\beta}{2}\,\frac{\Delta_1\Delta_2\sinh(\beta^* y^*)}{\Delta_3\Delta_4} \;, \tag{26}$$

$$Q^* = \frac{2}{a\beta^*}\left[1 - \frac{\tanh(\beta^*)}{(\beta^*)}\right] - \frac{2\Delta_9\Delta_2\cosh(\sqrt{\alpha^{*2}+a\beta^{*2}})}{\Delta_3\Delta_4} - \frac{2b\tanh(\beta^*)}{a\,\alpha^{*2}\,\beta^*}\,\frac{\Delta_8\sinh(\sqrt{\alpha^{*2}+a\beta^{*2}})}{\Delta_3\Delta_4}, \tag{27}$$

$$S^*|_{y^*=\pm 1} = \pm\left\{\frac{1}{a}\frac{\tanh(\beta^*)}{(\beta^*)} + \Delta_2\left[\frac{\Delta_4 + \sinh(\sqrt{\alpha^{*2}+a\beta^{*2}})}{a\,\Delta_3\Delta_4}\right]\right.$$

$$\left. + \left[\frac{b\tanh(\beta^*)}{a\,\alpha^*2}\;\frac{\Delta_8 \sinh(\sqrt{\alpha^{*2}+a\beta^{*2}})}{\Delta_3\Delta_4}\right]\right\} \tag{28}$$

Special Cases :

(i) **When $\alpha^* = 0$**

$$V^* = \frac{1}{a\beta^{*2}}\left[1 - \frac{\cosh(\sqrt{a\beta^*}y^*)}{\cosh(\sqrt{a\beta^*})}\right] , \tag{29}$$

$$\zeta^* = 0 , \tag{30}$$

$$Q^* = \frac{2}{a\beta^{*2}}\left[1 - \frac{\tanh(\sqrt{a\beta^*})}{(\sqrt{a\beta^*})}\right], \tag{31}$$

$$S^*|_{y^*=\pm 1} = \pm\frac{1}{a}\frac{\tanh(\sqrt{a\beta^*})}{(\sqrt{a\beta^*})} . \tag{32}$$

(ii) **When $\alpha^* \to \infty$**

$$V^* = \frac{1}{a\beta^{*2}}\left[1 - \frac{\cosh(\beta^* y^*)}{\cosh(\beta^*)}\right], \tag{33}$$

$$\zeta^* = \frac{1}{a\beta^{*2}}\left[\frac{\sinh(\beta^* y^*)}{\cosh(\beta^*)} - \tanh(\beta^*)\right], \tag{34}$$

$$Q^* = \frac{2}{a\beta^{*2}}\left[1 - \frac{\tanh(\beta^*)}{(\beta^*)}\right], \tag{35}$$

$$S^*|_{y^*=\pm 1} = \pm\frac{1}{a}\frac{\tanh(\beta^*)}{(\beta^*)} . \tag{36}$$

4. Discussion. Figures (1) – (12) show the distributions of velocity and micro-rotation, while Tables (1) and (2) show those of the mass flow and stress exerted by the fluid on the walls.

From figures (1) – (3), we infer that the velocity of the fluid particles increases with increase in microstructure parameter α^*; b and β^* being fixed, while it decreases with increase in wall porosity β^* and in creases with increase in microstructure parameter b. When $\alpha^* = 0$, we obtain the maximum effects of the microstructure, while the minimum effects corresponds to $\alpha^* \to \infty$. As $\beta^* \to 0$, our results coincide with those of Vidyanidhi and Sithapathi [16]. Similar features are observed for $A = \infty$, from figures (4) – (6).

Spin distribution is depicted in figures (7)–(12). From these figures we observed that the micro–rotation is distributed over the negative and positive *x-axis*. It is clear from figures (7)–(9), for which $A = 0$, that increase in any of the parameters α^*, β^* and b causes increase in magnitude of the micro–rotation. Similar features are observed from figures (10) – (12), micro-rotation being zero at the wall for $A = \infty$.

Tables (1) and (2) show the distributions of shear stress exerted by the fluid on the walls and mass flow, for $A = 0$ and ∞ respectively. From Table (2), we observe that with increase in any of the parameters α^* and b, the mass flow rate increases, while it decreases with increase in β^*.

APPENDIX

$$\Delta_1 = \frac{b}{a\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\,\frac{\cosh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)}{\cosh\left(\beta^*\right)},$$

$$\Delta_2 = \frac{b}{\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\,\frac{\tan h\left(\beta^*\right)}{\beta}\coth\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right),$$

$$\Delta_3 = \coth\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right) - \frac{b\beta^*}{\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\,\coth\left(\beta^*\right),$$

$$\Delta_4 = \frac{b\beta^*}{\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\,\tanh\left(\beta^*\right)\cosh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right) - \sin\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right),$$

$$\Delta_5 = \left(\alpha^{*2}+a\beta^{*2}\right)\left[1-\frac{b\beta^*}{\alpha^{*2}}\right]\cosh\left(\alpha^{*2}+a\beta^{*2}\right),$$

$$\Delta_6 = \frac{b}{a\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\,\frac{\sinh\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right)}{\sinh\left(\beta^*\right)},$$

$$\Delta_7 = 1-\frac{\sqrt{\alpha^{*2}+a\beta^{*2}}}{\beta^*}\tanh(\beta^*)\coth\left(\sqrt{\alpha^{*2}+a\beta^{*2}}\right),$$

$$\Delta_8 = \frac{b\beta^*}{\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\, coth\,(\beta^*) - coth\,(\sqrt{\alpha^{*2}+a\beta^{*2}}\,)\ ,$$

$$\Delta_9 = \frac{b}{\alpha^{*2}}\sqrt{\alpha^{*2}+a\beta^{*2}}\,\frac{tan\,h\,(\beta^*)}{\beta^*}\ ,$$

$$C_1 = \frac{sech\,(\beta^*)}{a\beta^*} + \frac{\Delta_1\left\{\Delta_2 A^{*2} + (\sqrt{\alpha^{*2}+a\beta^{*2}})\left[1 - \frac{b\beta^{2*}}{\alpha^{*2}}\right]\right\}}{A^{*2}\Delta_3\Delta_4 + \Delta_5}\ ,$$

$$C_2 = \frac{A^*\Delta_6\Delta_7}{A^{*2}\Delta_3\Delta_4 + \Delta_5}\ , \qquad C_3 = \frac{A^*\ \Delta_7}{2a\,(A^{*2}\Delta_3\Delta_4 + \Delta_5)}\ ,$$

$$C_4 = \frac{1}{2a}\,\frac{A^{*2}\Delta_8\frac{tanh\,(\beta^*)}{\beta^*} + (\sqrt{\alpha^{*2}+a\beta^{*2}})\left[1 - \frac{b\beta^{2*}}{\alpha^{*2}}\right]}{A^{*2}\Delta_3\Delta_4 + \Delta_5}\ .$$

Captions for Figures and Tables

Figure 1 : Velocity distribution for $b = 0.25, A = 0$

Figure 2 : Velocity distribution for $b = 0.50, A = 0$

Figure 3 : Velocity distribution for $b = 0.75, A = 0$

Figure 4 : Velocity distribution for $b = 0.25, A \rightarrow \infty$

Figure 5 : Velocity distribution for $b = 0.50, A \rightarrow \infty$

Figure 6 : Velocity distribution for $b = 0.75, A \rightarrow \infty$

Figure 7 : Spin distribution for $b = 0.25, A = 0$

Figure 8 : Spin distribution for $b = 0.50, A = 0$

Figure 9 : Spin distribution for $b = 0.75, A = 0$

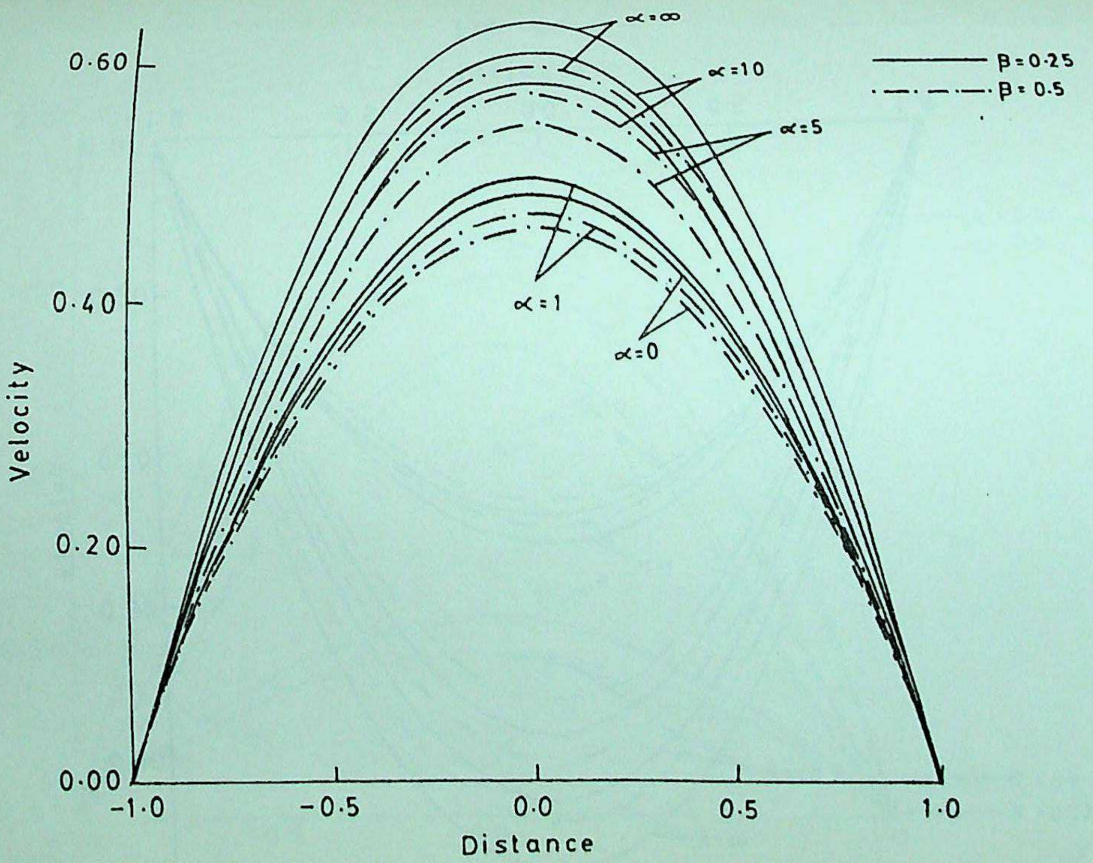
Figure 10 : Spin distribution for $b = 0.25, A \rightarrow \infty$

Figure 11 : Spin distribution for $b = 0.50, A \rightarrow \infty$

Figure 12 : Spin distribution for $b = 0.75, A \rightarrow \infty$

Table 1 : Distribution of shear stress and mass flow for $A = 0$

Table 2 : Distribution of shear stress and mass flow for $A \rightarrow \infty$



$\alpha=\infty$
$\alpha=10$
$\alpha=5$
$\alpha=1$
$\alpha=0$
$\beta=0.25$
$\beta=0.5$
0.60
0.40
0.20
0.00
Velocity
-1.0
-0.5
0.0
0.5
1.0
Distance

Figure 1

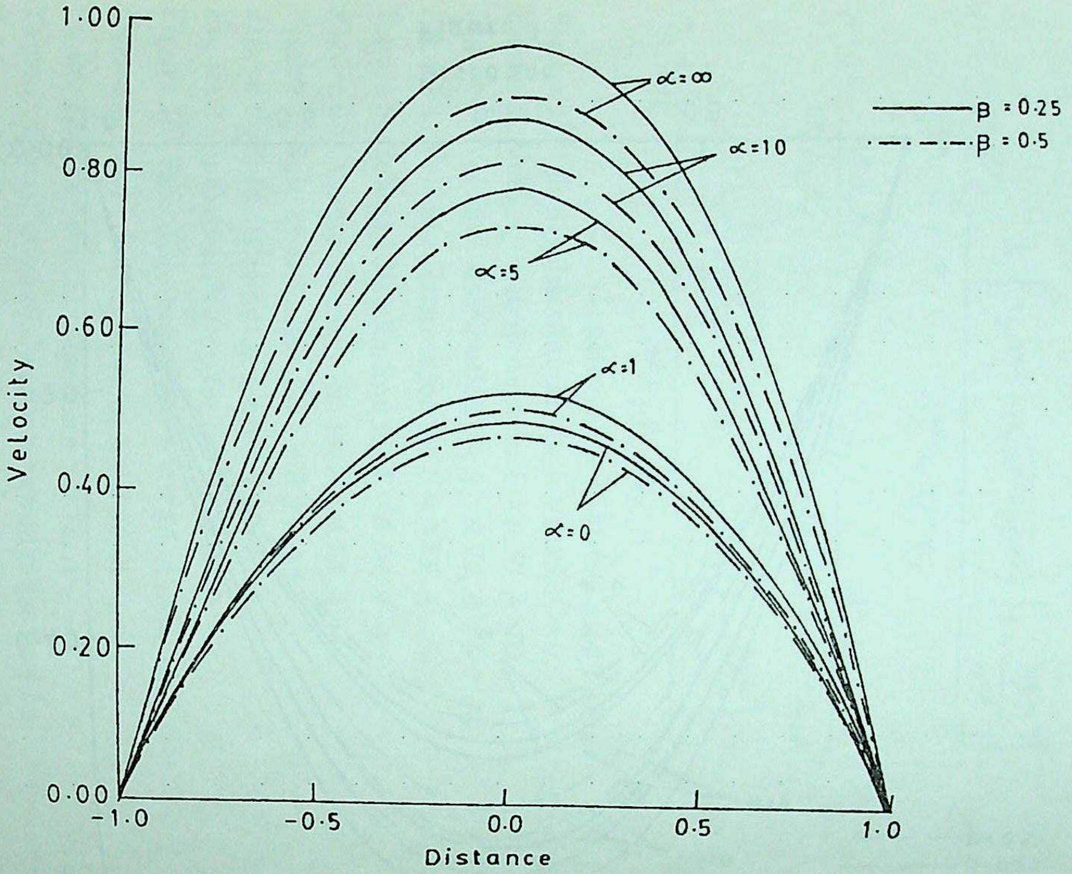

Velocity
1.00
0.80
0.60
0.40
0.20
0.00
-1.0
-0.5
0.0
0.5
1.0
Distance
β = 0.25
β = 0.5
α = ∞
α = 10
α = 5
α = 1
α = 0

Figure 2

Distance

Figure 2

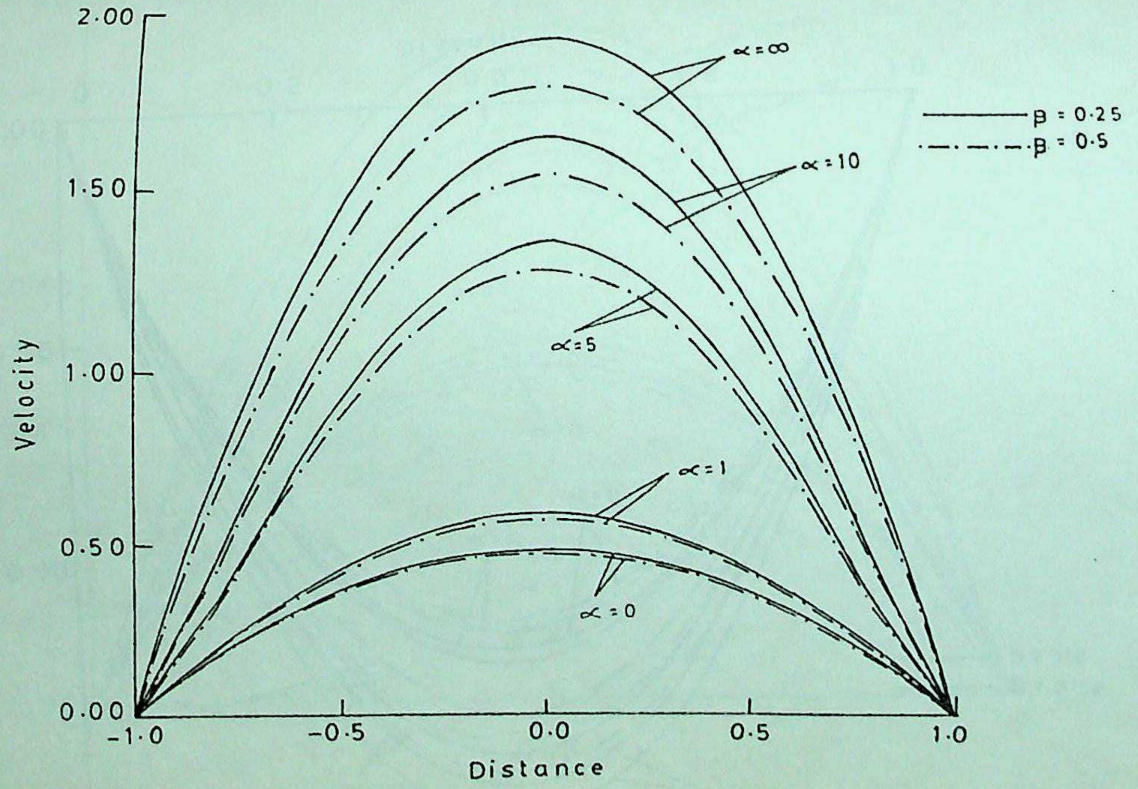


Figure 3

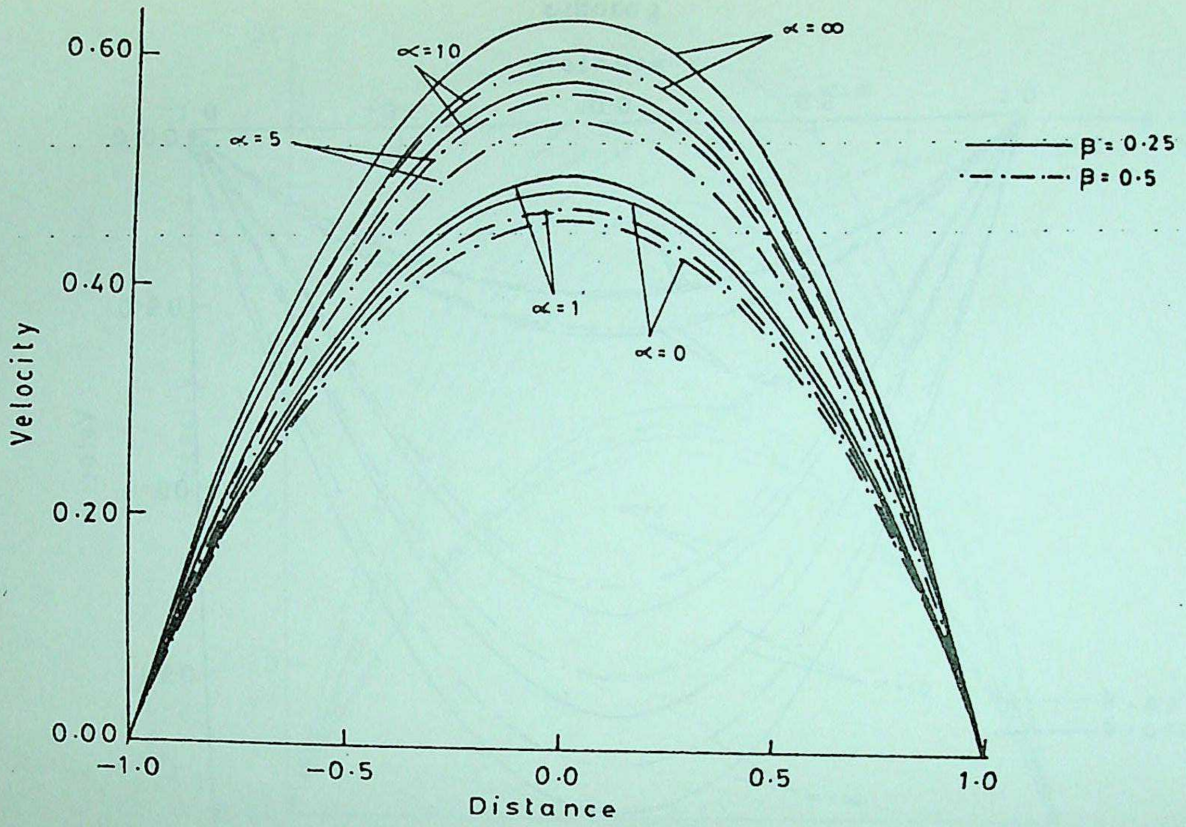

0·60
0·40
0·20
0·00
Velocity
−1·0
−0·5
0·0
0·5
1·0
Distance
$\alpha = \infty$
$\alpha = 10$
$\alpha = 5$
$\alpha = 1$
$\alpha = 0$
$\beta = 0{\cdot}25$
$\beta = 0{\cdot}5$

Figure 4

Figure 4

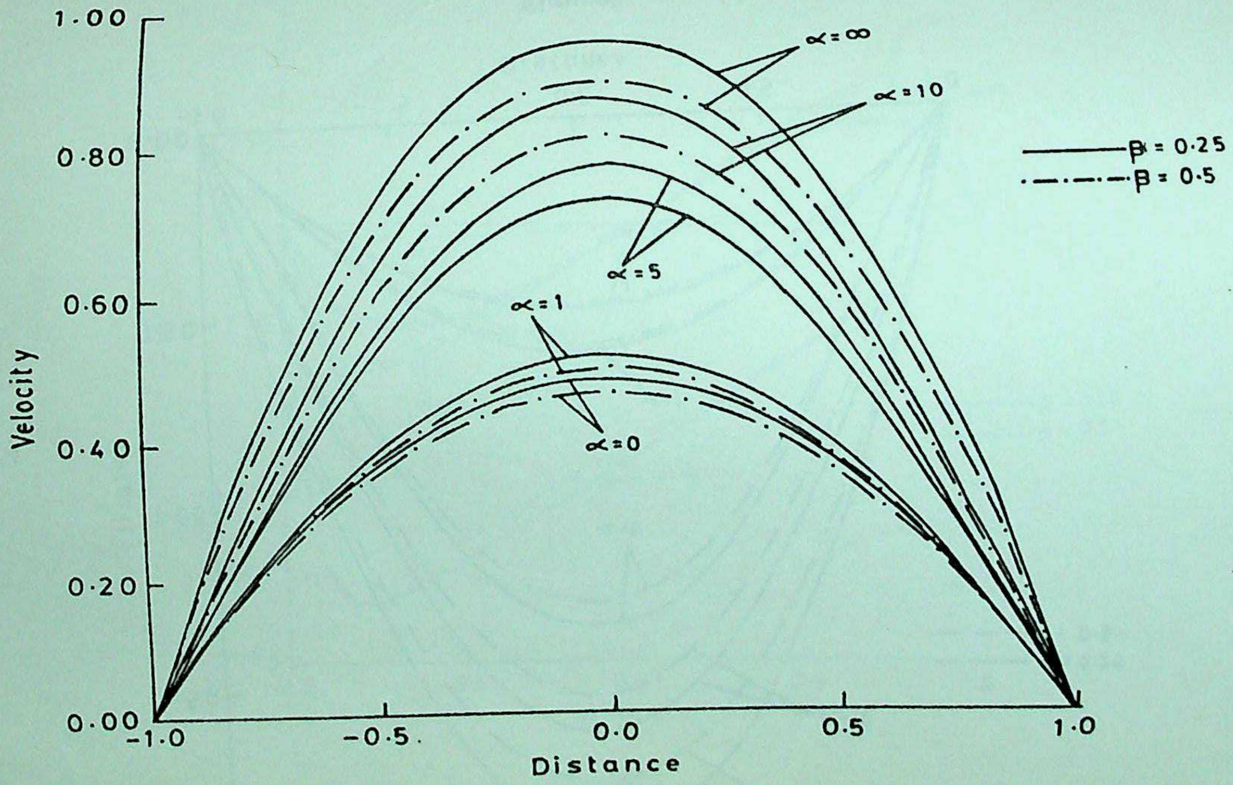


Figure 5

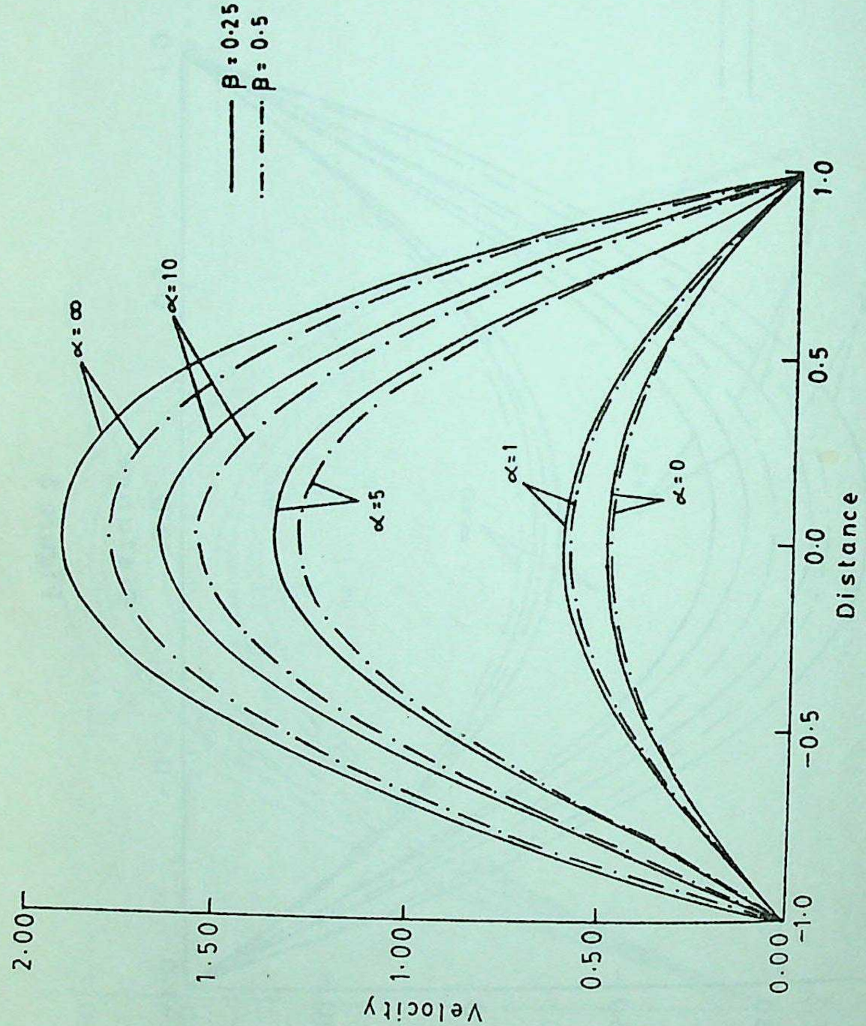

Velocity
0.00
0.50
1.00
1.50
2.00
-1.0
-0.5
0.0
0.5
1.0
Distance
α=∞
α=10
α=5
α=1
α=0
β = 0.25
β = 0.5

Figure6

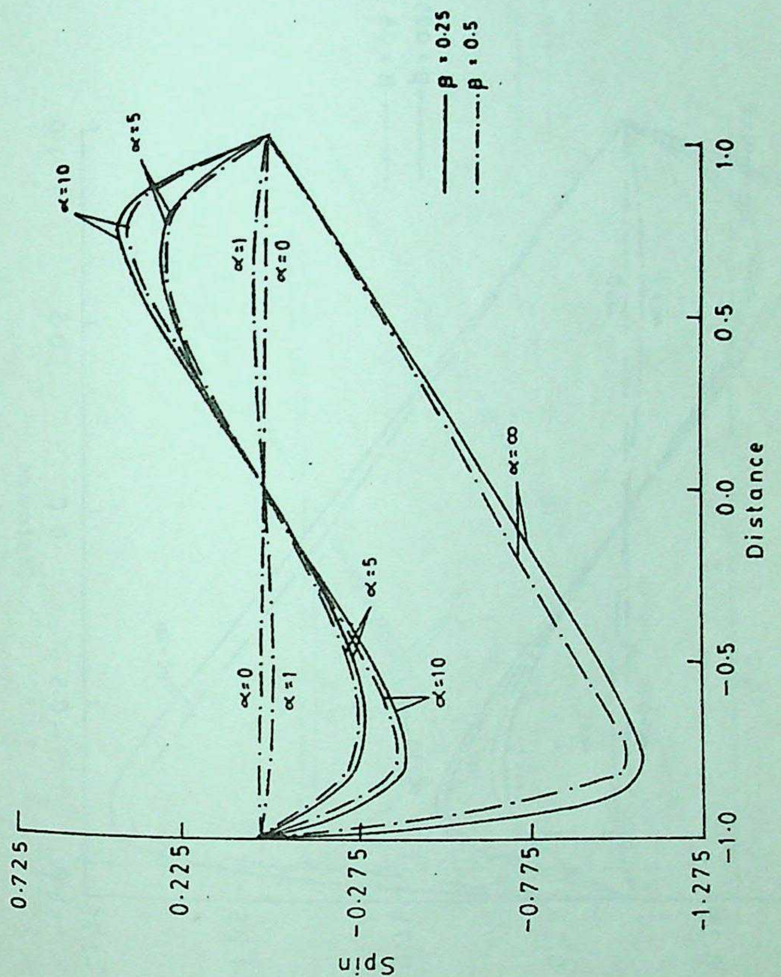

Spin
0·725
0·225
−0·275
−0·775
−1·275
−1·0
−0·5
0·0
0·5
1·0
Distance
α=10
α=5
α=1
α=0
α=5
α=10
α=0
α=1
α=∞
β = 0·25
β = 0·5

Figure 7

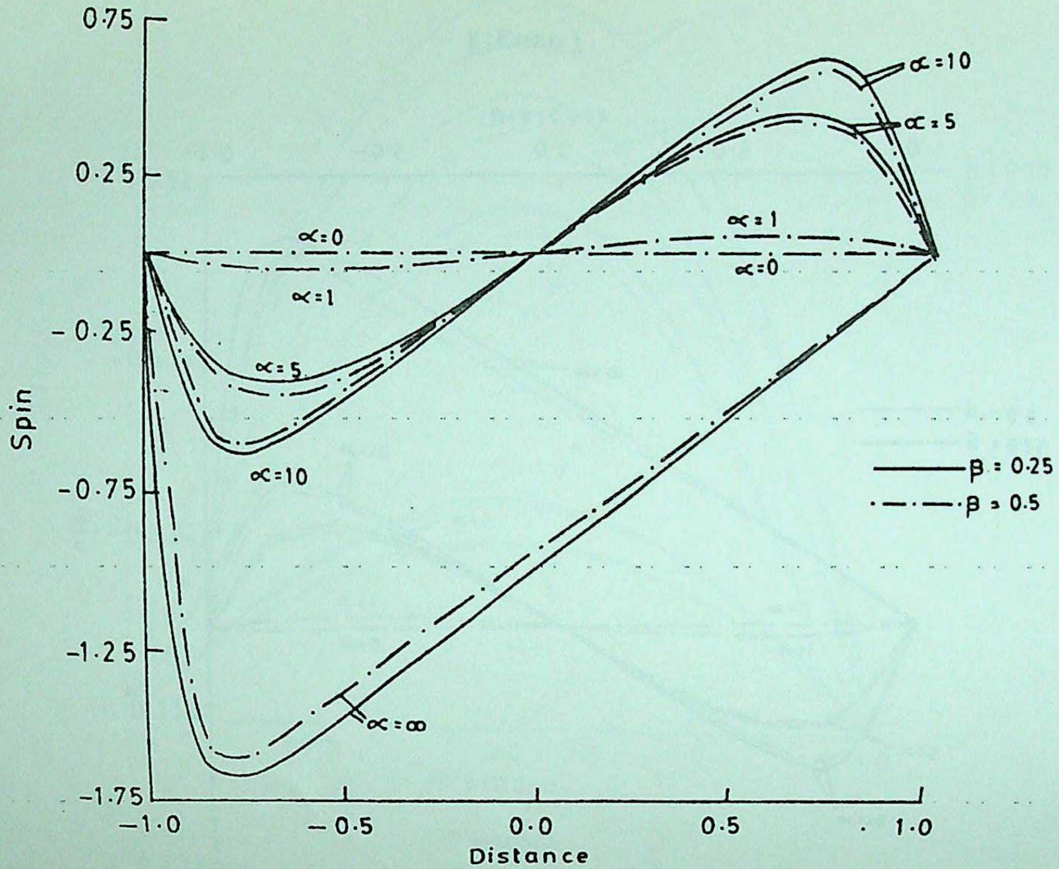

0·75
0·25
- 0·25
- 0·75
-1·25
-1·75
Spin
-1·0
- 0·5
0·0
0·5
1·0
Distance
∝=10
∝=5
∝=1
∝=0
∝=0
∝=1
∝=5
∝=10
∝=∞
β = 0·25
β = 0·5

Figure 8

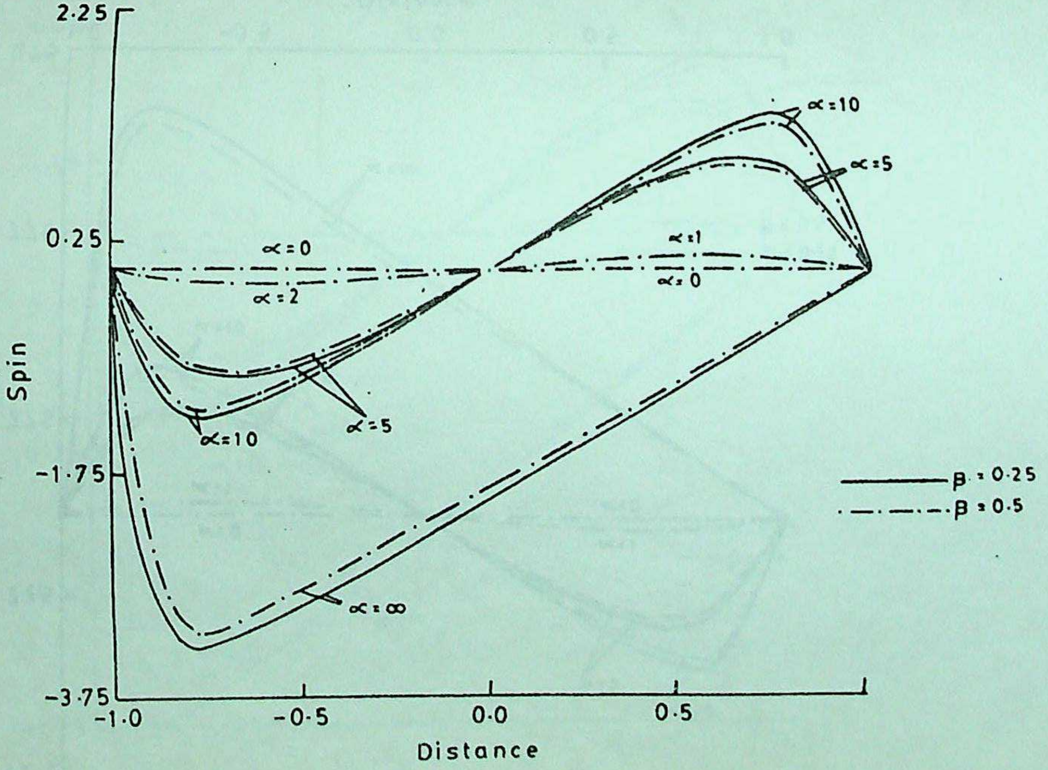

2.25
0.25
-1.75
-3.75
Spin
-1.0
-0.5
0.0
0.5
Distance
α=0
α=2
α=10
α=5
α=∞
α=1
α=0
α=10
α=5
β=0.25
β=0.5

Figure 9

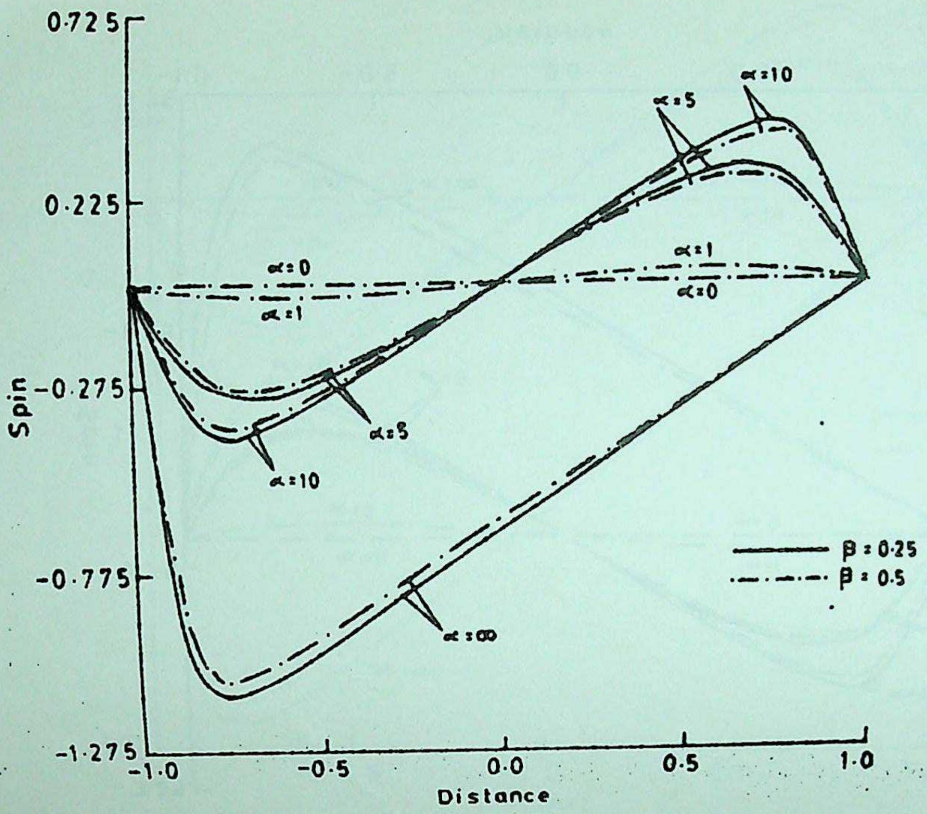

0·725
0·225
-0·275
-0·775
-1·275
Spin
-1·0
-0·5
0·0
0·5
1·0
Distance
α=10
α=5
α=1
α=0
α=0
α=1
α=5
α=10
α=∞
β = 0·25
β = 0·5

Figure 10

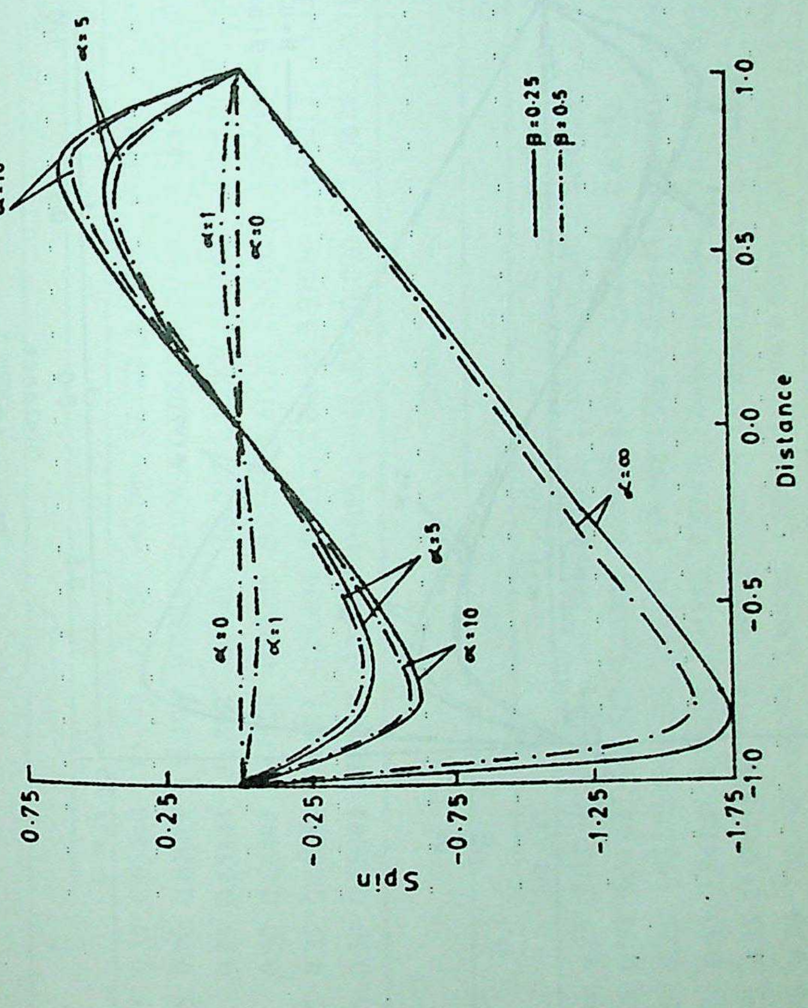

Spin
0·75
0·25
−0·25
−0·75
−1·25
−1·75
−1·0
−0·5
0·0
0·5
1·0
Distance
α=10
α=5
α=1
α=0
α=0
α=1
α=5
α=10
α=∞
β=0·25
β=0·5

Figure 11

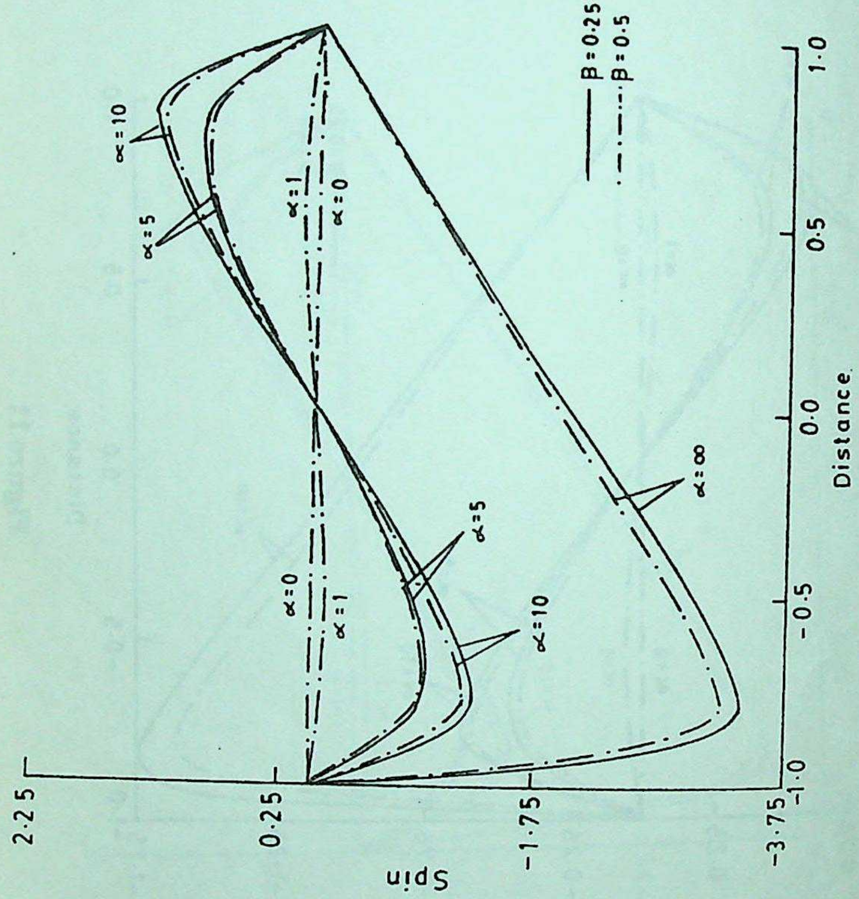

Spin
2·25
0·25
−1·75
−3·75
−1·0
−0·5
0·0
0·5
1·0
Distance
α = 10
α = 5
α = 1
α = 0
α = ∞
β = 0·25
β = 0·5

Figure 12

Table 1

b	β	S^*					Q^*				
		$\alpha = 0$	1	5	10	∞	0	1	5	10	∞
0.25	0.25	0.65440	- 23.938	-42.710	-42.680	82.722	1.31288	-1.31102	1.30688	-1.30642	1.30623
0.25	0.50	0.62019	1.86108	-0.02314	-0.00634	0.80817	1.25581	-1.24947	-1.23474	-1.23300	1.23231
0.5	0.25	0.65844	-36.398	-64.141	-64.039	1.30082	1.97942	-1.97376	-1.96130	-1.95990	1.95935
0.5	0.50	0.63494	2.47578	-0.07096	-0.01903	1.21225	1.12063	-1.90083	-1.85575	-1.85053	1.84847
0.75	0.25	0.66253	-75.041	-128.434	-128.117	2.60165	3.97930	-3.96212	-3.92456	-3.92035	3.91840
0.75	0.50	0.65041	4.28097	-0.21477	-0.05711	2.42450	3.91870	-3.85675	-3.71884	-3.70313	3.69694

Table 2

b	β	S^*					Q^*				
		$\alpha = 0$	1	5	10	∞	0	1	5	10	∞
0.25	0.25	5.25153	1.41664	1.31352	1.30905	1.30623	0.65440	0.66733	0.76493	0.80966	0.86722
0.25	0.50	2.51162	1.50527	1.25301	1.24103	1.23231	0.62019	0.63152	0.71743	0.75703	0.80817
0.5	0.25	7.91770	2.29726	1.98130	1.96782	1.95935	0.65843	0.69772	0.99313	1.12791	1.30082
0.5	0.50	3.84127	2.73739	1.91123	1.87476	1.84847	0.63494	0.67062	0.93725	1.05800	1.21225
0.75	0.25	15.91718	4.95316	3.98470	3.94416	3.91870	0.66253	0.78190	1.67609	2.08214	2.60165
0.75	0.50	7.83740	6.60785	3.88730	3.77626	3.69694	0.65041	0.76291	1.59097	1.95920	2.42450

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HYDROMAGNETIC THERMAL INSTABILITY OF RIVLIN-ERICKSEN COMPRESSIBLE FLUID IN THE PRESENCE OF HALL-CURRENT

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ABSTRACT

The thermal instability of elastico-viscous Rivlin-Ericksen compressible fluid is considered in the presence of uniform vertical magnetic field to include the Hall-current. It is found that the stability criterion is independent of the effects of viscosity and viscoelasticity and is dependent on the magnitude of the magnetic field and Hall-current. The magnetic field is found to stabilize a certain wavenumber range of the unstable configuration. The system is found to be stable for $(C_p\beta)/g<1$; Cp, β and g being specific heat at constant pressure, uniform adverse temperature gradient and acceleration due to gravity. For $(C_p\beta)/g>1$ the compressibility, magnetic field and Hall-current are found to having stabilizing effects. The uniform vertical magnetic field and Hall current introduce oscillatory modes in the system which were non-existent in their absence for $(C_p\beta)/g>1$. The sufficient conditions for the non-existence of overstability, in the presence of magnetic field and Hall-current on the problem are also derived.

1. Introduction. The theory of thermal instability of a fluid layer heated from below under varying hydromagnetic assumptions was summarized by Chandrasekhar [1]. A good account of hydrodynamic stability problems has also been given by Drazin and Reid [2] and Joseph [6]. Gupta [5] studied the thermal instability of fluid in the presence of Hall currents. The Boussinesq approximation was used in all the above studies. The approximation is well justified in the case of incompressible fluids.

When the fluids are compressible, the equations governing the system become quite complicated. To simlify them, Bousssinesq tried to justify the approximation for compressible fluids for density variations arising principally from thermal effects. Spiegel and Veronis [10] simplified the set of equations governing the flow of compressible fluids under the following assumptions :

a. The vertical dimension of the fluid is much less than any scale height, as defined by Spiegel and Veronis [10] ,

b. The motion-induced perturbations in density and pressure do not exceed, in order of magnitude, their total static variations.

Under the above approximations, Spiegel and Veronis [10] showed that the equations governing convection in a compressible fluid are formally equivalent to those for an incompressible fluid if the static temperature gradient is replaced by its excess over the adiabatic one and C_v is replaced by C_p ; where C_v and C_p are the specific heats at constant volume and constant pressure respectively.

Sharma [8] has studied the stability of a layer of electrically conducting Oldroyd fluid (i.e. fluid described by Oldroyd [7] constitutive relation) heated from below in the presence of magnetic field and has found that the magnetic field has a stablizing effect.

In another study the stability of the plane interface separating two viscoelastic (Oldroyd) superposed fluids of uniform densities. Fredricksen [3] has given a good review of non-Newtonian fluids whereas Joseph [6] has also considered the stability of viscoelastic fluids. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [7] constitutive relations. The Rivlin-Ericksen elastico-viscous fluid is one such fluid. Garg, Srivastava and Singh [4] have studied the drag on sphere oscillating in conducting dusty Rivlin-Ericksen elastico-viscous liquid. Sharma and Kumar [9] have also studied hydromagnetic stability of two Rivlin-Ericksen elastico-viscous superposed conducting fluids. It is this class of elastico-viscous fluids we are interested particularly to study the effect of Hall-current on the thermal instability of Rivlin-Ericksen compressible fluid pervaded by a uniform horizontal magnetic field in addition to a constant gravity field; which is important in ground water hydrology, chemical engineering, modern technology and industries. This aspect forms the subject of the present paper.

2. Formulation of the Problem and perturbation equations.

Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, p, q_i , x_i, μ and μ' denote respectively the stress tensor, shear stress tensor, rate of strain tensor, kronecker delta, scalar pressure, velocity position vector, viscosity and viscoelasticity. Then the Rivlin-Ericksen elastico-viscous fluid is described by the constitutive relations

$$\left.\begin{aligned} T_{ij} &= -p\delta_{ij} + \tau_{ij} \\ \tau_{ij} &= 2\left[\mu + \mu' \frac{d}{dt}\right] e_{ij} \\ e_{ij} &= \frac{1}{2}\left[\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right] \end{aligned}\right\}, \tag{1}$$

Consider an infinite horizontal compressible elastico-viscous and finitely (electrically) conducting Rivilin-Ericksen fluid layer of depth d in which a uniform temperature gradient, $\beta = |\, dT/dz|$ is maintained. Consider certesian coordinates (x,y,z) with origin on the lower boundary $z=0$ and the z-$axis$ perpendicular to it along the vertical. A gravitational field $\vec{g} = (0, 0, -g)$ and uniform vertical magnetic field $\vec{H} = (0, 0, H)$ pervade the system.

Let $\rho, p, \rho_m, T, \alpha, \kappa', \kappa = \kappa'/(\rho_m C_p), g, \upsilon = \mu/\rho_m, \upsilon' = \mu'/\rho_m, \bar{q}\,(u, v, w), \eta, N_e$ and e denote respectively the density, scalar pressure, constant spatial average of density, temperature, thermal coefficient of expansion, thermal conductivity, thermal diffusivity, gravitational acceleration, kinematic viscosity, kinematic viscoelasticity, fluid velocity, resistivity, electron number density and electron charge.

Then the equations expressing the conservation of momentum, mass, heat and the equation of state are

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\,.\nabla)\vec{q} = -\frac{1}{\rho_m}\nabla p + \vec{g}\,\frac{\rho}{\rho_m} + \frac{1}{4\pi\rho_m}(\nabla \times \vec{H}) \times \vec{H} + \left[\upsilon + \upsilon'\frac{\partial}{\partial t}\right]\nabla^2 \vec{q}, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + (\vec{q}\,.\nabla)\rho + \rho\,(\nabla.\,\vec{q}\,) = 0, \tag{3}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\,.\nabla)\,T = \kappa\,\nabla^2 T, \tag{4}$$

$$\rho = \rho_m\,[1 - \alpha\,(T - T_m)], \tag{5}$$

where T_m is the temperature of which $\rho = \rho_m$. The magnetic permeability has been taken to be unity.

The Maxwell's equations yield

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{q}.\vec{H}) + \eta \nabla^2 \vec{H} - \frac{1}{4\pi N_e e} \nabla \times [(\nabla \times \vec{H}) \times \vec{H}] \tag{6}$$

$$\nabla . \vec{H} = 0. \tag{7}$$

Initially, $\vec{q} = (0,0,0)$, $\vec{H} = (0,0,H)$, $T = T(z)$, $p = p(z)$, $\rho = \rho(z)$, where, following Spiegel and Veronis [10], we have

$$\left.\begin{aligned} &T(z) = -\beta z + T_0,\ p(z) = p_m - g\int_0^z (\rho_m + \rho_0)dz,\ K_m = \left[\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right]_m \\ &\rho(z) = \rho_m\,[1 - \alpha_m (T - T_m) + K_m (p - p_m)],\ \alpha_m = -\left[\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right]_m (= \alpha, say), \end{aligned}\right\} \tag{8}$$

Spiegel and Veronis (10) expressed any state variable, say X, in the form
$X = X_m + X_0(z) + X'(x,y,z,t)$,
where X_m is the constant spatial distribution of X, X_0 is the variation in X in the absence of motion and $X'(x, y, z, t)$ is the perturbations in X due to motion of the fluid. Thus p_m and ρ_m are the constant spatial distributions of p and ρ, and ρ_0 and T_0 are the density and temperatue of the fluid at the lower boundary $z = 0$. The pressure and temperature have been shown to be related by relations of the form (9) (Spiegel and Veronis [10], which have been obtained from the basic equations by integration.

Let δp, $\delta\rho$, θ, $\vec{h} = (h_x, h_y, h_z)$ and $\vec{q} = (u,v,w)$ denote respectively the perturbations in pressure p, density ρ, temperature T, magnetic field $\vec{H}(0,0,H)$ and velocity $(0,0,0)$. The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature, is given by

$$\delta\rho = -\rho_m \alpha\theta. \tag{9}$$

Then the linearized hydromagnetic perturbation equations appropriate for the problem, under the Spiegel and Veronis [10] approximations, are

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_m}\nabla\delta p \times \frac{1}{4\pi\rho_m}(\nabla \times \vec{h}) \times \vec{H} - g\alpha\theta + \left[\upsilon + \upsilon'\frac{\partial}{\partial t}\right]\nabla^2 \vec{q}, \tag{10}$$

$$\nabla . \vec{q} = 0, \tag{11}$$

$$\frac{\partial \theta}{\partial t} = \left[\beta - \frac{g}{C_p}\right] w + \kappa \nabla^2 \theta, \tag{12}$$

$$\frac{\partial \vec{h}}{\partial t} = \nabla \times (\vec{q}.\vec{H}) + \eta \nabla^2 \vec{h} - \frac{1}{4\pi N_e e} \nabla \times [(\nabla \times \vec{H}) \times \vec{H}], \tag{13}$$

$$\nabla . \vec{h} = 0, \tag{14}$$

where g/C_p is the adiabatic gradient, Under the above mentioned approximations, as shown by Spiegel and Veronis [10], the equations governing convection in a compressible fluid have been written as formally equivalent to those for an incompressible fluid, except that the static temperature gradient β is replaced by $\beta - g/C_p$.

3. Dispersion relation. Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, h_z, \zeta, \xi] = [W(z), \Theta(z), K(z), Z(z), X(z)] \, exp\, (ik_x x + ik_y y + nt) \tag{15}$$

where k_x and k_y are the wave numbers along the x-and y-directions. $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number and n is, in general, a complex constant. $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ and $\xi = \dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}$ are the z-components of the vorticity and current density respectively.

Expressing the coordinates (x,y,z) in the new unit of length d and letting $a = kd$, $\sigma = nd^2/\upsilon$, $p_1 = \upsilon/\kappa$, $p_2 = \upsilon/\eta$, $\upsilon'' = \upsilon'/d^2$ and $D = d/dz$ equaions (10)-(14), with the help of (9) and (15), in non-dimensional form become

$$[\sigma - (1+\upsilon''\sigma)(D^2-a^2)](D^2-a^2) W + \frac{g\alpha d^2}{\upsilon} a^2 \Theta - \frac{Hd}{4\pi\rho_m \upsilon}(D^2-a^2) DK = 0, \tag{16}$$

$$[\sigma - (1+\upsilon''\sigma)(D^2-a^2)]Z = \frac{Hd}{4\pi\rho_m \upsilon} DX, \tag{17}$$

$$(D^2-a^2-p_2\sigma)X = -\frac{Hd}{\eta} DZ - \frac{H}{4\pi N_e e \eta d}(D^2-a^2) DK, \tag{18}$$

$$(D^2-a^2-p_2\sigma)K = -\frac{Hd}{\eta} DW - \frac{H}{4\pi N_e \eta} DX, \tag{19}$$

$$(D^2-a^2-p_1\sigma)\Theta = -\frac{d^2}{\kappa}\left(\beta - \frac{g}{C_p}\right) W. \tag{20}$$

Eliminating Θ, K, X and Z between (17)-(20) we obtain

$$[(D^2-a^2-p_2\sigma)^2 \{\sigma - (1+\upsilon''\sigma)(D^2-a^2)\} + QD^2 (D^2-a^2-p_2\sigma) + MD^2 (D^2-a^2)$$
$$\{\sigma - (1+\upsilon''\sigma)(D^2-a^2)\}] \times [(D^2-a^2)(D^2-a^2-p_1\sigma)\{\sigma - (1+\upsilon''\sigma)(D^2-a^2)\} - Ra^2$$
$$\left(\frac{G-1}{G}\right)] W + Q (D^2-a^2)(D^2-a^2-p_1\sigma) + \Big[[\{\sigma - (1+\upsilon''\sigma)(D^2-a^2)\}(D^2-a^2-p_2\sigma)$$

$$+QD^2\}]\, D^2 \Big]\, W = 0, \tag{21}$$

where $Q = \dfrac{H^2 d^2}{4\pi\rho_m \upsilon\eta}$ is the Chandrasekhar number, $R = \dfrac{g\alpha\beta d^2}{\upsilon\kappa}$ is the Rayleigh number, $M = \dfrac{H}{4\pi N_e e\eta}$ is the non-dimensioanal number accounting for Hall current and $G = \dfrac{C_p\beta}{g}$. Consider the case where both the boundaries are free and the medium adjoining the fluid is non-conducting. The appropriate boundary conditions for this case are {Chandrasekhar [1]}.

$$W = D^2W = 0,\ \Theta = 0,\ DZ = 0,\ X = 0 \text{ at } z = 0 \text{ and } 1 \tag{22}$$

and $\vec{h}$ are continuous.

The case of two free boundaries, although rather artificial, is the most appropriate for the stellar atmospheres {Spiegel [11]}. Using the boundary conditions (22), one can show that all the even-order derivatives of W must vanish for $z = 0$ and 1, and hence the proper solution of (21) characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{23}$$

where W_0 is a constant. Substituting (23) into (21) and letting, $a^2 = \pi^2 b^2$, $R_1 = R/\pi^4$, $Q_1 = Q/\pi^4$, $\bar{\upsilon} = \pi^2\upsilon''$ and $i\sigma_1 = \sigma/\pi^2$ we obtain the dispersion relation

$$R = \left[\frac{G}{G-1}\right]\Bigg[\frac{1+b}{b}\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\}(1 + b + ip_1\sigma_1) + \frac{1}{b}\{Q_1(1+b)$$
$$(1+b+ip_1\sigma_1)\quad [\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\}(1 + b + ip_2\sigma_1) + Q_1]\} \times [(1+b+ip_2\sigma_1)^2$$
$$\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\} + Q_1(1 + b + ip_2\sigma_1) + M(1+b)\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\}]^{-1}\Bigg] \tag{24}$$

4. Stationary convection. When instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relations (24) reduces to

$$R_1 = \frac{G}{(G-1)}\left[\frac{(1+b)^3}{b} + \frac{1}{b}\{Q_1(1+b)((1+b)^2 + Q_1)\} \times \{(1+b)^2 + Q_1 + M(1+b)\}^{-1}\right] \tag{25}$$

which expresses the modified Rayleigh number R_l as a function of the dimensionless wave number b and the parameters Q_l, M and G for fixed Q_l and M, let G (accounting for the compressibility effect)also be kept fixed. Then we find that

$$\bar{R}_c = \frac{G}{G-1}\, R_c\,, \tag{26}$$

where $\bar{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and absence of compressibility .The effct of compressibility is thus to postpone the onset of thermal instability. Hence compressibility has a stabilizing effect . $G > 1$ is relevant here. The cases $G < 1$ and $G = 1$ correspond to negetive and infinite values of the critical rayleigh numbers in the presenc of compressibility which are not relevant in the present study.

It follows from (25) that

$$\frac{dR_1}{dQ_1} = \frac{G}{(G-1)\,b}\, [(1+b)\{(1+b)^2+Q_1\}+Q_1(1+b)]\times[\,((1+b)^2 + Q_1 + M\,(1+b)]^{-2} \tag{27}$$

This shows that the magnetic field has a stabilizing effect for all wavenumber range for $G < 1$ and destabilizing effect for $G > 1$. in the presence of Hall current for $G > 1$, the magnetic field may have stabilizing or destabilizing effect .

Equation (25) also yields

$$\frac{dR_1}{dM} = \frac{G}{(1-G)b}\, Q_1\,(1+b)^2\{(1+b)^2+Q_l\}\{(1+b)^2+Q_l + M(1+b)\}^{-2}\,. \tag{28}$$

It is evident from (28), that dR_l/dM is always positive for $G < 1$, implying thereby the stabilizing effect of Hall current and dR_l/dM is always negative for $G > 1$, implying thereby the destabilizing effect of Hall current for all wavenumbers.

We thus conclude that the presence of viscoelasticity does not effect the stability or instability of the system.

5. Stability of The System and Oscillatory Modes

Multiplying by W^*, the complex conjugate of W, integrating over the range of z, and making use of (17) - (20) , we obtain

$$[\sigma I_1 + (1+\upsilon''\sigma)I_2] + \frac{C_p \alpha \kappa a^2}{\upsilon\,(1-G)}\,(I_3 + p_1\sigma^* I_4) + \frac{\eta d^2}{4\pi\,\rho_m \upsilon}\,(I_5 + p_2\sigma I_6)\; +$$

$$\frac{\eta}{4\pi\,\rho_m \upsilon}\,(I_7 + p_2\sigma^* I_8) + d^2\,[\sigma^* I_{10} + (1+\upsilon''\sigma^*)I_9] = 0, \tag{29}$$

where

$$I_1 = \int_0^1 (|DW|^2 + a^2\,|W|^2)\,dz, \qquad I_2 = \int_0^1 (|D^2W|^2 + a^4\,|W|^2 + 2a^2\,|DW|^2)\,dz,$$

$$I_3 = \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2)\, dz, \quad I_4 = \int_0^1 |\Theta|^2\, dz \quad I_5 = \int_0^1 (|DX|^2 + a^2 |X|^2)\, dz,$$

$$I_6 = \int_0^1 |X|^2\, dz, \qquad I_7 = \int_0^1 (|D^2K|^2 + 2a^2 |DK|^2 + a^4 |K|^2)\, dz,$$

$$I_8 = \int_0^1 (|DK|^2 + a^2 |K|^2)\, dz, \qquad I_9 = \int_0^1 (|DZ|^2 + a^2 |Z|^2)\, dz,$$

$$I_{10} = \int_0^1 |Z|^2\, dz, \tag{30}$$

which are all positive definite .The real and imaginary parts of (29) give

$$\sigma_r \left[I_1 + \upsilon'' I_2 + \frac{C_p \alpha \kappa a^2}{\upsilon (1-G)} p_1 I_4 + \frac{\eta p_2}{4\pi \rho_m \upsilon} (I_8 + d^2 I_6) + d^2 (I_{10} + \upsilon'' I_9) \right]$$

$$= - \left[I_2 + \frac{C_p \alpha \kappa a^2}{\upsilon (1-G)} I_3 + \frac{\eta}{4\pi \rho_m \upsilon} (I_7 + d^2 I_5) + d^2 I_9 \right], \tag{31}$$

$$\sigma_i \left[I_1 + \upsilon'' I_2 - \frac{C_p \alpha \kappa a^2}{\upsilon (1-G)} p_1 I_4 + \frac{\eta p_2}{4\pi \rho_m \upsilon} (d^2 I_6 - I_8) + d^2 (-I_{10} + \upsilon'' I_9) \right] = 0 \tag{32}$$

It follows from (31) that σ_r is negative if $G < 1$. The system is therefore stable for $G < 1$. It is evident from (32) that σ_i may be zero and non zero . Thus the modes may be non -oscillatory or oscillatory .The oscillatory modes are introduced owing to the presence of a magnetic field (and hence Hall current).In the absence of the magnetic field , the oscillatory modesare not allowed for $G > 1$, but the presence of a magnetic field and Hall currents introduce oscillatory modes in the system.

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ON COMMON FIXED POINT THEOREM IN BANACH SPACE

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ABSTRACT

In this paper we prove a common fixed point theorem in Banach space, which is generalization of result of Sharma [5].

1. Introduction. Goebel and Zlotkiewicz [2] have studied the aspect of fixed point in Banach Space. Khan and Imdad [4] extended the result of Goebel and Zlotkiewicz [2] on common fixed point for two mappings.

Recently Dubey [1] generalized the concept of common fixed point for these mappings and established the following

Theorem A : Let C be a closed and convex subset of X. Let E, F and $G: C \to C$ satisfying the following conditions :

(i) $EF = FE, \quad FG = GF$ and $EG = GE$

(ii) $E^2 = I, \quad F^2 = I, \quad G^2 = I$,

where I denotes the identity map and

$$\|Ex - Ey\| \le (\alpha/2)\, max\,\{ \|FGx-FGy\|,\ 1/2\,\|FGx-Ex\|,\ 1/2\,\|FGy-Ey\|\ 1/2\,\|FGx-Ey\|,\ 1/2\,\|FGx-Ex\| \}$$

for all $x,y \in C$, where $0 \le \alpha < 2$. Then E, F and G have a unique common fixed point.

Sharma [5] extended the result of Dubey [1] and proved the following

Theorem B : Let E, F and G be mappings of X into itself, satisfying the following conditions :

(i) $EF = FE, \quad FG = GF$ and $EG = GE$

(ii) $E^2 = I, \quad F^2 = I, \quad G^2 = I$,

where I denotes the identity map and

(iii) $$\|Ex - Ey\| \le \alpha\,(\{ \|FGx-FGy\| + \beta\,(\|FGx-Ex\| + \|FGy-Ey\|) + r\,(\|FGx-Ey\| + \|FGy-Ex\| \}$$

for every $x,y \in X$, where $0 < \alpha, \beta, r$ and $\alpha + 2r < 1$, $1/2 \le \alpha+\beta+r < 3/2$, then E, F and G have a unique common fixed point.

2. Main Result . We now extend result of Sharma [5] for four mappings. In what follows we prove the following

Theorem 1 : Let C be a closed convex subset of X and E, F, G and $H : C \to C$ satisfy the following conditions :

(i) $EF = FE,\ FG = GF,\ GH = HG,\ EG = GE,\ EH = HE,\ FH = HF$

(ii) $E^2 = I, \qquad F^2 = I, \qquad G^2 = I \qquad H^2 = I,$

where I denotes the identity map and

(iii) $\| Ex - Ey \| \le (\alpha/2) \max \{ \| FGHx - FGHy \|, 1/2 \| FGHx - Ex \|, 1/2 \| FGHy - Ey \|, 1/2 \| FGHx - Ey \|, 1/2 \| FGHy - Ex \| \}$

for all $x, y \in C$, where $0 < \alpha < 1$. Then E, F, G and H have a unique common fixed point.

Proof. from (i) and (ii) it follows that $(EFGH)^2 = I$ and (ii) and (iii) we have

$$\| EFGHFx - EFGHFy \| \le (\alpha/2) \max \{ \| (FGH)^2 Fx - (FGH)^2 Fy \|, 1/2 \| (FGH)^2 Fx - (EFGH)Fx \|, 1/2 \| (FGH)^2 Fy - (EFGH)Fy \|, 1/2 \| (FGH)^2 Fx - (EFGH)Fy \| 1/2 \| (FGH)^2 Fy - (EFGH) Fx \| \}$$

$$\le (\alpha/2) \max \{ \| Fx - Fy \|, 1/2 \| Fx - (EFGH)Fx \|, 1/2 \| Fy - (EFGH)Fy \|, 1/2 \| Fx - (EFGH) Fy \|, 1/2 \| Fx - (EFGH)Fx \| \}.$$

Now If we put $Fx = z$ and $Fy = w$, we get

$$\| EFGHz - EFGHw \| \le (\alpha/2) \max \{ \| z - w \|, 1/2 \| z - (EFGH)z \|, 1/2 \| w - (EFGH)w \|, 1/2 \| z - (EFGH)w \|, 1/2 \| w - (EFGH) z \| \}$$

where $(EFGH)^2 = I$ and $0 \le \alpha \le 1$. So by theorem of Khan and Imdad [4] $(EFGH)$ has at least one fixed point, say x_0 in C i.e.

$$EFGHx_0 = x_0 \tag{iv}$$

$$FGH\,(EFGH)x_0 = FGHx_0 \text{ or } Ex_0 = FGHx_0 \tag{v}$$

$$H\,(EFGH)\,x_0 = Hx_0 \text{ or } EFG\,x_0 = Hx_0 \tag{vi}$$

Now using (i), (ii), (iii), (iv), (v) and (vi), we have

$$\| Hx_0 - x_0 \| = \| EFGx_0 - E^2 x_0 \| = \| E(FGx_0) - E(Ex_0) \|$$

$$\le (\alpha/2) \max \{ \| FGH\,(FGx_0) - FGH(Ex_0) \|, 1/2 \| FGH\,(FGx_0) - E(FGx_0) \|, 1/2 \| FGH\,(Ex_0) - E(Ex_0) \|, 1/2 \| FGH\,(FGx_0) - E(Ex_0) \|, 1/2 \| FGH\,(Ex_0) - E(FGx_0) \| \}$$

$$\| Hx_0 - x_0 \| \le (\alpha/2) \max \{ \| Hx_0 - x_0) \|, 1/2 \| Hx_0 - Hx_0) \|, 1/2 \| x_0 - x_0 \|, 1/2 \| H\,x_0 - x_0 \|, 1/2 \| x_0 - Hx_0 \| \}$$

$$\le (\alpha/2) \| Hx_0 - x_0 \| .$$

Since $\alpha < 1$ it follows that $Hx_0 = x_0$, *i.e.* x_0 is the fixed point of H thus we have from our previous condition

$$EFGx_0 = x_0 \tag{vii}$$

$$FG\,(EFGx_0) = FGx_0 \text{ or } Ex_0 = FGx_0 \tag{viii}$$

$$G(EFGx_0) = Gx_0 \text{ or } EF\,x_0 = Gx_0\ . \qquad \text{(ix)}$$

Now using (i), (ii), (iii), (vii), (viii) and (ix), we have

$$\begin{aligned}
\|Gx_0 - x_0\| = \|EFx_0 - E^2x_0\| &= \|E(Fx_0) - E(Ex_0)\| \\
&\le (\alpha/2)\ max\ \{\|FGH\ (Fx_0) - FGH(Ex_0)\|,\ 1/2\,\|FGH\ (Fx_0) - \\
&\quad E(Fx_0)\|,\ 1/2\,\|FGH\ (Ex_0) - E(Fx_0)\|,\ 1/2\,\|FGH\ (Fx_0) - E(Ex_0)\|, \\
&\quad 1/2\,\|FGH\ (Ex_0) - E(Fx_0)\|\,\} \\
\|Gx_0 - x_0\| &\le (\alpha/2)\ max\ \{\|Gx_0 - x_0)\|,\ 1/2\,\|Gx_0 - Gx_0)\|,\ 1/2\,\|x_0 - x_0\|, \\
&\qquad 1/2\,\|Gx_0 - x_0\|,\ 1/2\,\|x_0 - Gx_0\| \\
&\le (\alpha/2)\,\|Gx_0 - x_0\|\ .
\end{aligned}$$

Since $\alpha < 1$ it follows that $Gx_0 = x_0$, *i.e.* x_0 is the fixed point of G. thus we have from (viii) that $Ex_0 = Fx_0$.Again

$$\begin{aligned}
\|Ex_0 - x_0\| = \|Ex_0 - E^2x_0\| &= \|Ex_0 - E(Ex_0)\| \\
&\le (\alpha/2)\ max\ \{\|FGH\ x_0 - FGH(Ex_0)\|,\ 1/2\,\|FGH\ x_0 - Ex_0\|, \\
&\quad 1/2\,\|FGH\ (Ex_0) - E(Ex_0)\|,\ 1/2\,\|FGH\ x_0 - E(Ex_0)\|, \\
&\quad 1/2\,\|FGH\ (Ex_0) - Ex_0\|\,\} \\
&\le (\alpha/2)\ max\ \{\|Fx_0 - x_0)\|,\ 1/2\,\|Fx_0 - Ex_0)\|,\ 1/2\,\|x_0 - x_0\|, \\
&\qquad 1/2\,\|Fx_0 - x_0\|,\ 1/2\,\|x_0 - Ex_0\|\,\}.
\end{aligned}$$

We have from (viii) that $Ex_0 = Fx_0$ and therefore we have

$$= (\alpha/2)\,\|Ex_0 - x_0\| = (\alpha/2)\,\|Fx_0 - x_0\|$$

A contradiction, since $\alpha < 1$. Hence it follows that $Fx_0 = x_0$.

But $Fx_0 = Ex_0$, we have $Ex_0 = Fx_0 = Gx_0 = Hx_0 = x_0$

i.e. x_0 is the common fixed point of E, F, G and H.

In order to prove the uniqueness of x_0, let us consider y_0 be another common fixed point of E, F, G and H.

Now using (i), (ii), (iv), (v) and (vi), we have

$$\begin{aligned}
\|x_0 - y_0\| = \|E^2x_0 - E^2y_0\| &= \|E(Ex_0) - E(Ey_0)\| \\
&\le (\alpha/2)\ max\ \{\|FGH\ (Ex_0) - FGH(Ey_0)\|,\ 1/2\,\|FGH\ (Ex_0) - \\
&\quad E(Ex_0)\|,\ 1/2\,\|FGH\ (Ey_0) - E(Ey_0)\|,\ 1/2\,\|FGH\ (Ex_0) - E(Ey_0)\|, \\
&\quad 1/2\,\|FGH\ (Ey_0) - E(Ex_0)\|\,\} \\
&\le (\alpha/2)\ max\ \{\|x_0 - y_0)\|,\ 1/2\,\|x_0 - x_0\|,\ 1/2\,\|y_0 - y_0\|, \\
&\quad 1/2\,\|x_0 - y_0\|,\ 1/2\,\|y_0 - x_0\|\,\} \\
&\le (\alpha/2)\,\|x_0 - y_0\|.
\end{aligned}$$

Since $\alpha < 1$ it follows that $x_0 = y_0$, proving the uniqueness of x_0.

Remark 1. If we put $H = I$(I denotes the identity mapping) in our theorem we get Theorem of Dubey [1].

Remark 2 . If we put $G = I$ and $H = I$ in our theorem, we get Theorem

2.3 of Khan and Imadad [4].

Remark 3. If we put $F = I$, $G = I$ and $H = I$ in our theorem, we get Theorem 2.1 of Khan and Imdad [4].

Theorem 2. Let E, F, G and H be mappings of X into itself, satisfying the following conditions :

(i) $EF = FE$, $FG = GF$, $GH = HG$, $EG = GE$, $EH = HE$ and $FH = HF$.

(ii) $E^2 = I$, $F^2 = I$, $G^2 = I$, $H^2 = I$,

where I denotes the identity map and

(iii) $$\| Ex - Ey \| \le \alpha \{ \| FGHx-FGHy \| + \beta (\| FGHx-Ex \| + \| FGHy-Ey\|) + r(\| FGHx-Ey \| + \| FGHy-Ex \|)$$

for every x, y in X, where $0 < \alpha,\beta,r$ and $\alpha + 2r < 1$, $1/2 \le \alpha+\beta+r < 3/2$, then E, F, G and H have a unique common fixed point.

Proof. From (i) and (ii) it follows that $(EFGH)^2 = I$ and (ii) and (iii) we have

$$\begin{aligned} \| EFGHFx - EFGHFy \| &\le \alpha (\| (FGH)^2 Fx-(FGH)^2Fy \|) + \beta (\| (FGH)^2Fx \\ &\quad -(EFGH)Fx \| + \| (FGH)^2Fy-(EFGH)Fy \|) + r (\| (FGH)^2Fx \\ &\quad - (EFGH)Fy \| + \| (FGH)^2Fy-(EFGH) Fx \|) \\ &\le \alpha (\| Fx-Fy \| + \beta (\| Fx- (EFGH)Fx \| + \| Fy- (EFGH)Fy \|) \\ &\quad + r \| Fx-(EFGH) Fy \| + \| Fy -(EFGH)Fx \|). \end{aligned}$$

Now if we put $Fx = z$ and $Fy = w$, we get

$$\begin{aligned} \| EFGHz - EFGHw \| &\le \alpha (\| z-w \|) + \beta (\| z-(EFGH)z \| + \| w-(EFGH)w \| \\ &\quad +r (\| z - (EFGH)w \| + \| w-(EFGH) z \| \} \end{aligned}$$

where $(EFGH)^2 = I$ and $\alpha+2r <1$.

Now by theorem of Khan and Imdad [4], $(EFGH)$ has at least one fixed point, say x_0 in X i.e.

$$EFGHx_0 = x_0 \tag{iv}$$

$$FGH (EFGH)x_0 = FGHx_0 \text{ or } Ex_0 = FGHx_0 \tag{v}$$

$$H (EFGH) x_0 = Hx_0 \text{ or } EFG\, x_0 = Hx_0 \tag{vi}$$

Now using (i), (ii), (iii), (iv), (v) and (vi), we have

$$\begin{aligned} \| Hx_0 - x_0 \| = \| EFGx_0-E^2x_0 \| &= \| E(FGx_0-E(Ex_0) \| \\ &\le \alpha (\| FGH (FGx_0)- FGH(Ex_0) \| + \beta (\| FGH (FGx_0) -E(FGx_0)\| \\ &\quad + \| FGH (Ex_0)-E(Ex_0) \|)+r (\| FGH (FGx_0)-E(Ex_0)\| + \| FGH \\ &\quad (Ex_0)-E(FGx_0) \|) \\ \| Hx_0 - x_0 \| &\le \alpha (\| Hx_0- x_0 \|) + \beta (\| Hx_0 -Hx_0) \| + \| x_0-x_0 \|) \\ &\quad + r (\| Hx_0-x_0\| + \| x_0-Hx_0 \|) \\ &\le (\alpha+2r) \| Hx_0- x_0 \| \end{aligned}$$

which is a contradiction, Since $\alpha + 2r < 1$, it follows that $Hx_0 = x_0$, *i.e.* x_0 is a fixed point of H, thus we have from conditions .

$$EFGx_0 = x_0 \qquad \text{(vii)}$$

$$FG\,(EFGx_0) = FGx_0 \text{ or } Ex_0 = FGx_0 \qquad \text{(viii)}$$

$$G(EFGx_0) = Gx_0 \text{ or } EF\,x_0 = Gx_0 \qquad \text{(ix)}$$

Now using (i), (ii), (iii), (vii), (viii) and (ix), we have

$$\begin{aligned}
\|Gx_0 - x_0\| = \|EFx_0 - E^2x_0\| &= \|E(Fx_0) - E(Ex_0)\| \\
&\le \alpha\,(\|FGH\,(Fx_0) - FGH(Ex_0)\|) + \beta\,(\|FGH\,(Fx_0) - E(Fx_0)\| \\
&\quad + \|FGH\,(Ex_0) - E(Ex_0)\|) + r\,(\|FGH\,(Fx_0) - E(Ex_0)\| + \\
&\quad \|FGH\,(Ex_0) - E(Fx_0)\|) \\
&\le \alpha(\|Gx_0 - x_0\|) + \beta\,(\|Gx_0 - Gx_0\| + \|x_0 - x_0\|) + r\,(\|Gx_0 - x_0\| \\
&\quad + \|x_0 - Gx_0\|) \\
\|Gx_0 - x_0\| &\le (\alpha + 2r)\,\|Gx_0 - x_0\|
\end{aligned}$$

a contradiction, since $\alpha + 2r < 1$, it follows that $Gx_0 = x_0$, *i.e.* x_0 is the fixed point of G, thus we have from (viii) that $Ex_0 = Fx_0$.

Again

$$\begin{aligned}
\|Ex_0 - x_0\| = \|Ex_0 - E^2x_0\| &= \|Ex_0 - E(Ex_0)\| \\
&\le \alpha\,(\|FGH\,x_0 - FGH(Ex_0)\|) + \beta\,(\|FGH\,x_0 - Ex_0\| + \\
&\qquad \|FGH\,(Ex_0) - FGH(Ex_0)\|) + r(\|FGH\,x_0 - E(Ex_0)\| + \\
&\qquad \|FGH\,(Ex_0) - Ex_0\|) \\
&\le \alpha(\|Fx_0 - x_0\|) + \beta\,(\|Fx_0 - Ex_0\| + \|x_0 - x_0\|) + r\,(\|Fx_0 - x_0\| + \\
&\qquad \|x_0 - Ex_0\|)\ .
\end{aligned}$$

We have from (viii) that $Ex_0 = Fx_0$ and therefore we have

$$= (\alpha + 2r)\,\|Ex_0 - x_0\| = (\alpha + 2r)\,\|Fx_0 - x_0\| \ ,$$

a contradiction, since $\alpha + 2r < 1$. Hence it follows that $Fx_0 = x_0$. But $Fx_0 = Ex_0$, we have $Ex_0 = Fx_0 = Gx_0 = Hx_0 = x_0$ i.e. x_0 is the common fixed point of E, F, G and H.

To prove the uniqueness of x_0, let us consider y_0 be another common fixed point of E, F, G and H.

Now using (i), (ii), (iii), (iv), (v) and (vi), we have

$$\begin{aligned}
\|x_0 - y_0\| = \|E^2x_0 - E^2y_0\| &= \|E(Ex_0) - E(Ey_0)\| \\
&\le \alpha\,(\|FGH\,(Ex_0) - FGH(Ey_0)\|) + \beta\,\|FGH\,(Ex_0) - E(Ex_0)\| \\
&\quad + \|FGH\,(Ey_0) - E(Ey_0)\| + r(\|FGH\,(Ex_0) - E(Ey_0)\| \\
&\quad + \|FGH\,(Ey_0) - E(Ex_0)\|). \\
&= \alpha\,(\|x_0 - y_0\|) + \beta\,(\|x_0 - x_0\| + \|y_0 - y_0\|) + r\,(\|x_0 - y_0\| \\
&\quad + \|y_0 - x_0\|)
\end{aligned}$$

$$= (\alpha+2r) \| x_0 - y_0 \|$$

a contradiction. Since $(\alpha+2r) < 1$, it follows that $x_0 = y_0$, and so the common fixed point x_0 is unique.

Example . Let $x = [-1, 1]$ be the Banach space with the usual norm $\| x-y \| = | x - y|$ for all $x, y \in X$. Consider the mappings E, F, G and H on X into X defined as $Ex = 1/6\, x$, $Fx = 2/3\, x$, $Gx = 1/2\, x$ and $Hx = 3/4\, x$. for all $x \in X$.

This example satisfies the condition (iii) of our theorem (2).

Remark 1. If we put $F = I$, $G = I$ and $H = I$ and $r = 0$ we get the result of Iseki [3].

Remark 2. For $F = I$, $G = I$ and $H = I$ and $\beta = r = 0$ our theorem reduces to the theorem of Goebel and Zlotkiewicz [2].

Example . Let $X = [-1, 1]$ be the Banach space with the usual norm $\| x-y \| = | x - y|$ for all $x, y \in X$. Consider the mappings E, F, G and H on X into X defined as $Ex = 1/6\, x$, $Fx = 2/3\, x$, $Gx = 1/2\, x$ and $Hx = 3/4\, x$, for all $x \in X$.

Solution. $-1 \le x \le 1$

$\Rightarrow -1/6 \le x/6 \le 1/6$ or $-1/6 \le y/6 \le 1/6$ or $-1/6 \le Ex \le 1/6$, $-1/6 \le Ey \le 1/6$

$\Rightarrow |Ex - Ey| \le 1/3.$

We know that

$$\|Ex - Ey\| = max\, | Ex - Ey | \le 1/3 \tag{1}$$

$FGx = 2/3\, [1/2\ x] = 1/3\, x$ and $FGHx = 1/3\, [3/4\, x] = 1/4\, x - 1/4 \le FGHx \le 1/4$ and $-1/4 \le FGHy \le 1/4$

$$\Rightarrow \quad \|FGHx - FGHy\| \le 1/2 . \tag{2}$$

Similarly

$$\Rightarrow \quad \|FGHx - Ex\| \le 5/12 \tag{3}$$

$$\Rightarrow \quad \|FGHy - Ex\| \le 5/12 \tag{4}$$

$$\Rightarrow \quad \|FGHx - Ey\| \le 5/12 \tag{5}$$

$$\Rightarrow \quad \|FGHy - Ey\| \le 5/12 \tag{6}$$

Putting this value in (iii), we obtain for theorem 2 .

$$1/3 \quad \le \alpha\, (1/2) + \beta\, (5/12 + 5/12) + \gamma\, (5/12 + 5/12)$$

$0.333 \le 0.348$, where $\alpha = 1/4$, $\beta = 1/7$ and $\gamma = 1/8$

$0.333 \le 0.361$, where $\alpha = 1/6$, $\beta = 1/6$ and $\gamma = 1/6$

$0.333 \le 0.571$, where $\alpha = 2/3$, $\beta = 1/7$ and $\gamma = 1/7$

$0.333 \le 0.335$, where $\alpha = 1/4$, $\beta = 1/7$ and $\gamma = 1/9$

$0.333 \le 0.338$, where $\alpha = 1/8$, $\beta = 1/7$ and $\gamma = 1/4$

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(*Dedicated to Professor J.N. Kapur on his Platinum Jubilee Celebrations*)

PHASE SHIFT INVOLVING MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

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ABSTRACT

Many years ago, there have been a great development in many fields of the general theory of scattering and experimental studies in the nuclear and atomic collision. The problem of evaluating a potential from the observed phase shifts has led to many mathematical investigaions. However, it may be of great interest to find the phase shifts from a given potential function. Recently, a number of investigators [3,5] have determined the phase shifts making an application of Tietz's method. Here, in the present paper, we introduce a gradient of potential to evaluate the generalized formulae for the phase shifts involving Srivastava and Daoust function [8,9] using that Tiet'z method [6]. The theoretical value of cross-section is also obtained with the help of phase shifts ΔN_L where L is the angular momentum Quantum number.

1. Introduction. For the radial Schrodinger equation

$$\frac{d^2 U_L(r)}{dr^2} + [K^2 - V(r) - \frac{L(L+1)}{r^2}]\, U_L(r) = 0 \qquad \ldots (1.1)$$

Tietz [6] has derived following approximate formulae for the phase shift difference ΔN_L :

$$N_L - N_{L+1} = \frac{\pi}{K} \int_0^\infty r \frac{dv}{dr} J_{L+1/2}(Kr).\, J_{L+3/2}(Kr)\, dr \qquad \ldots (1.2)$$

and

$$N_{L-1} - N_{L+1} = \frac{(L+1/2)\pi}{K^2} \int_0^\infty \frac{dv}{dr} J^2_{L+1/2}(Kr)\, dr \qquad \dots (1.3)$$

where, $J\,(Kr)$ is the Bessel function of the first kind, K^2 and $V(r)$ denote the terms for the total energy and potential energy respectively and it is provided that L is very small.

Now, making an appeal to Luke [2]

$$J_u(az).\, J_v(bz) = \frac{(az/2)^u (bz/2)^v}{\Gamma(u+1)\Gamma(v+1)} \sum_{n=0}^{\infty} \frac{(-1)^n (az/2)^{2n}}{n!\,(u+1)_n}\; {}_2F_1\left[\begin{matrix}-n; -u-n \\ v+1;\end{matrix}\; b^2/a^2\right]$$

where $Re\,(u + v+1) > 0.$ ... (1.4)

and using the formula due to Rainiville [4] as putting $a = b = K$ and then replacing z by r, we find

$$J_u(Kr).\, J_v(Kr) = \frac{(Kr)^{u+v}}{\Gamma(u+1)\Gamma(v+1)}\; {}_2F_3\left[\begin{matrix}\frac{u+v+1}{2}, \frac{u+v+2}{2}; \\ u+1,\ v+1,\ u+v+1;\end{matrix}\; -K^2r\right] \qquad \dots (1.5)$$

provided that $Re\,(u + v+ 1) > 0$.

Again, making an appeal to (1.2) and (1.5) with the aid of $u = L + 1/2$, $v = L + 3/2$, we obtain the phase shifts

$$N_{L-1} - N_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+3}\,\Gamma(L+3/2)\Gamma(L+5/2)} \int_0^\infty \frac{dv}{dr}\, r^{2L+3}\; {}_1F_2\left[\begin{matrix}L+2; \\ L+5/2, L+3/2;\end{matrix}\; -K^2r^2\right]$$

where $Re\,(2L +3) > 0.$ ... (1.6)

Futher, putting $u = v= L + 1/2$ in (1.5) and then setting it in (1.3), we get the phase difference

$$N_{L-1} - N_{L+1} = \frac{\pi K^{2L-1}}{2^{2L+3}\,\Gamma(L+1/2)\Gamma(L+3/2)} \int_0^\infty \frac{dV}{dr}\, r^{2L+3}\; {}_1F_2\left[\begin{matrix}L+1; \\ 2L+3,\ 2L+2;\end{matrix}\; -K^2r^2\right] dr$$

whre $Re\,(2L + 2\,) > 0$ ... (1.7)

Recently, Bhattacharjie and Sudarshan [1] has constructed a solvable potential of the form

$$V(r) = \frac{B\,e^{\alpha r}}{1-e^{\alpha r}} \qquad \dots (1.8)$$

for the s-wave Schrodinger equation by transforming second order Gauss equation where B and α are parameters. Obviously, (1.8) gives a gradient of radial potential function as

$$\frac{dV}{dr} = \sum_{n=1}^{\infty} C_n\, e^{n\alpha r} \qquad \dots (1.9)$$

Motivated by this work, in the present paper, we define a new gradient of radial potential function analogous to (1.9) by which we evaluate the phase shifts involving Srivastava and Daoust function which is due to Srivastava and Daoust [8,9] (Also, see Srivastava and Manocha [10] p. 64. 1.7 (18),(19) and (20)), defined as

$$F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\ \dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\ \dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1,\dots,z_n\right]$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\dots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\dots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\dots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\dots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\ \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!}. \qquad \dots(1.10)$$

where

$$\begin{cases}\theta_j^{(k)},\ j=1,\dots,A; & \phi_j^{(k)},\ j=1,\dots,B^{(k)};\\ \psi_j^{(k)},\ j=1,\dots,C; & \delta_j^{(k)},\ j=1,\dots,D^{(k)};\ \forall\ k\in\{1,\dots,n\}\end{cases} \qquad \dots(1.11)$$

are real and positive and (a) abbreviates the array of A parameters $a_1, \dots a_A$; $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters.
$b_j^{(k)}, j=1, \dots, B^{(k)}; \forall\ k \in \{1,\dots,n\}$ with similar interpretations for (C) and $(d^{(k)})$, $k=1,\dots n$ etcetra.

2. Gradient of Potential Function.

In our investigation, we define a new gradient of radial potential function for the s-wave Schrokdinger equation such that

$$\frac{dv}{dr}=\sum_{\iota=1}^{\infty}(\alpha_1)^{\lambda\beta_1+1}\dots(\alpha_\iota)^{\lambda\beta_\iota+1}e^{-\mu\iota r}\left(\alpha_1-\frac{r}{\lambda}\right)^{-\lambda\beta_1}\dots\left(\alpha_\iota-\frac{r}{\lambda}\right)^{-\lambda\beta_\iota}, \qquad \dots(2.1)$$

provided that $\lambda > 0$ and $\mu, \alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_\iota$; are real or complex numbers.

Evidently, for $\lambda \to \infty$, (2.1) gives

$$\frac{dv}{dr}=\sum_{\iota=1}^{\infty}(\alpha_1\dots\alpha_\iota)\,e^{\beta_1/\alpha_1+\dots+\beta_\iota/\alpha_\iota-\mu\iota) r}. \qquad \dots(2.2)$$

Particularly, setting $\alpha_1=\alpha_2=\dots=\alpha_\iota=\alpha$ and $\beta_1=\beta_2=\dots=\beta_\iota=\beta$ in (2.2), we get the gradient of radial potential equivalent to (1.9) as

$$\frac{dv}{dr}=\sum_{\iota=1}^{\infty}(\alpha)^{\iota}\,e^{(\beta/\alpha-\mu)\iota r} \qquad \dots(2.3)$$

3. Phase Shifts Involving Srivastava and Daoust Function.

In this section, we evaluate the phase shifts as making an appeal to (1.6) and (2.1) as

$$N_L - N_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+3}\Gamma(L+3/2)\Gamma(L+5/2)} \sum_{t=1}^{\infty} (\alpha_1 \ldots \alpha_t) \int_0^{\infty} e^{-\mu t r} \left(1-\frac{r}{\lambda\alpha_1}\right)^{-\lambda\beta_1}$$

$$\ldots \left(1-\frac{r}{\lambda\alpha_t}\right)^{-\lambda\beta_t} r^{2L+3} {}_1F_2\left[\begin{matrix} L+2; \\ 2L+3,\ L+5/2; \end{matrix} -K^2 r^2\right] dr \qquad \ldots(3.1)$$

provided that the conditions of (1.6) and (2.1) are satisfied

Now, after doing some simplifications and then making an appeal to well known result

$\int_0^{\infty} e^{-\alpha x} x^n \, dx = \frac{\Gamma(n+1)}{\alpha^{n+1}}$ in (3.1), we find

$$N_L - N_{L+1} = \frac{\pi K^{2L+1}\Gamma(2L+4)}{2^{2L+3}\Gamma(L+3/2)\Gamma(L+5/2)} \sum_{t=1}^{\infty}\sum_{m=0}^{\infty}\sum_{s_1=0}^{\infty} \cdots \sum_{s_t=0}^{\infty} \frac{\alpha_1 \ldots \alpha_t}{(\mu t)^{2L+4}}$$

$$\frac{(L+2)_m (\lambda\beta_1)_{s_1} \ldots (\lambda\beta_t)_{s_t} (2L+4)_{2m+s_1+\ldots+s_t}}{(L+5/2)_m (2L+3)_m} \frac{(-k^2/\mu^2 t^2)^m}{m!}$$

$$\frac{(1/\lambda\alpha_1 \mu t)^{s_1}}{s_1!} \cdots \frac{(1/\lambda\alpha_t \mu t)^{s_t}}{s_t!} \qquad \ldots (3.2)$$

and by making an application of (1.10), (1,11) and (3.2) , we get the phase shifts involving Srivastava and Daoust function

$$N_L - N_{L+1} = \frac{\pi K^{2L+1}\Gamma(2L+4)}{2^{2L+3}\Gamma(L+3/2)\Gamma(L+5/2)} \sum_{t=1}^{\infty} \frac{\alpha_1 \ldots \alpha_t}{(\mu t)^{2L+4}} F^{1:1;1;\ldots;1}_{0:2;0;\ldots;0}$$

$$\left[\begin{matrix} [2L+4:2,1,\ldots 1]:[L+2:1];[\lambda\beta_1:1];\ldots;[\lambda\beta_t:1]; \\ -\quad : \qquad [L+5/2,\ 2L+3:1];-;\ldots;-; \end{matrix} \frac{-K^2}{(\mu t)^2}, \frac{1}{\lambda\mu t\alpha_1}, \ldots, \frac{1}{\lambda\mu t\alpha_t}\right]$$

provided that the conditions stated in (1.6) and (1.11) are satisfied. (3.3)

Again using above techniques with the aid of (1.7) and (2.1) we obtain another formula for phase shifts involving Srivastava and Daoust function

$$N_L - N_{L+1} = \frac{\pi K^{2L-1}\Gamma(2L+2)}{2^{2L+1}\Gamma(L+1/2)\Gamma(L+3/2)} \sum_{t=1}^{\infty} \frac{\alpha_1 \ldots \alpha_t}{(\mu t)^{2L+2}} F^{1:1;1;\ldots;1}_{0:2;0;\ldots;0}$$

$$\left[\begin{matrix} [2L+2:2,1,\ldots 1]:[L+1:1],[\lambda\beta_1:1];\ldots;[\lambda\beta_t:1]; \\ -\quad : \qquad [L+3/2,\ 2L+2:1];-;\ldots;-; \end{matrix} \frac{-K^2}{(\mu t)^2}, \frac{1}{\lambda\mu t\alpha_1}, \ldots, \frac{1}{\lambda\mu t\alpha_t}\right]$$

provided that the conditions stated in (1.7) and (1.11) are satisfied. (3.4)

4. Particular Cases.

Setting $\alpha_1 = \alpha_2 = \ldots = \alpha_t = \alpha$ and $\beta_1 = \beta_2 = \ldots = \beta_t = \beta$ in (3.3) and (3.4), and taking λ → ∞ , then after using the formula due to Srivastava [7] (Also see Srivastava and Manocha [10] 1.6, p.52 (3))

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}=\sum_{M=0}^{\infty} f(M)\,\frac{(x_1+\ldots+x_n)^M}{M!}\,,$$

we find following formulae for phase shifts

$$N_L-N_{L+1}=\frac{\sqrt{\pi}\,K^{2L+1}\,\Gamma(L+2)}{\Gamma(L+3/2)}\sum_{t=1}^{\infty}(\alpha)^t/(\mu t-\beta t/\alpha)^{2L+4}\;{}_2F_1\left[\begin{matrix}L+2,\ L+2;\\ 2L+3\,;\end{matrix}\ \frac{-4k^2}{(\mu t-\beta t/\alpha)^2}\right] \qquad \ldots (4.1)$$

provided that $Re\ (2l+3)>0$ and $Re\{(\mu t-\beta t/\alpha)\pm 2iK\}>0$;

and

$$N_{L-1}-N_{L+1}=\frac{\sqrt{\pi}\,K^{2L+1}\,\Gamma(L+1)}{\Gamma(L+1/2)}\sum_{t=1}^{\infty}(\alpha)^t/(\mu t-\beta t/\alpha)^{2L+2}\;{}_2F_1\left[\begin{matrix}L+1,\ L+1;\\ 2L+2\,;\end{matrix}\ \frac{-4k^2}{(\mu t-\beta t/\alpha)^2}\right] \qquad \ldots (4.2)$$

provided that $Re\ (2L+2)>0$ and $Re\{(\mu t-\beta t/\alpha)\pm 2iK\}>0$;

respectively

5. Verification of Formulae (3.3) and (3.4).

Setting $L=0$ in the explicit form of (3.3) that is in (3.2), we deduce it in the following form

$$N_0-N_1=\frac{\pi K\,\Gamma(4)}{2^3\,\Gamma(3/2)\,(5/2)}\sum_{t=1}^{\infty}\sum_{m=0}^{\infty}\sum_{s_1=0}^{\infty}\cdots\sum_{s_t=0}^{\infty}(\alpha_1\ldots\alpha_t)\;\frac{(4)_{2m+s_1+\ldots+s_t}\,(3+m)\,(2)_m\,(\lambda\beta_1)_{s_1}\cdots(\lambda\beta_t)_{s_t}}{3\,(5/2)_m\,(4)_m\,(\mu t)^4}\,\frac{(-K^2/\mu^2t^2)^m}{m!}\;\frac{(1/\lambda\alpha_1\mu t)^{s_1}}{s_1!}\cdots\frac{(1/\lambda\alpha_t\mu t)^{s_t}}{s_t!} \qquad \ldots (5.1)$$

Further putting $L=1$ in (3.2) and then making an appeal to

$$(\lambda-1)\,(\lambda)_{m-1}=(\lambda-1)_m$$

and

$(\lambda-2)\,(\lambda-1)\,(\lambda)_{m-2}=(\lambda-2)_m$ in it, we get

$$N_1-N_2=\frac{\pi K\,\Gamma(4)}{2^3\,\Gamma(3/2)\,(5/2)}\sum_{t=1}^{\infty}\sum_{m=0}^{\infty}\sum_{s_1=1}^{\infty}\cdots\sum_{s_t=1}^{\infty}\alpha_1\ldots\alpha_t\;\frac{(4)_{2m+s_1+\ldots+s_t}\,m\,(2)_m\,(\alpha\beta_1)_{s_1}\cdots(\alpha\beta_t)_{s_t}}{3\,(5/2)_m\,(4)_m\,(\mu t)^4}\,\frac{(-K^2/\mu^2t^2)^m}{m!}\;\frac{(1/\lambda\alpha_1\mu t)^{s_1}}{s_1!}\cdots\frac{(1/\lambda\alpha_t\mu t)^{s_t}}{s_t!} \qquad \ldots (5.2)$$

Now, adding (5.1) and (5.2), we evaluate

$$N_0 - N_2 = \frac{\pi K \Gamma(4)}{2^3 \Gamma(3/2) \Gamma(5/2)} \sum_{t=1}^{\infty} \sum_{m=0}^{\infty} \sum_{s_1=0}^{\infty} \cdots \sum_{s_t=0}^{\infty} \alpha_1 \ldots \alpha_t$$

$$\frac{(4)_{2m+s_1+\ldots+s_t} (2)_m (\alpha\beta_1)_{s_1} \ldots (\alpha\beta_t)_{s_t}}{(5/2)_m (4)_m (\mu t)^4} \frac{(-K^2/\mu^2 t^2)^m}{m!}$$

$$\frac{(1/\lambda\alpha_1 \mu t)^{s_1}}{s_1!} \cdots \frac{(1/\lambda\alpha_t \mu t)^{s_t}}{s_t!} . \qquad \ldots (5.3)$$

Again with the aid of (1.10), (1.11) and (5.3), we find

$$N_0 - N_2 = \frac{\pi K \Gamma(4)}{2^3 \Gamma(3/2)(5/2)} \sum_{t=1}^{\infty} (\alpha_1 \ldots \alpha_t)/(\mu t)^4 \, F^{1:1;1;\ldots;1}_{0:2;0;\ldots;0}$$

$$\left[\begin{matrix} [4:2,1,\ldots,1]: [2:1]; [\lambda\beta_1:1]; \ldots; [\lambda\beta_t:1] ; \\ — : \quad [5/2, 4:1]; - ;\ldots; - ; \end{matrix} \; \frac{-K^2}{(\mu t)^2}, \frac{1}{\lambda \mu t \alpha_1}, \ldots, \frac{1}{\lambda \mu t \alpha_t} \right] \ldots (5.4)$$

Now, setting $L=1$ in (3.4), we also get exactly same result (5.4) for $N_0 - N_2$. If N_0 is known, all other values of the phase shifts can easily be evaluated from the relation (3.3) and (3.4).

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STABILITY OF WALTERS' (MODEL B') SUPERPOSED FLUIDS IN POROUS MEDIUM

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ABSTRACT

The stability of Walters'(model B') viscoelastic superposed fluids in porous medium is considerd. The effect of a uniform rotation and a uniform horizontal magnetic field, separately, are also considerd . The system is found to be stable /unstable for bottom heavy configurations, as in Newtonion viscous fluid. For exponentially varying density, viscosity, viscoelasticity , medium porosity and medium permeability ;the system becomes stable (or unstable)for the stable (or unstable) stratifications . however, the growth retes decrease with the increase in kinematic viscosity and kinematic viscoelasticity whereas the growth rates increase with the increase in medium permeability. A uniform rotation does not affect the stability or instability as was the situation in the absence of rotation of Walters' (model B') viscoelastic superposed fluids in porous medium .A uniform magnetic field stabilizes certain wave number band whereas system was unstable for all wave- numbers in the absence of magnetic field , for the potentially unstable configuration .

1. Introduction. A comprehensive account of the Rayleigh-Taylor instability , under varying assumptions of hydrodynamics and hydromagnetic , has been given by Chandrasekhar [1]. The medium has been assumed to be non - porous .

The flow through a porous medium has been considerable interest in recent years particularly among geophysical fluid dynamicists. A macroscopic equation which describes incompressible flow of a Newtonian fluid of viscoisity through a macroscopically homogeneous and istropic porous medium of permeabilly k_1 is the well-known Darcy's equation. The usual viscous term in the eqations of viscoelastic fluid motion is replaced by the resistance term $-\frac{1}{k_1}\left[\mu-\mu'\frac{\partial}{\partial t}\right]\vec{q}$ where $\vec{q}$ is the filter velocity of the fluid and μ, μ' are the viscosity and viscoelasticity of the Walters' (model B') fluid. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considerd by Wooding [5].

Sharma [3] has studied the stability of a layer of electrically conducting Oldroyd fluid (i.e. fluid described Oldroyd [4] constitutive relation) heated from below in the presence of magnetic field and has found that the magnetic field has satabilizing effect . there are many elastico - viscous fluids that can not be characterized by the Maxwell 's constitutive relation or Oldroyd 's constitutive relation. One such class of elastico -viscous fluids is Walters' (model B') fluid . Sharma and Kumar [2] have studied the "stability of two superposed Walters' (model B') viscoelastic fluid". Yadav and Ray [6] have studied the unsteady flow of n-immisable viscoelastic [Walters' (model B')] fluids through a porous medium between two perallel plates in the presence of a transverse magnetic field . It is this class of elastico - viscous fluids we are interested in .

The present paper ,therefore, attempts to study the stability of Walters' (model B') viscoelastic superposed fluids in porous medium .Owing to the importance of uniform magnetic field and uniform rotation in geophysics (e.g.Earth's magnetic field and Earth's rotation) and there possible influence on stability, the present paper also considers the stability of Walters'(model B') superposed fluids in porous medium , separately , in the presence of uniform rotation and uniform magnetic field.

2.Formulation of the problem and perturbation equations.

The initial stationary state whose stability we wish to examine is that of an incompressible Walters'(model B')fluid of variable density, kinematic viscosity and kinematic viscoelasticity arranged in a horizontal strata in porous medium of variable porosity and permeability . The charactr of the equilibrium of this initial static state is determined by supposing that the system is slightly disturbed and then following its further evolution .

Let ρ, υ, υ', p and $\vec{V}(0,0,0)$ denote , respectively ,the density, the kinematic viscosity,the kinematic viscoelasticity ,the pressure and the velocity of the fluid. Let $\vec{g}\ (0,0,-g)$ and $\in$ stand for acceleration due to gravity and medium porosity respectively.Then the equations of motion , continuity and incompressibility for the fluid through a porous medium are given by

$$\frac{\rho}{\epsilon}\left[\frac{\partial\vec{V}}{\partial t}+\frac{1}{\epsilon}(\vec{V}.\nabla)\vec{V}\right]=-\nabla p+\rho\vec{g}-\frac{\rho}{k_1}\left[\upsilon-\upsilon'\frac{\partial}{\partial t}\right]\vec{V}, \tag{1}$$

$$\nabla.\vec{V}=0, \tag{2}$$

$$\epsilon\frac{\partial\rho}{\partial t}+(\vec{V}.\nabla)\rho=0. \tag{3}$$

Let $\delta\rho$, δp and $\vec{V}(u,v,w)$ denote, respectively, the perturbation in density ρ, pressure p and velocity (0,0,0). Then the linearized perturbation equations relevant to the problem are

$$\frac{\rho}{\epsilon}\frac{\partial\vec{V}}{\partial t}=-\nabla\delta p+\rho\,\vec{g}\,\delta-\frac{\rho}{k_1}\left[\upsilon-\upsilon'\frac{\partial}{\partial t}\right]\vec{V}, \tag{4}$$

$$\nabla.\vec{V}=0, \tag{5}$$

$$\epsilon\frac{\partial}{\partial t}(\delta\rho)=-w\frac{d\rho}{dz}. \tag{6}$$

Analysing the disturbances into normal modes, we seek the solutions whose dependence on x,y and t is of the form

$$exp\,(ik_x x+ik_y y+nt). \tag{7}$$

where n is, in general, a complex constant, $k=(k_x^2+k_y^2)^{1/2}$ is the resultant wave number and k_x, k_y are the horizontal wave numbers.

Equations (4)-(6) using (7), give

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right]u=-ik_x\ \delta p, \tag{8}$$

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right]v=-ik_y\ \delta p, \tag{9}$$

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right]w=-D\delta p-g\,(\delta p), \tag{10}$$

$$ik_x u+ik_y v+Dw=0, \tag{11}$$

$$\epsilon n\delta\rho=-wD\rho, \tag{12}$$

where $D=d/dz$.

Eliminating u, v and δp from equation (8) – (12), we obtain

$$D\left[\rho\left\{n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right\}Dw\right]-k^2\rho\left\{n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right\}w+\frac{gk^2(D\rho)\,w}{n}=0. \tag{13}$$

3. Two Uniform Fluids Separated by a Horizontal Boundary.

Let two uniform fluids of densities ρ_1 and ρ_2 be separated by a horizontal boundary at $z = 0$. Then in each region of constant $\rho, \upsilon, \upsilon'$, and k_1, equation (13) reduces to

$$(D^2 - k^2)\, w = 0. \tag{14}$$

The boundary conditions to be satisfied are:

(i) w must be continuous across an interface between two fluids. (15)

(ii) Intergrating (13) between $Z_0 - \delta$ and $Z_0 + \delta$ and passing to the list $\delta = 0$ we obtain in view of (15), the jump condition

$$\Delta_0 \left[\rho \left\{ n + \frac{\in}{k_1} (\upsilon - \upsilon' n) \right\} Dw \right] = - \frac{g k^2 \Delta_0(\rho)\, w_0}{n}, \tag{16}$$

while the equation valid everywhere else $(Z \neq Z_0)$ is

$$D \left[\rho \left\{ n + \frac{\in}{k_1} (\upsilon - \upsilon' n) \right\} Dw \right] - k^2 \rho \left\{ n + \frac{\in}{k_1} (\upsilon - \upsilon' n) \right\} w + \frac{g k^2 (D\rho)\, w}{n} = 0. \tag{17}$$

Applying the boundary condition (i) and the fact that w cannot increase exponentially on either side of the interface, the solutions appropriate for the two regions are

$$w_1 = A e^{+kz}, \ (z < 0) \tag{18}$$

$$w_2 = A e^{-kz}, \ (z > 0) \tag{19}$$

Appyling the boundary condition (16) to the solutions (18) and (19), we obtain the dispresen relation

$$\left[1 - (\alpha_1 \upsilon_1' + \alpha_2 \upsilon_2') \frac{\in}{k_1} \right] n^2 + \frac{\in}{k_1} (\alpha_1 \upsilon_1 + \alpha_2 \upsilon_2)\, n - gk\ (\alpha_2 - \alpha_1) = 0, \tag{20}$$

where $\upsilon_1 = (\mu_1/\rho_1)$, $\quad \upsilon_2 = (\mu_2/\rho_2)$ and

$\upsilon_1' = (\mu_1'/\rho_1) \quad \upsilon_2' = (\mu_2'/\rho_2)$

are the kinematic viscosities and kinematic viscoelasticities of fluids 1 and 2 respectively and $\alpha_{1,2} = \dfrac{\rho_{1,2}}{(\rho_1 + \rho_2)}$. The medium porosity $\in$ and medium

permeabitity k_1 in both the regions are assumed to be the same .

Equation (20) yields

$$n = \frac{\left[\frac{\in}{k_1} (\alpha_1 \upsilon_1 + \alpha_2 \upsilon_2) \pm \left[\left\{ \frac{\in}{k_1} (\alpha_1 \upsilon_1 + \alpha_2 \upsilon_2) \right\}^2 + 4 \left\{ 1 - \frac{\in}{k_1} (\alpha_1 \upsilon_1' + \alpha_2 \upsilon_2') \right\} gk(\alpha_2 - \alpha_1) \right]^{1/2}}{2 \left\{ 1 - \frac{\in}{k_1} (\alpha_1 \upsilon_1' + \alpha_2 \upsilon_2') \right\}} \tag{(21)}$$

When $\rho_2 > \rho_1$ and $k_1/\in > (\alpha_1 \upsilon'_1 + \alpha_2 \upsilon'_2)$,one of the values of n is positive which means that perturbation grow with time and so the system is unstable . When $\rho_1 > \rho_2$ and $k_1/\in > (\alpha_1 \upsilon'_1 + \alpha_2 \upsilon'_2)$, both values of n are either real, negative or complex cojugates wih negative real parts . The system is , therefore, stable; where as for $\rho_1 > \rho_2$ and $k_1/\in > (\alpha_1 \upsilon'_1 + \alpha_2 \upsilon'_2)$ one of the values of n is positive which means that perturbations grow with time and so the system is unstable ,which is in contrast to the Newtonian fluid where the system is always stable for $\rho_1 > \rho_2$.

4. The Case of Exponentially Varying Stratifications. Assume the stratifications in fluid density, fluid kinematic viscosity , fluid kinematic viscoelasticity , medium porosity and medium permeability of the forms

$$\rho = \rho_0 e^{\beta z}, \quad \mu = \mu_0 e^{\beta z}, \ \mu' = \mu'_0 e^{\beta z}, \ \in = \in_0 e^{\beta z}, \quad k_1 = k_{10} e^{\beta z}. \tag{22}$$

where $\rho_0, \mu, \mu_0', \in_0, k_{10}$ and β are constants and so the kinematic viscosity $\upsilon = \mu/\rho = \mu_0/\rho_0 = (\upsilon_0)$ and the kinematic viscoelasticity $\upsilon' = \mu'/\rho = \mu'_0/\rho_0 = (\upsilon'_0)$ are constant everywhere . We also assume that $\beta d < < 1$, i.e. thevariation of density at two neighbouring points in the velocity field which is much less than average density has a negligible effect on the velocity of the fluid.

Using stratification of the form (22), equation (17) transforms to

$$\left[n + \frac{\in_0}{k_{10}} (\upsilon_0 - n\upsilon'_0) \right] (D^2 - k^2)\, w + \frac{gk^2 \beta w}{n} = 0. \tag{23}$$

The general solution of equation (23) is

$$w = Ae^{m_1 z} + Be^{m_2 z}, \tag{24}$$

where A, B are two arbitrary constants; m_1, m_2 are given by

$$m_{1,2} = \pm k \left[1 - \frac{g\beta}{nL} \right]^{1/2}, \tag{25}$$

and $$L = \left[n + \frac{\in_0}{k_{10}} (\upsilon_0 - n\upsilon'_0) \right] \ . \tag{26}$$

Here we consider the fluid to be confined between two rigid planes at $z = 0$ and $z = d$. The boundary conditions for the case of two rigid surfaces are

$$w = 0 \ \text{ at } \ z = 0 \text{ and } z = d. \tag{27}$$

The proper solution of equation (23), satisfying (27),is

$$0 = A\,(e^{m_1 d} + e^{m_2 d}),$$

i.e. $$(m_1 - m_2)\, d = 2is\pi, \tag{28}$$

where s is an integer.

Inserting the values of m_1, m_2 from (25) in equation (28), we obtain

$$1-\frac{g\beta}{k_{10}}=2\,\frac{-s^2\pi^2}{k^2d}\,, \tag{29}$$

which on simplification gives

$$\left[1-\frac{\epsilon_0\upsilon'_0}{k_{10}}\right]n^2+\frac{\epsilon_0\upsilon_0}{k_{10}}n-\frac{g\beta\,k^2d^2}{k^2d^2+s^2\pi^2}=0. \tag{30}$$

Equation (30) is the dispersion relation governing the stability of stratified fluid (exponentially varying fluid density, fluid viscosity, fluid viscoelasticity) in a stratified porous medium (exponentially varying medium porosity and medium permeability).

If $\beta<0$ (stable stratification) and $k_{10}<\epsilon_0\upsilon'_0$, equation (30) does not admit any positive root of n and so the system is stable for disturbances of all wave numbers. However,the system d unstable for $k_{10}<\epsilon_0\upsilon'_0$. Thus for stable stratifications , the system is stsble for $k_{10}>\epsilon_0\upsilon'_0$ and unstable for $k_{10}<\epsilon_0\upsilon'_0$. This is in contrast to the newtonian fluid where the system is always stable for stable stratifications [Chandrasekhar [1]].

If $\beta>0$ (unstable stratification) and $k_{10}>\epsilon_0\upsilon'_0$, equation (30) has one positive root and so the system is unstable for all waves number .However ,the system is stable for $k_{10}<\epsilon_0\upsilon'_0$. This is in contrast to the Newtonian fluid where the system is always unstable for unstable stratification.

Thus viscoelasticity has damping as well as enhancing effect on the growth rates of the system.

Let n_0 denote the positive root of equation (30).Then

$$\left[1-\frac{\epsilon_0\upsilon}{k_1}\right]n_0^{\,2}+\frac{\epsilon\upsilon}{k_1}n_0-\frac{g\beta\,k^2d^2}{k^2d^2+s^2\pi^2}=0\,, \tag{31}$$

as $\upsilon_0=\dfrac{\mu_0}{\rho_0}=\dfrac{\mu}{\rho}=\upsilon$, $\upsilon'_0=\dfrac{\mu'_0}{\rho_0}=\dfrac{\mu'}{\rho}=\upsilon'$, and $\dfrac{\epsilon_0}{k_{10}}=\dfrac{\epsilon_0}{k_1}$.

To find the roles of fluid kinematic viscosity, fluid kinematic viscoelasticity and medium permeability on the growth rate of unstable modes, we examine the natures of $dn_0/d\upsilon$, $dn_0/d\upsilon'$ and dn_0/dk_1 analytically.

Equation (31) yields

$$\frac{dn_0}{d\upsilon} = -\frac{n_0 \in}{2\, n_0 (k_1 - \in\upsilon') + \in\upsilon}, \tag{32}$$

$$\frac{dn_0}{d\upsilon'} = -\frac{n_0^2 \in}{2\, n_0 (k_1 - \in\upsilon') + \in\upsilon}, \tag{33}$$

$$\frac{dn_0}{dk_1} = -\frac{n_0 \in (-n_0\, \upsilon' + \upsilon)}{k_1\, [2n_0(k_1 - \in\upsilon') + \in\upsilon]}, \tag{34}$$

It is evident from equatin (32) and (33) that $dn_0/d\upsilon$ and $dn_0/d\upsilon'$ are both negative or positive depending upon whether $k_1 > \in\upsilon'$ or $k_1 < \in\upsilon'$. The growth rates , therefore, decrease as well as increase with the increase in kinematic viscosity and kinematic viscoelaticity of the fluid.

It is clear from equation (34) that in adition to the condition $k_1 > \in\upsilon'$, we have the condition $\upsilon > n\upsilon'$,for which dn_0/dk_{10} is negative. Thus the growth rates decrease with the incease in medium permeability due to the presence of viscoelasticity .

i.e.for $\dfrac{k_1}{\in} < \upsilon' < \dfrac{\upsilon}{n}$. (35)

However ,the growth rates increase with the increase in medium permeability for

$k_1 < \in\upsilon'$ and $\upsilon < \eta\upsilon'$ i.e. $\upsilon' > \upsilon/\eta$ and $\upsilon' > k_1/\eta$,

or $k_1 > \in\upsilon'$ and $\upsilon > \eta\upsilon'$,

i.e. $\upsilon' > \upsilon/\eta$ and $\upsilon' > k_1/\eta$, (36)

We thus conclude the whole analysis with the following statements. The criteria determining stability or instability are independent of the effects of viscosity , viscoelasticity and medium permeability . The viscosity, visvcoelacity and the medium permeability have damping as well as enhancing effects on the growth rates .

5. Effect of Rotation.

5.1 Perturbation Equation. Here we concider the effect of rotation on the character of the equilibrium of a stratified viscoelastic Walters' (model B') fluid . The fluid is acted on by a uniform rotation with an angular velocity $\vec{\Omega}\,(0,0,\Omega)$ and gravity force $\vec{g}\ (0,0,-g)$. Then the linearized pertubation equations of motion are

$$\frac{\rho}{\in}\left[\frac{\partial}{\partial t} + \frac{1}{k_1}\left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\right]\vec{V} = -\nabla\delta p + \vec{g}\,(\delta\rho) - \frac{2\rho}{\in}\,(\vec{V}\times\vec{\Omega})\,, \tag{37}$$

together with equations (5) and (6).

Equations (5), (6) and (37), using expression (7), give

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon' n)\right]u=-ik_x\ \delta\rho+\frac{2\rho}{\epsilon}\Omega\, v\,, \tag{38}$$

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon' n)\right]v=-ik_y\ \delta\rho+\frac{2\rho}{\epsilon}\Omega\, u, \tag{39}$$

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon' n)\right]w=-D\delta\rho-g\,(\delta p), \tag{40}$$

$$ik_x u+ik_y v+Dw=0, \tag{41}$$

$$\epsilon n\delta\rho=-\,wD\rho, \tag{42}$$

Eliminating u, v and δp from equations(38)-(40)and using equations(41), (42), after a little algebra, we get

$$D\left[\frac{\rho}{\epsilon}\left\{n+\frac{\epsilon}{k_1}(\upsilon-\upsilon' n)\right\}\left\{1+\frac{4\Omega^2}{\left\{n+\frac{\epsilon}{k_1}\overline{\upsilon-\upsilon' n}\right\}^2}\right\}Dw\right]$$

$$=\frac{\rho}{\epsilon}\left\{n+\frac{\epsilon}{k_1}\ (\upsilon-\upsilon' n)\right\}\ k^2w-gw\,(D\rho)k^2/\epsilon n\,. \tag{43}$$

5.2 Two Uniform Fluids Separated by a Horizontal Boundary. Consider two uniform fluids of constant densities ρ_1 and ρ_2 superposed one over the other separated by a horizontal baundary $z=0$ .Then in each region of constant ρ, υ and υ'; equation (43) reduces to

$$(D^2-\kappa^2)\,w=0, \tag{44}$$

where $\kappa^2=\dfrac{k^2}{\left[1+\dfrac{4\Omega^2}{\left\{n+\frac{\epsilon}{k_1}\overline{\upsilon-\upsilon' n}\right\}^2}\right]}.$

Integrating(43) between $z_0-\delta$ and $z_0+\delta$ and passing to the limit $\delta=0$, we obtain in view of (15), the jump condition

$$\Delta_0\left[\frac{\rho}{\epsilon}\left\{n+\frac{\epsilon}{k_1}(\upsilon-\upsilon' n)\right\}\left\{1+\frac{4\Omega^2}{\left\{n+\frac{\epsilon}{k_1}\overline{\upsilon-\upsilon' n}\right\}^2}\right\}Dw\right]=-\frac{gk^2\Delta_0(\rho)}{\epsilon n}\,w_0 \tag{45}$$

while the equation valid everywhere else $(z \neq z_0)$ is

$$D\left[\frac{\rho}{\in}\left\{n+\frac{\in}{k_1}(\upsilon-\upsilon' n)\right\}\left\{1+\frac{4\Omega^2}{\left\{n+\frac{\in}{k_1}\overline{\upsilon-\upsilon' n}\right\}^2}\right\}Dw\right]$$

$$=\frac{\rho}{\in}\left\{n+\frac{\in}{k_1}\ (\upsilon-\upsilon' n)\right\}\ k^2 w-\frac{gk^2\ (D\rho)}{\in n}\,w. \tag{46}$$

Applying the boundary condition (15), the solution appropriate for the two regions are

$$w_1 = Ae^{+kz}\ ,\ (z<0) \tag{47}$$

$$w_2 = Ae^{+kz}\ ,\ (z>0) \tag{48}$$

Applying the condition (45) to the solutions (47) and (48), we obtain the dispersion relation,

$$\left[1-\frac{\in}{k_1}(\alpha_1\upsilon_1'+\alpha_2\upsilon_2')\right]n^2+\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\,n-\frac{gk\,(\alpha_2-\alpha_1)}{\left[1+\frac{4\Omega^2}{\left\{n+\frac{\in}{k_1}\overline{\upsilon-\upsilon' n}\right\}^2}\right]^{1/2}}$$

$$=0. \tag{49}$$

If we let $n=n_0$ when $\Omega = 0$, equation (49), yields

$$\left[1-(\alpha_1\upsilon_1'+\alpha_2\upsilon_2')\frac{\in}{k_1}\right]\ n_0^2+\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\,n_0-gk\ \ (\alpha_2-\alpha_1)\ . \tag{50}$$

Using (50) in equation (49), we get, after some simplifications

$$\begin{aligned}&\left[\left(1-\frac{\in}{k_1}\upsilon'\right)\left\{\left(1-\frac{\in}{k_1}\overline{\alpha_1\upsilon_1'+\alpha_2\upsilon_2'}\right)n+\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\right\}^2\right]n^4\\&+\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\left[\frac{2\in}{k_1}\left(1-\frac{\in\upsilon'}{k_1}\right)n+\left(\frac{\in^2\upsilon^2}{k_1^2}+4\Omega^2\right)\right]\left[2n\ \left\{1-\frac{\in}{k_1}\right.\right.\\&\quad\left.\left.(\alpha_1\upsilon_1'+\alpha_2\upsilon_2')\right\}+\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\right]n^2-n_0^2\left[n_0\left\{1-\frac{\in}{k_1}\ \ (\alpha_1\upsilon_1'+\alpha_2\upsilon_2')\right\}\right.\\&\left.+\ \frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\right]=0.\end{aligned} \tag{51}$$

When $n_0^2 < 0$ and $k_1 < \in(\alpha_1\upsilon_1'+\alpha_2\upsilon_2')$ there is no change of sign in equation (51). Therefore, equation (51) does not involve a positive root of n i.e. all the values of n are either real, negative or complex conjugates with negative real parts. The systems is, therefore, stable. However, the system is unstable for $k_1 > \in (\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$.

When $n_0^2 > 0$ and $k_1 < \in(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$ the constant term in equation (51) is negative so there is at least one change of sign in equation (51) and allows at least one positive root of n, meaning there by instability of the system. However, the system is stable for $k_1 > \in(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$.

5.3. The Case of Exponentially Varying Stratifications

Using the strtifications of the form (22), equation (43) becomes

$$\left[n + \frac{\in_0\upsilon_0}{k_{10}} - \frac{\in_0\upsilon'_0}{k_{10}} n \right] (D^2 - \kappa^2) + \frac{g\kappa^2\beta}{n} w = 0. \tag{52}$$

The appropriate solution of equation (52) is given by

$$w = Ae^{m_1z} + Be^{m_2z}, \tag{53}$$

where A, B are arbitrary constants; m_1, m_2 are given by

$$m_{1,2} = \pm\,\kappa \left[1 - \frac{g\beta}{nL} \right]^{1/2}, \tag{54}$$

and $L = \left[n \left(1 - \frac{\in_0\upsilon'_0}{k_{10}} \right) + \frac{\in_0\upsilon_0}{k_{10}} \right]$.

Inserting the values of m_1, m_2 from (54) in equation (28), we obtain

$$1 - \frac{g\beta}{nL} = -\frac{s^2\pi^2}{\kappa^2 d^2}, \tag{55}$$

which on simplification gives

$$n^3 + \frac{2\upsilon\in}{(k_1 - \upsilon'\in)} n^2 + \left[\frac{\upsilon^2\in^2}{(k_1 - \upsilon'\in)^2} + \frac{s^2\pi^2}{k^2d^2 + s^2\pi^2} \frac{4\Omega^2 k_1^2}{(k_1 - \upsilon'\in)^2} - \frac{g\beta k^2 d^2}{k^2d^2 + s^2\pi^2} \frac{k_1^2}{(k_1 - \upsilon'\in)^2} \right] n - \frac{g\beta\upsilon\in}{k_1} = 0. \tag{56}$$

If $\beta < 0$ and $k_1 > \in\upsilon'$ (stable stratification), equation (56) does not allow any positive root of n and so the system is always stable for disturbances of all wave numbers and the system is stable for $k_1 > \in\upsilon'$ and unstable for $k_1 < \in\upsilon'$. This is in contrast to the Newtonian fluid where the system is always stable for stable stratifications (Chandrasekhar [1]).

if $\beta > 0$ and $k_1 > \in\upsilon'$ (unstable stratification), equation (56) has one change of sign. Therefore, the system is unstable for disturbances of all wave numbers.

6. Fffect of uniform magnetic field.

6.1 Perturbation equations

The initial stationary state whose stability we wish to examine is that of an incompressible, electrically conducting Walters' (model B') fluid through a homogeneous porous medium. A uniform horizontal magnetic field $\vec{H}$ $(H,0,0)$ pervades the system. The relevant equations governing the fluid and Maxwell's equations through a porous medium, are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \vec{V}}{\partial t}+\frac{1}{\epsilon}(\vec{V}.\nabla)\vec{V}\right]=-\nabla p+\frac{\rho}{k_1}\left[\upsilon-\upsilon'\frac{\partial}{\partial t}\right]\vec{V}+\rho\vec{g}+\frac{\mu_e}{4\pi}(\nabla\times\vec{H})\times\vec{H}, \tag{57}$$

$$\nabla.\vec{H}=0, \tag{58}$$

$$\epsilon\frac{\partial H}{\partial t}=\nabla\times(\vec{V}.\ H)\ \vec{V}\ , \tag{59}$$

together with equations (2) and (3).

Let $\vec{V}(u,\upsilon,w)$, $\vec{h}\,(h_x, h_y, h_z)$, $\delta\rho$ and δp denote the perturbations in velocity (initially zero), magnetic field $\vec{H}$, pressure p and density ρ respectively. Then the linearized perturbation equations governing the motion of the fluid and Maxwell's equations are given by

$$\frac{\rho}{\epsilon}\frac{\partial \vec{V}}{\partial t}=-\nabla\delta p+\frac{\rho}{k_1}\left[\upsilon-\upsilon'\frac{\partial}{\partial t}\right]\vec{V}+\vec{g}\,\delta\rho+\frac{\mu_e}{4\pi}(\nabla\times\vec{h})\times\vec{H}, \tag{60}$$

$$\nabla.\vec{h}=0, \tag{61}$$

$$\epsilon\frac{\partial\vec{h}}{\partial t}=(\nabla.\vec{H})\ \vec{V}, \tag{62}$$

togethr with equations (5) and (6), μ_e stands for the magnetic permeability. Equations (5), (6) and (60)-(62), using expression (7), give

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right]u=-ik_x\ \delta p, \tag{63}$$

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right]\upsilon=-ik_y\ \delta p+\frac{\mu_e H}{4\pi}\left[\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}\right] \tag{64}$$

$$\frac{\rho}{\epsilon}\left[n+\frac{\epsilon}{k_1}(\upsilon-\upsilon'n)\right]w=-D\delta p-g\,(\delta\rho)+\frac{\mu_e H}{4\pi}\left[\frac{\partial h_z}{\partial x}-\frac{\partial h_x}{\partial z}\right] \tag{65}$$

$$\frac{\partial h_x}{\partial x}-\frac{\partial h_y}{\partial y}+Dh_z=0, \tag{66}$$

$$\epsilon n\,h_x=ik_x Hu, \tag{67}$$

$$\epsilon n\,h_y=ik_x H\upsilon, \tag{68}$$

$$\in n\, h_z = ik_x Hw. \tag{69}$$

Eliminating u, v, h_x, h_y, h_z and δp from equation (63) – (65) and using equations (66)–(69), after a little algebra, we get

$$D\left[\frac{\rho}{\in}\left\{\left(1-\frac{\in\upsilon'}{k_1}\right)n+\frac{\in\upsilon}{k_1}\right\}Dw\right]-k^2\left[\frac{\rho}{\in}\left\{\left(1+\frac{\in\upsilon'}{k_1}\right)n+\frac{\in\upsilon}{k_1}\right\}w\right]$$

$$+\frac{\mu_e k_x^2 H^2}{4\pi\in n}(D^2-k^2)\,w=-\frac{gk^2}{\in n}(D\rho)\,w\,. \tag{70}$$

6.2 Two Uniform Fluids Separated by a Horizontal Boundary.

Let two uniform fluids of densities ρ_1 and ρ_2 be saparated by a horizontal boundary at $z = 0$. Then in each region of constant $\rho, \upsilon, \in$, k_1 and υ' equation (70) reduces to

$$(D^2-k^2)\,w=0. \tag{71}$$

Equation (69) implies that the continuity of w ensures the continuity of h_z.

(72)

Integrating (70) between $z_0 - \delta$ and $z_0 + \delta$ and passing to the limit $\delta = 0$, we obtain, in view of (72), the jump condition

$$\Delta_0\left[\frac{\rho}{\in}\left\{\left(1-\frac{\in\upsilon'}{k_1}\right)n+\frac{\in\upsilon}{k_1}\right\}Dw\right]+\frac{\mu_e k_x^2 H^2}{4\pi\in n}\Delta_0(Dw)=-\frac{gk^2}{\in n}\Delta_0(\rho)\,w_0. \tag{73}$$

Applying the solutions given by equations (18) and (19) to (73), we obtain the dispersion relation

$$n^2\left[1-\frac{\in}{k_1}(\alpha_1\upsilon_1'+\alpha_2\upsilon_2')\right]+n\left[\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\right]+2k_x^2\,V_A^2-gk\,(\alpha_2-\alpha_1)=0, \tag{74}$$

where $V_A^2=\dfrac{\mu_e H^2}{4\pi(\rho_1+\rho_2)}$ is the square of the Alfven veloctiy.

Equation (74) yields

$$n=\frac{-\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\pm\left[\left\{\frac{\in}{k_1}(\alpha_1\upsilon_1+\alpha_2\upsilon_2)\right\}^2-4\left\{1-\frac{\in}{k_1}(\alpha_1\upsilon'_1+\alpha_2\upsilon'_2)\right\}\right.}{2}$$

$$\frac{\left.\{2k_x^2\,V_A^2-gk(\alpha_2-\alpha_1)\}\right]^{1/2}}{\left\{1-\frac{\in}{k_1}(\alpha_1\upsilon'_1+\alpha_2\upsilon'_2)\right\}}. \tag{75}$$

Several cases of interest are now considered.

(i) When $k_x = 0$, equation (75) gives

$$n = \frac{-\frac{\epsilon}{k_1}(\alpha_1\upsilon_1 + \alpha_2\upsilon_2) \pm \left[\left\{\frac{\epsilon}{k_1}(\alpha_1\upsilon_1 + \alpha_2\upsilon_2)\right\}^2 - 4\left\{1 - \frac{\epsilon}{k_1}(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)\right\}gk(\alpha_2 - \alpha_1)\right]^{1/2}}{2\left[1 - \frac{\epsilon}{k_1}(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)\right]}.$$

When $\rho_2 > \rho_1$ and $k_1 > \epsilon(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$, one of the values of n is positive which means that perturbations grow with time and so the system is unstable. When $\rho_2 < \rho_1$ and $k_1 > \epsilon(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$ both values of n are either real, negative or complex conjugates with negative real parts.

The system is therefore stable. However, the system is unstable for $k_1 < \epsilon(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$. Thus for potentially stable configuration, the system is stable for $k_1 > \epsilon(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$ and ustable for $k_1 < \epsilon(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$. This is in contrast to the Newtonian fluid where the system is always stable for potentially stable configuration.

For the potentially unstable configuration $\rho_2 > \rho_1$ and $k_1 < \epsilon(\alpha_1\upsilon'_1 + \alpha_2\upsilon'_2)$, the system is stable or unstable according as

$$2k_x^2\, V_A^2 \gtrless gk(\alpha_2 - \alpha_1)$$

$$\text{i.e. } \rho_2 - \rho_1 \lessgtr \frac{\mu_e k_x^2 H^2}{4\pi gk}. \tag{77}$$

The system is clearly unstable in the absence of a magnetic field. However, the system can be completely stabilized by large enough magnetic field as can be seen from equation (75), if

$$\rho_2 - \rho_1 < \frac{\mu_e k_x^2 H^2}{4\pi gk} \; :$$

The magnetic field, thus, completely stablizes the wave number band $k > k^*$,

$$k^* = \frac{2\pi g(\rho_2 - \rho_1)}{\mu_e H^2}\sec^2\theta, \tag{78}$$

where θ is the inclination of the wave-vector $\vec{k}$ to the direction of magnetic field $\vec{H}$.

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RECIPROCAL CONTINUITY AND FIXED POINT

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ABSTRACT

The aim of this paper is to obtain common fixed point theorems for reciprocally continuous mappings. We studied some contractive definitions which are strong enough to generate a common fixed point, but which does not force the mappings to be continuous at the common fixed points. These theorems will thus provide a common fixed point when all the mappinsgs may be discontinuous.

1. Introduction. In recent years a number of interesting results concerning four mappings or a sequence of mappings have been obtained by various authors. The common fixed point theorems for four mappings require each mapping to be compatible or weakly commuting with one or more mappings. Similarly, the theorems concerning sequences of mappings generally require each mapping to satisfy a compatibility condition, a condition on its range and a strong type continuity condition. The most generally common fixed point theorems concerning sequences of mappings are those due to Jachymaski [2] and Pant [7].

In 1986 Jungck [3] introduced the notion of compatibility.

If (X,d) be a metric space, two self maps A and S of X are called compatible if $Lim_n\, d(ASx_n, SAx_n) = 0$ when $\{x_n\}$ is a sequence such that $lim_n Ax_n = lim_n Sx_n = t$ for some t in X.

Two selfmaps A and S are said to be R-weakly commuting [6] at a point x in X if $d\,(ASx, SAx) \leq R\, d(Ax, Sx)$ for some $R > 0$. It is obvious that A and S can fail to pointwise R-weakly only if there is some x in X such that $Ax = Sx$ but $ASx \neq SAx$.

We discuss problem when there exists a contractive definition which is strong enough to generate a fixed point but which does not force the maps to be continuous is still problem [5]. It may be observed in this context that due to Kannan [5] there exist maps which have a discontinuity in their domain but they have fixed points. However, in all the cases maps involved were continous at the fixed point. However, in all the cases maps involved were continuous at the fixed points. In the context of common fixed point op pairs of mappings there is an approach to deal with this problem. This approach is to weaken the notion of continuity and define a condition which is applicable to discontinuous mappings also.

This approach has been adopted by Pant ([6], [7]) in his recent studies, and he introduced the notion of reciprocally continuous mappings.

Two self maps A and S of a metric space (X,d) is called reciprocally continuous if $lim_n\ ASx_n = At$ and $lim_n\ SAx_n = St$ whenever $\{x_n\}$ is a sequence such that $lim_n\ Ax_n = lim_n\ Sx_n = t$ for some t in X.

In the setting of common fixed point theorems for compatible mappings satisfying contractive conditions, continuity of one of the mappings A and S implies their reciprocal continuity but not conversely. We now prove common fixed point theorems for four mappings A,B,S and T for the purpose of this theorem we have the main result

$$M\ (x,y) = max\ \{d\ (Sx,\ Ty),\ d(Ax,\ Sx),\ d(By,\ Ty)\}.$$

Theorem 1 : Let (A,S) and (B,T) be pointwise R-weakly commuting pairs of selfmappings of a complete metric space (X,d) such that

(i) $AX \subset TX,\ BX \subset SX$

(ii) $d\ (Ax,By) \leq h\ M\ (x,y),\ 0 \leq h < 1,\ x,y \in X.$

Suppose that one of (A,S) and (B,T) be a compatible pair of reciprocally continuous mappings. Then A,B,S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \tag{1}$$

This can be done by virtue of (i). Then using (ii) we obtain

$$d(y_{2n}, y_{2n-1}) \leq hd(y_{2n-1}, y_{2n}),\ d\ (y_{2n-1}, y_{2n}) \leq hd\ (y_{2n-2}, y_{2n-1})$$

that is, $d\ (y_n, y_{n+1}) \leq\ hd\ (y_{n-1}, y_n) \leq\ ...\ \leq h^n d\ (y_0, y_1)$

Moreover, for any integer $p > 0$, we get

$$\begin{aligned} d\ (y_n, y_{n+p}) &\leq d\ (y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+p+1}, y_{n+p}) \\ &\leq (1+h+h^2+...+h^{p-1})\ d(y_n, y_{n+1}) \\ &\leq (1/(1-h))\ h^n d(y_0, y_1) \end{aligned}$$

This means that $d\ (y_n, y_{n+p}) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point z in X such that $y_n \to z$. Also $y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$. (2)
Suppose A and S are compatible and reciprocally continuous mappings. Then, reciprocal continuity of A and S implies that $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$. Compatibility of A and S yields $lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$, that is, $d(Az,Sz) = 0$. Hence $Az = Sz$. (3)
Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. Using (ii) we now get

$$d(Az, Bw) \leq h(M(z,w)).$$
$$= h(max\{d(Sz, Tw), d(Az, Sz), d(Bw, Tw)\}),$$
$$= h(d(Az, Bw)),$$

that is $Az = Bw$. Thus $Sz = Az = Tw = Bw$. Compatibility of A and S implies that $ASz = SAz$ and, therefore, $AAz = ASz = SAz = SSz$. Similarly, pointwise R-*weak* commutativity of B and T implies that there exists $R > 0$ such that $d(BTw, TBw) \leq R\ d(Bw, Tw) = 0$, that is, $BTw = TBw$ and $BBw = BTw = TBw = TTw$. Using (ii) we get

$$d(Az, AAz) = d(Bw, AAz) = d(AAz, Bw) \leq h\, M(Az, w)$$
$$< h(d\{(SAz, Tw), d(SAz, AAz)\, d(Bw, Tw)\}),$$
$$d(Az, AAz) \leq hd(Az, AAz),$$

$(1-h)\, d(Az, AAz) \leq 0$,. Hence $Az = AAz$ and $Az = AAz = SA\,z$. Thus Az is a common fixed point of A and S. Similarly, by using (ii) we obtain that Bw is a common fixed point of B and T. But $Az = Bw$. Hence Az is a unique common fixed point of A,S,B, and T. The proof is similar when B and T are assumed compatible and reciprocally continuous. This completes the proof of the theorem.

If $\{A_i\}$, $i = 1,2,3,\ldots,S$ and T be selfmappings of a metric space (X,d) in the sequel for each $i > 1$. We shall denote

$$M_{1i}(x,y) = max\{d(Sx, Ty), d(A_1x,Sx), d(A_iy, Ty)\}.$$

Corollary 1.1 . Let $\{A_i\}$, $i = 1,2,3, \ldots ,S$ and T be self mappings of a complete metric space (X,d) such that

(i)′ $A_1X \subset Tx$ and $A_2X \subset SX$

(ii)′ $d(A_1x, A_2y) \leq h(max\{d(Sx, Ty), d(A_1x, Sx), d(A_2y, Ty)\}, 0 \leq h<1$,

(iii)′ $d(A_1x, A_iy) < max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty)\}$,

Let S be compatible with A_1 and T be compatible with A_2. If the mappings in one of the compatible pairs (A_1,S) or (A_2,T) be reciprocally continuous then all the A_i, S and T have a unique common fixed point.

Proof. In theorem 5.1 we have discussed $\{x_n\}$ and $\{y_n\}$ in X are Cauchy sequences. Using the same argument in Theorem 5.1, $Sz = Az = Tw = Bw$. Now if $A_1z \leq A_iw$ for some $i > z$, (iii)$'$ yields

$d(A_1z, A_iw) < max\,(d(Sz, Tw), d(A_1z, Sz), d(A_iw, Tw)) = d\,(A_iw, A_1z)$,

a contradiction. Hence $A_1z = A_iw = Tw$. Compatibility of A_1 and S implies that $A_1Sz = SA_1z$ and, therefore, $A_1A_1z = A_1Sz = SA_1z = SSz$. Similarly, compatibility of A_2 and T implies that $A_2A_2w = A_2Tw = TA_2w = TTw$. If $A_1z \neq A_1A_1z$, Using (iii)$'$ we get

$$d(A_1z, A_1A_1z) = d\,(A_1A_1z, A_iw) < max\,(d(SA_1z, Tw)\, d(A_1A_1z, SA_1z), d(A_iw, Tw))$$
$$= d(A_1A_1z, A_iw),$$

a contradiction. Hence $A_1A_1z = A_1Sz = SA_1z = SSz = A_1z$, that is A_1z is a common fixed point of A_1 and S. Similarly A_2w is a common fixed point of A_2 and T. Hence $A_1z = A_2\,w$ is a common fixed point of A_1, A_2, S and T. Moreover, if $A_2w \neq A_iA_2w$ for some $i > 2$, using (iii)$'$ we have

$$d\,(A_1z, A_iA_2w) < d(A_1z, A_iA_2w),$$

a contradiction. Hence $A_1z = A_2w$ is a common fixed point of all $\{A_i\}$, S and T. Uniqueness of the common fixed point follows easily. The proof is similar when A_2 and T assumed reciprocally continuous.

We now give an example to illustrate Theorem 1.

Example 1.1 . Let $X = [2,20]$ and d be the usual metric on X. Define A, B, S and $T : X \to X$ by

$A2 = 2,$ $\quad Ax = 3 \text{ if } x > 2$
$S2 = 2,$ $\quad Sx = 6 \text{ if } x > 2$
$Bx = 2 \text{ if } x = 2 \text{ or } > 5,$ $\quad Bx = 6 \text{ if } 2 < x \leq 5$
$T2 = 2,\ Tx = 12 \text{ if } 2 < x \leq 5,\ Tx = x-3 \text{ if } x > 5$.

Then A, B, S and T satisfy all the conditions of Theorem 1 and have a unique common fixed point $x = 2$. It may be noted in this example that A and S are reciprocally continuous compatible maps. But neither A nor S is continuous, not even at their common fixed point $x = 2$. The mappings B and T are *R-weakly* commuting since they commute at their coincidence points. The ranges of A, B, S and T are complete subspaces of X. It is obvious in this example that all the mappings are discontinuous at the common fixed point $x = 2$.

In the next theorem we obtain a generalization of the above theorem replacing the Banach type contractive condition by a more general contractive condition that employs a contractive gauge function ϕ.

If A, B, S and T be self mappings of a metric space (X,d) we shall denote

$$M(x,y) = max\ \{d(Sx, Ty), d(Ax,Sx), d(By, Ty)\}.$$

Also, let $\phi : R_+ \to R_+$ denote an upper semicontinuous function such that $\phi(t) < t$ for each $t > 0$.

Theorem 2. Let (A,S) and (B,T) be pointwise *R-weakly* commuting pairs of self mappings of a complete metric space (X,d) such that

(i) $AX \subset TX$, $BX \subset SX$

(ii) $d(Ax,By) \leq \phi\ (M(x,y))$ whenever $M(x, y) > 0$.

Let S be compatible with A and T be compatible with B. If the mappings in one of the compatible pairs (A,S) or (B,T) be reciprocally continuous then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by

$$y_{2n} = Ax_{2n} = Tx_{2n+1},\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of (i), Then, using (ii) we obtain

(i) $d(y_{2n}, y_{2n+1}) \leq \phi\ (d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n})$ and

(ii) $d(y_{2n-1}, y_{2n}) \leq \phi\ (d(y_{2n-2}, y_{2n-1}) < d(y_{2n-2}, y_{2n-1})$.

We thus see that $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \geq 0$. Suppose $r > 0$. Then relation (i) on making $n \to \infty$ and in view of upper semicontinuity of ϕ yields $r \leq \phi\ (r) < r$, a contradiction.

Hence $r = lim_{n\to\infty}\ d(y_n,y_{n+1}) = 0$. We thus show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist an $\in > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\in$. Since $lim_{n\to\infty}\ d(y_n, y_{n+1}) = 0$, there exist integers m_i satisfying $n_i < m_i < n_i+1$ such that $d(y_{n_i}, y_{m_i}) \geq \in$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}}-1) + d(y_{n_{i+1}}-1, y_{n_{i+1}})$$
$$< \in + d(y_{n_{i+1}}-1, y_{n_{i+1}}) < 2\in,$$

a contradiction. If m_i be the smallest integer such that $d(y_{n_i},y_{m_i}) \geq \in$, then

$$\in \leq d(y_{n_i},y_{m_i}) \leq d(y_{n_i},y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$
$$< \in + d(y_{m_{i-2}},y_{m_{i-1}}) + d(y_{m_{i-1}}, y_{m_i}).$$

That is there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that

$d(y_{n_i}, y_{m_i}) \geq \in$ and

(3) $\lim_{n_i \to \infty} d(y_{n_i}, y_{m_i}) = \in$.

Without loss of generality we can assume that n_i is odd and m_i even. Now, by virtue of (1), we have

$$d(y_{n_{i+1}}, y_{m_{i+1}}) \leq \phi(\max\{d(y_{n_i}, y_{m_i})\} .$$

Now on letting $n_i \to \infty$ and in view of (3) and upper semicontinuity of ϕ, the above relation yields $\in \leq \phi(\in) < \in$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is Complete, there exists a point z in X such that $y_n \to z$. Also

$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$.

Suppose A and S are compatible and reciprocally continuous mappings. Then, reciprocal continuity of A and S implies that $ASx_n \to Az$ and $SAx_n \to Sz$. Compatibility of A and S yields $\lim_n d(AS_n, SAx_n) = 0$, that is, $d(Az,Sz) = 0$. Hence $Az = Sz$. Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. Using (ii) we get

$$d(Az, Bw) \leq \phi(\max\{d(Sz,Tw), d(Az,Sz), d(Bw, Tw)\}$$
$$\leq \phi(d(Bw, Az) < d(Bw, Az),$$

a contradiction. Hence $Az = Bw$ and $Az = Bw = Sz = Tw$. Compatibility of A and S implies that $ASz = SAz$ and, therfore $AAz = ASz = SAz = SSz$. Similarly, pointwise *R-weak* commutativity of B and T implies that there exists $R > 0$ such that $d(BTw, TBw) \leq R\, d(Bw, Tw) = 0$, that is, $BTw = TBw$ and $BBw = BTw = TBw = TTw$. Using again (ii) we get

$$d(Az, AAZ) = d(Bw, AAz) = d(AAz, Bw)$$
$$\leq \phi(\max\{d(SAz, Tw), d(AAz, SAz), d(Bw, Tw\})$$
$$= \phi(d(AAz,Az)) < d(AAz, Az),$$

a contradiction. Hence $Az = AAz = ASz$. Thus Az is a common fixed point of A and S. Similarly by using (ii) we obtain Bw is a common fixed point of B and T. Hence $Az = Bw$ is a unique common fixed point of A,B,S and T. The proof is similar when B and T are assumed compatible and reciprocally continuous. This completes the proof of the theorem. To consider an example for this theorem, we may define (X, d), A, B,S and T as in the above example and let $\phi(t) = ht$, $0 \leq h < 1$, then we obtain the Theorem 1 as a special case of the present theorem.

Discussion

Theorem 3.3 of Jachymaski [2] assumes one of the mappings to be continuous while Theorem 5.1 of the same paper assumes S and T to be continuous. The main theorem of Rhoades *et. al.* [8] and Jungck *et al.* [4] requires S and T to be continuous. Likewise, the theorems of Fisher [1] and Pant [6], [7] assume on of the mappings to be continuous. Here it is clear from the above example 1.1 that the present theorems do not require one of the mappings in a compatible pair or any other mapping to be continuous.

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COINCIDENCE POINTS AND FIXED POINTS FOR HYBRID CONTRACTION

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ABSTRACT

Introducing a new class of mappings, sufficient conditions for the existance of coincidence and fixed point for hybrid contraction i.e. contraction involving single-valued and multivalued mappings are obtained. Our results extend, unify and generalize some results of Naidu [2] and Sastry, Naidu and Prasad [6].

Introduction. Hybrid contraction *viz.* cotractive conditon involving single-valued and multi valued mappings have recently been studied by Mukharjee [1], Naidu [2], Naimpally et. al. [3]. Rhoades et. al. [4]. Sastry et. al. [6] and Singh et. al. [7]. In this paper we consider a very general type of condition involving two pairs of mappings, where each pair consists of a single-valued and a multi-valued mapping (known as hybrid contraction) and proved some coincidence point theorems. Further we also give answer to the problem arose in [2]. Our results extend and generalize the results of Naidu [2] and Sastry et. al. [6].

Throughout this paper we adopt the following notations : (X,d) is a metric space, $CB(X)$ is the collection of all nonempty, bounded, closed subsets of X, h is the Hausdorff metric on X. i.e. for A, B in $CB(X)$, $h\ (A,B) = max \{\underset{x \in A}{Sup}\ d(x.\ B),\ \underset{y \in B}{Sup}\ d(y,A)\}$. $F.G$ are maps from X into $CB(X)$, for an integer $n \geq 2$ and $x \in X$, $F^n x = F(F^{n-1}x)$, S, T are self maps of X. R^+ is the set of all nonnegative real numbers and $\phi : R^+ \to R^+$ is an increasing function with $\phi(t^+) < t$ for all t in $(0,\infty)$.

First we give following definitions :

Definition 1. If for some $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that the sequence $\{y_n\}$ defined by $y_{2n+2} = Sx_{2n+1}$, $y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$, $(n = 0,1,2, \dots)$, then $O_{S,T}(x_0) = \{Tx_1, Sx_2, Tx_3, \dots\}$ is said to be orbit for $(F,G: S,T)$ at x_0.

Definition 2 . (X,d) is said to be $(F, G: S,T)$ orbitally complete with respect to x_0 or simply $(F, G : S. T, x_0)$- orbitally complete, iff every Cauchy sequence of the form $O_{S,T}(x_0)$ converges in X.

If we substitute $S = T$ in definitions 1 and 2 these definitions reduce respectively to definitions 1.1 and 1.2 of Singh et. al. [7] and if $S = T$ and $F = G$ then definitions 1 and 2 are the definitions 4 and 5 of Rhoades et. al. [4]. Following two simple Lemma will be used in proof of the main results of this paper.

Lemma 1. Every $(F, G : S,T, x_0)$-orbitally complete metric space is $(F; S, x_0)$- orbitally complete as well as $(G : T, x_0)$ - orbitally complete, but not conversely.

Lemma 2 . If X is $(F, G : S, T, x_0)$-orbitally complete then $CB(X)$ is also $(F,G:S,T, x_0)$-orbitally complete, i.e. every Cauchy sequence of the form $\{A_n\}$ where

$Tx_{2n-1} \in Fx_{2n} = Ax_{2n}$ and $Sx_{2n+1} \in Gx_{2n+1} = Ax_{2n+1}$, $(n = 0,1,2, \dots)$ Converges in $CB(X)$.

Example

Let $X = (0,1]$, $Sx = x^2$. $\qquad TX = x \ \forall\ x \in X.$

$Gx = [1/2, 1] \ \forall\ x \in (0,1/2]$, $\qquad Fx = [1/2, 1] \ \forall x \in (0,1/\sqrt{2}]$

$\quad = [x-1/2, 1] \ \forall\ x \in (1/2, 1]$ $\qquad = [x^2-1/2, 1] \ \forall\ x \in (1/\sqrt{2},1]$.

Let $x_0 = 1$, $x_n = 2n-1/2n$ for all $n = 1,2,3 \dots$. Then $O_{S,T}(x_0) = \{1, 1/2, 9/16, 5/6 \dots)$ clearly $O_{S,T}(x_0) \to 1$, X is $(F,G;S,T,x_0)$ -orbitally complete. Obviously X is not complete. It is also clear that X is $(G; S,x_0)$–orbitally complete as well as $(F;T,x_0)$-orbitally complete.

Main Results

Theorem 1. Suppose that $\lim (t - \phi(t)) = +\infty$, there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in X such that $Sx_{n+1} \in Fx_n$ and $Ty_{n+1} \in Gy_n$ $(n=0,1,2,\dots)$, X is $(F;S)$-orbitally complete and $(G;T)$ orbitally complete at x_0. Further suppose the inequality,

$$h(Fx, Gy) \leq \phi(max\{d(Sx, Gy). d(Ty, Fx)\}) \ldots \qquad (1.1)$$

holds. for all x,y in X. Then the sequences $\{Fx_n\}$ and (Gy_n) converge in $CB(X)$ to the same limit A for some A in $CB(X)$, $\{Sx \mid x \in X$ and $Sx \in Fx\}$ $= S(X) \cap A$ and $\{Ty \mid y \in X$ and $Ty \in Gy\} = T(X) \cap A$. Futher $Fx = A$ whenever $Sx \in Fx$ and $Gy = A$ whenever $Ty \in Gy$.

Proof. For a positive integer n let

$$\beta_n = Sup\{h(Fx_i, Gy_j) \mid 1 \leq i, j \leq n\}.$$

Let $\delta = max\{h(Fx_0, Fx_1), h(Gy_0, Gy_1)\}$. For positive integers i, j from inequality (1.1) we have

$$\begin{aligned} h(Fx_i, Gy_j) &\leq \phi(max\{d(Sx_i, Gy_j), d(Ty_j, Fx_i)\}) \\ &\leq \phi(max\{d(Fx_{i-1}, Gy_j), h(Gy_{j-1}, Fx_i)\}) \qquad (1.2) \end{aligned}$$

Hence $\beta_n \leq \phi(\beta_n + \delta)$ for all $n = 1,2\ldots$ Hence $(\beta_n + \delta) - \phi(\beta_n + \delta) \leq \delta$ for all $n = 1, 2, \ldots$ Since $t - \phi(t) \to +\infty$ as $t \to +\infty$, it follows that $\{\beta_n\}$ is bounded. Hence $Sup\{h(Fx_i, Gy_j) \mid i,j$ are positive integers$\}$ is finite. For a positive integer n, let

$$\gamma_n = Sup\{h(Fx_i, Gy_j) \mid i,j \geq n\}.$$

From inequality (1.2) it is evident that $y_n \leq \phi(\gamma_{n-1})$ for all $n = 1,2 \ldots$ Hence $y_n \leq \phi^n(\gamma_0)$ for all $n = 1,2 \ldots$ Since $\phi(t^+) < t$ for all $t \in (0,\infty)$ and $\phi(0) = 0$ it follows that $\phi^n(\gamma_0) \to 0$ as $n \to \infty$. Hence $\{\gamma_n\}$ converges to zero.

We have $h(Fx_i, Fx_j) \leq h(Fx_i, Gx_i) + h(Gx_i, Fx_j)$ for all positive integers i and j. Since $\{\gamma_n\}$ converges to zero, the R.H.S. of the above inequality tends to zero as i and j tend to $+\infty$. Hence $\{Fx_n\}$ is Cauchy. Since $CB(X)$ is orbitally complete, $\{Fx_n\}$ is convergent in CB(X). In a similar manner it can be shown that $\{Gy_n\}$ is also convergent in $CB(X)$. Since $\{h(Fx_n, Gy_n\}$ converges to zero, it follows that sequences $\{Fx_n\}$ and $\{Gy_n\}$ converge in $CB(X)$ to the same limit A for some A in $CB(X)$.

Let $u \in X$ be such that $Su \in A$. Then from inequality (1.1) it follows that $h(Fu_i, Gy_n) \leq \phi(max\{h(A, Gy_n), h(Gy_{n-1}, Fu)\})$ for all $n = 1,2 \ldots$ Taking limit as $n \to \infty$ in the above inequality we obtain $h(Fu, A) \leq \phi(h(Fu, A)+)$. Since $\phi(t+) < t$ for all t in $(0,\infty)$, it follows that $h(Fu, A) = 0$. Hence $Fu=A$. Suppose now that $\upsilon \in A$ is such that $S\upsilon \in F\upsilon$. Then from inequality (1.1) we have

$$h(F\upsilon, Gy_n) \leq \phi(max\{h(F\upsilon, Gy_n), h(Gy_{n-1}, F\upsilon)\})$$

for all $n = 1,2 \ldots$. Taking limit on both sides of the above inequality as

$n \to \infty$ we obtain $h(F\upsilon, A) \le \phi(h(F\upsilon, A)+)$. Hence $h(F\upsilon, A) = 0$. Hence $F\upsilon = A$ Thus we have shown that $\{Sx \mid x \in X$ and $Sx \in Fx\} = S(X) \cap A$ whenever $Sx \in Fx$. Similarly it can be shown that $\{Tx \mid x \in X$ and $Tx \in Gx\} = T(X) \cap A$ and $Tx = A$ whenever $Tx \in Gx$.

Corollary 1. Suppose that $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$, $S(X)$ and $T(X)$ are closed subsets of X. $Fx \subseteq Sx$ and $Gx \subseteq Tx$ for all x in X and inequality (1.1) holds for all x, y in X.

Then $\{Sx \mid x \in X$ and $Sx \in Fx\} = A = \{Tx \mid x \in X$ and $Tx \in Gx\}$ for some A in $CB(X)$ and $Fx = A = Gy$ for all x in $S^{-1}(A)$ and for all y in $T^{-1}(A)$.

Remark 1. Example 10 of Sastry, Naidu and Prasad [6] shows that in Corollary 1 (Theorem 1) condition $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$, cannot be dropped even when $\sum_{n=1}^{\infty} \phi^n(t) < +\infty$ for all t in R^+, $F = G$ and S, T are identity maps on X. It also shows that in Theorem 1 one cannot prove the convergence of the sequence $\{Fx_n\}$ in $CB(X)$ in the absence of this condition on ϕ.

Theorem 2 . Suppose that

$$h(Fx, Gy) \le \phi(\max\{d(Sx, Fx), d(Ty, Gy), d(Sx, Gy), d(Ty, Fx)\}) \quad (2.1)$$

for all x, y, in X. Suppose also that there is a sequence $\{X_n\}_{n=1}^{\infty}$ in X such that $Tx_{2n+1} \in FX_{2n}$, $SX_{2n+2} \in GX_{2n+1}$ $(n = 0, 1, 2, \ldots)$ such that X is (F, G, S, T, x_0) orbitally complete and the sequence $\{h(A_n, A_{n+1}\}$ converges to zero, where $A_{2n} = Fx_{2n}$ and $A_{2n+1} = Gx_{2n+1}$ $(n = 0, 1, 2, \ldots)$. Then $\{A_n\}$ converges in $CB(X)$ to some A in $CB(X)$, $\{Sx \mid x \in X$ and $Sx \in Fx\} = S(X) \cap A$ and $\{Tx \mid x \in X$ and $Tx \in Gx\} = T(X) \cap A$. Further $Fx = A$ whenever $Sx \in Fx$, and $Gy = A$ whenever $Ty \in Gy$.

Proof. If possible suppose that $\{A_n\}$ is not Cauchy in $CB(X)$. Then there exist a positive real number and strictly increasing sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that $n(k) < m(k), h(A_{n(k)}, A_{m(k)}) \ge \varepsilon$ and $h(A_{n(k)}, A_{m(k)-1}) < \varepsilon$ $(k = 1, 2, \ldots)$. Such a selection is possible since $\{h(A_n, A_{n+1})\}$ converges to zero. It is clear that $\lim_{k \to \infty} h(A_{n(k)}, A_{m(k)}) = \varepsilon = \lim_{k \to \infty} h(A_{n(k)}, A_{m(k)-1})$. Thus we have strictly increasing sequences of positive integers $\{p(k)\}$ and $\{q(k)\}$ such that $\lim_{k \to \infty} h(A_{2p(k)}, A_{2q(k)+1}) = \varepsilon$. For a positie integer k let.

$$\alpha_k = \max\{d(Sx_{2p(k)}, Fx_{2p(k)}), d(Tx_{2q(k)+1}, Gx_{2q(k)+1}) \\ d(Sx_{2p(k)}, Gx_{2p(k)+1}), d(Tx_{2q(k)+1}, Fx_{2p(k)})\}$$

and

$$\beta_k = max\,\{h(Fx_{2p(k)}, Gx_{2p(k)-1}), h(Fx_{2q(k)}, Gx_{2q(k)+1}), \\ h(Fx_{2p(k)}, Gx_{2q(k)+1}), h(Fx_{2q(k)}, Gx_{2q(k)+1})\}$$

Then $\alpha_k \le \beta_k$ for all $k = 1,2 \ldots$ and $\{\beta_k\}$ converges to ε. In view of inequality (2.1) we have

$$h\,(A_{2p(k)}, A_{2q(k)+1}) \le \phi\,(\alpha_k) \le \alpha_k \le \beta_k$$

for all $k = 1,2 \ldots$. Since $\{h\,(A_{2p(k)}, A_{2q(k)+1})\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ converge to ε, it follows that both $\{\phi\,(\alpha_k)\}$ and $\{\alpha_k\}$ converge to ε. This is a contradiction since ε is a positive real number and $\phi\,(t+) < t$ for all t in $(0,\infty)$. Hence $\{A_n\}$ is Cauchy in $CB(X)$. Since (X,d) is orbitally complete, $CB(X)$ is also orbitally complete. Hence $\{A_n\}$ converges in $CB(X)$ to some A in $CB(X)$. Let $u \in X$ be such that $Tu \in A$. Taking $x = x_{2n}$ and $y = u$ in inequality (2.1) we see that $h\,(A_{2n}, Gu) \le \phi\,(max\,\{h(A_{2n}, A_{2n-1}), h\,(A, Gu), h\,(A_{2n-1}, Gu), h\,(A, A_{2n})\})$ for all $n = 1,2, \ldots$. Taking limit $n \to \infty$ in the above inequality we obtain $h\,(A, Gu) \le \phi\,(h(A,Gu)^+)$. Since $\phi(t+) < t$ for all t in $(0,\infty)$, it follows that $h(A, Gu) = 0$. Hence $Gu = A$. Hence $Tu \in Gu$.

Suppose now that $\upsilon \in X$ is such that $T\upsilon \in G\upsilon$. Taking $x = x_{2n}$ and $y = \upsilon$ in inequality (2.1) we see that

$h\,(A_{2n}, G\upsilon) \le \phi\,(max\,\{h(A_{2n}, A_{2n-1}), h\,(A_{2n-1}, G\,\upsilon), h\,(G\,\upsilon, A_{2n})\})$

for all $n = 1,2 \ldots$. Taking limit $n \to \infty$ in the above inequality we obtain $h\,(A, G\upsilon) \le \phi\,(h(A,G\upsilon)+)$. Hence $h\,(A,G\upsilon) = 0$. Hence $G\upsilon = A$. Hence $T\upsilon \in A$.

Thus we have shown that $\{Tx \mid x \in X$ and $Tx \in Gx\} = T(x) \cap A$ and $Gx = A$ whenever $Tx \in Gx$. In a similar manner it can be shown that $\{Sx \mid x \in X$ and $Sx \in Fx\} = S(X) \cap A$ and $Fx = A$ whenever $Sx \in Fx$.

Remark 2. Example 5 of Sastry and Naidu [5] shows that in Theorem 2 the condition "$\{h\,(A_n, A_{n+1}\}$ converges to zero" cannot be dropped even if $\phi(t) = t/2$ for all t in R^+, F and G are single valued maps, S and T are identity maps on X and (X,d) is a finite metric space. Example 6 of Sastry and Naidu [5] shows that the condition cannot be dropped even if one of the expressions $d(Sx,Fx)$, $d(Ty, Gy)$ on the R.H.S. of inequality (2.1) is deleted $\phi(t) = (3/4)t$ for all t in R^+, F and G are single valued, S and T are identity maps on X and (X, d) is a finite metric space.

Theorem 3. Suppose that $\lim\limits_{t \to +\infty} [t - \phi(t)] = +\infty$ and

$$h(Fx, Fy) \le \phi(max\,\{d(Sx, Fx), d(Sy, Fy), d(Sx, Fy), d(Sy, Fx)\}) \quad (3.1)$$

for all x,y in X. Suppose also that there is a sequence $\{X_n\}$ in X such that $Sx_n \in Fx_{n-1}$ $(n = 1,2 \ldots)$ such that X is $(F; S,x_0)$-orbitally complete. Then $\{Fx_n\}$ converges in $CB(X)$ to A for some A in $CB(X)$, $\{Sx \mid x \in X$ and $Sx \in Fx\} = S(X) \cap A$ and $Fx = A$ whenever $Sx \in Fx$. If further S and F are weakly commutative (i.e., $\underset{y \in SFx}{Sup}\, d(Y, FSx) \le d(Sx, Fx)$ for all x in X), then $S^n z \in Fz$ for all $n = 0, 1, 2, \ldots$ and for all z in $S(X) \cap A$.

Proof. For a positive integer n let

$$\beta_n = Sup\,\{h(Fx_i, Fx_j) \mid 1 \le i \le j \le n\}.$$

Clearly $\{\beta_n\}$ is an increasing sequence of nonnegative real numbers. For $1 \le i, j \le n$ from inequality (3.1) we have

$$h(Fx_i, Fx_j) \le \phi(max\,\{h(Fx_{i-1}, Fx_i), h(Fx_{j-1}, Fx_j), h(Fx_{i-1}, Fx_j), h(Fx_{j-1}, Fx_i), \quad (3.2)$$

Hence $h(Fx_i, Fx_j) \le \phi(\beta_n + h(Fx_0, Fx_1)$ for all $i, j \in \{1, \ldots, n)$. Hence $\beta_n \le \phi(\beta_n + h(Fx_0, Fx_1))$ for all $n = 1,2, \ldots$. Hence $[\beta_n + h(Fx_0, Fx_1)] - \phi[\beta_n + h(Fx_0, Fx_1)] \le h(Fx_0, Fx_1)$ for all $n = 1, 2, \ldots$. Since $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$, it follows that $\{\beta_n\}$ is bounded. Hence $Sup\{h(Fx_i, Fx_j) \mid i, j = 0, 1,2,\ldots\} < +\infty$ For a non-negative integers n let

$\gamma_n = sup\,\{h(Fx_i, Fx_j) \mid i, j \ge n\}$.

Then $\{\gamma_n\}$ is a decreasing sequence of non-negative real numbers and hence converges to some non-negative real number γ . From inequality (3.2) it is evident that $\gamma_n \le \phi(\gamma_{n-1})$ for all $n = 1,2, \ldots$ .Taking limits on both sides of the above inequality as $n \to \infty$ we obtain $\gamma_n \le \phi(\gamma+)$. Since $\phi(t+) < t$ for all t in $(0, \infty)$, it follows that $\gamma = 0$. Hence $\{Fx_n\}$ is a Cauchy sequence in $CB(X)$. Since (X,d) is orbitally complete, $CB(X)$ is also orbitally complete. Hence there exist $A \in CB(X)$ such that $\{Fx_n\}$ converges in $CB(X)$ to A.

As in the proof of Theorem 2 it can be shown that $\{Sx \mid x \in X$ and $Sx \in Fx\} = S(X) \cap A$ and $Fx = A$ whenever $Sx \in Fx$..

Suppose now that F and S are weakly commutative. Let $z \in S(X) \cap A$. Then there exists $x \in X$ such that $z = Sx$ and $Sx \in Fx$. From the weakly commutativity of F and S it follows that $S^2x \in FSx$. *i.e.* $Sz \in Fz$. Hence $Fz = A = Fx$. Hence $z \in Fz$ and S maps $S(X) \cap A$ into itself. Hence $S^n Z \in$

$A = Fz$ for all $n = 1, 2, \ldots$

Remark 3. The last part of Theorem 3 is valid in the case of Theorem 1 and 2 also. Following question is raised in [2]. Does F admit a fixed point if

$$h(Fx, Fy) \leq \alpha\,(max\{d(x,y),\ d(x, Fy),\ d(y, Fx)\} \tag{4.1}$$

for all x, y in X and for some constant in $[0,1)$?

We observe that without some additional condition, F does not admit a fixed point under (4.1). To give a sufficient condition for existance of a fixed point, we first define-

Defination 3. Let $\{x_n\}$ be a sequence in X such that $x_n \in Fx_{n-1}$. Then $\{x_n\}$ is said to be asymptotically F-regular if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Theorem 4. Let X is T-orbitally complete [i.e. every sequence of the form $\{x_{n(i)} : x_{n(i)} \in Tx_{n(i)-1}\}$converges in X] and F be multi-valued mapping satisfying (4.1). Then T has a fixed point if there exists a sequence $\{x_n\}$ which is asymptotically F-regular.

Proof. It is nothing hard to prove, if $\{x_n\}$ is a sequence which is asymptotically F-regular, then $\{x_n\}$ is a Cauchy sequence. Since X, is T-orbitally complete, $x_n \to z$ now

$$\begin{aligned} d(x_n, Fz) \leq h\,(Fx_{n-1}, Fz) &\leq \alpha\ max\,\{d\,(x_{n-1},z),\ d(x_{n-1}, Fz),\ d(z, Fx_{n-1})\} \\ &\leq \alpha\ max\,\{d\,(x_{n-1},z),\ d(x_{n-1}, Fz),\ d(z, x_n)\} \end{aligned}$$

Letting $n \to \infty$. we have $d(z, Fz) \leq \alpha\, d(z, Fz) < d(z,Fz)$ a contradiction. Hence $z \in Fz$. This completes the proof.

Following definition is known :

Definition 4. A topological space is said to be Pseudo-compact if every real valued continuous function on it is bunded.

Every real valued continuous function on Psedo-compact topological space attains its bounds.

We have following corollaries to Theorem 2 and 3.

Crollary 2. Suppose that $X \times X$ is Pseudo-compact, the functions which map (x,y) to $h(Fx, Fy)$, $max\,\{d(Sx,Gy),\ d(Ty,Fx)\}$ are continuous on $X \times X$, and $h(Fx, Gy) < max\,\{d(Sx,Gy),\ d(Ty,Fx)\}$

for all x,y in X for which $Sx \notin Fx$ and $Ty \notin Gy$. Further suppose that there is a nonempty subset A of X such that $S(A)$ and $T(A)$ are closed and $Fx \subseteq S(A)$ and $Gx \subseteq T(A)$ for all x in A. Then either F and S or G and T have a coincidence point.

Corollary 3. Suppose that $X \times X$ is Pseudo-compact, F and S are continuous on X and $h\ (Fx, Fy) < max\ \{d(Sx, Fx), d(Sy, Fy), d(Sx, Fy), d(Sy, Fx)\}$ whenever x,y in X such that $Sx \notin Fx$, $Sy \notin Fy$. Suppose also that there is a nonempty subset A of X such that SA is closed and $Fx \subseteq SA$ for all x in A. Then F and S have a coincidence point.

Proof. Suppose that F and S have no coincidence point. Define $f\colon X \times X \to R$ as

$$f\ (x,y) = h\ (Fx, Fy)\ /max\ \{d\ (Sx, Fx), \ldots, d\ (Sy, Fx)\}.$$

Then f is continuous on $X \times X$ and $f(x,y) < 1$ for all x, y in X. Since $X \times X$ is Pseudo compact, f attains its supremum, say α over $X \times X$. Then $\alpha \in [0,1)$ and

$$h\ (Fx, Fy) \leq \alpha\ max\ \{d(Sx, Fx), \ldots., d(Sy, Fx)\}$$

for all x, y in X. The corollary is now evident from Theorem 3.

Remark 4. Theorem 1, 2 and 3 are the theorem 2 , 3 and 4 of Naidu [2] in complete metric spaces.

Remark 5. Theorem 4 is a suitable answer to the problem 1, raised in [2], under weaker assumptions.

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ON COMMON FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

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ABSTRACT

Defining a new class of mappings, an attempt is made for proving more general fixed points theorems in metric space. Our results extend and generalize essentially known results of Das and Naik, Fisher, Jungck, Rhoades, Sessa, Khan and Swaleh, Singh and Tiwari and several others.

1. Introduction. Rhoades, Sessa, Khan and Swaleh [13] proved following theorem:

Theorem 1. Let A, S and T be three selfmaps of a complete metric space (X, d) satisfying

$$d(Sx, Ty) \leq a_1 d(Sx, Ax) + a_2 d(Tx, Ax) + a_3 d(Sy,Ay) + a_4 d(Ty,Ay) + a_5 d(Sx,Ay) + a_6 d(Tx,Ay) + a_7 d(Sy,Ax) + a_8 d(Ty,Ax) + a_9 d(Ax,Ay). \quad \ldots(1.1)$$

for all x, y in X where $a_i = a_i(x,y)$, $i = 1, 2, \ldots, 9$, are nonnegative funcitons such that

$$\max\{\sup_{x,y\in X}(a_1+a_2+a_5+a_6), \sup_{x,y\in X}(a_3+a_4+a_7+a_8), \sup_{x,y\in X}(a_5+a_6+a_7+a_8+a_9)\} < 1 \quad (1.2)$$

A weakly commutes with S and T, (1.3)

there exists a sequence which is asymptotically S-regular and T-regular with respect to A, (1.4)

A is continuous (1.5)

then A, S and T have a unique common fixed point.

The authors also gave an example to show that the condition of continuity of the mapping A can not be dropped. But our claim is that the continuity of A is not required on the whole space. Hence, continuity of A can be replaced by asymptotic continuity which we have to define like orbital continuity due to Ciric [2].

Definition 1. (appears in [12]) Let A and S be two selfmaps of X and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be asymptotically S-regular with respect to A if

$d(Ax_n, Sx_n) \to 0$ as $n \to \infty$.

If A is the identity map of X, definition 1 becomes that of Engl [4].

Now we define.

(2) $0(A, x_n) = \{Ax_1, Ax_2, \ldots, Ax_n, \ldots\}$ is called asymptotic orbit of A, where $\{x_n\}$ is a sequence which is asymptotically S-regular with respect to A.

(3) A is said to be asymptotically continuous mapping if it is continuous on closure of $0(A, x_n)$, denoted by $\overline{0(A, x_n)}$.

Clearly continuous mapping is asymptotically continuous, but converse is not necessarily true. [See example 1].

(4) X is said to be A-asymptotically complete if every Cauchy sequence of the form $\{Ax_n\}$ converges in X.

(5) Sessa [14] : Two selfmaps A and S of X are said to be weakly commuting mappings if $d(SAx, ASx) \leq d(Ax, Sx)$ for all x in X.

Now we present our main results.

2. Main Results

Theorem 2. Let A, S and T be three selfmaps of a metric space (X, d) satisfying (1.1), (1.2), (1.3), (1.4) and

A is asymptotically continuous and X is A-asymptotically complete.. (2.1)

Then A, S and T have a unique common fixed point.

Proof. Let $\{x_n\}$ be a sequence satisfying (1.4). Using (1.1),

$$\begin{aligned} d(Ax_n, Ax_m) &\leq d(Ax_n, Sx_n) + d(Sx_n, Tx_m) + d(Tx_m, Tx_m) \\ &\leq d(Ax_n, Sx_n) + a_1 d(Sx_n, Ax_n) + a_2 d(Tx_n, Ax_n) + a_3 d(Sx_m, Ax_m) \\ &\quad + a_4 d(Tx_m, Ax_m) + a_5 d(Sx_n, Ax_m) + a_6 d(Tx_n, Ax_m) + a_7 d(Sx_m, Ax_n) \\ &\quad + a_8 d(Tx_m, Ax_n) + a_9 d(Ax_n, Ax_m) + d(Tx_m, Ax_m) \end{aligned}$$

where $a_i = a_i(x_n, x_m)$. With the use of triangle inequality, rearranging the terms and then taking the *Lim Sup* of both sides, we get $\{Ax_n\}$ is Cauchy sequence. Since X is asymptotically complete, Let $z = Lim\ Ax_n$. We also have, $Sx_n \to z$ and $Tx_n \to z$. Also using (2.1), $A^2x_n \to Az$, $ASx_n \to Az$ and $ATx_n \to Az$. Now

$d(SAx_n, Az) \leq d(SAx_n, ASx_n) + d(ASx_n, Az)$ whence $SAx_n \to Az$.

Similarly from (1.3) $TAx_n \to Az$ further, from (1.1) with $a_i = a_i(Ax_n, z)$.

$$\begin{aligned} d(Az, Tz) &\leq d(Az, SAx_n) + d(SAx_n, Tz) \\ &\leq d(Az, SAx_n) + a_1 d(SAx_n, A^2x_n) + a_2 d(TAx_n, A^2x_n) + a_3 d(Sz, Az) \\ &\quad + a_4 d(Tz, Az) + a_5 d(SAx_n, Az) + a_6 d(TAx_n, Az) + a_7 d(Sz, A^2x_n) \\ &\quad + a_8 d(Tz, A^2x_n) + a_9\, d(A^2x_n, Az) \\ &\leq d(Az, SAx_n) + a_1 d(SAx_n, A^2x_n) + a_2 d(TAx_n, A^2x_n) \\ &\quad + (a_3+a_4+a_7+a_8)\, max\{d(Az, Sz), d(Az, Tz)\} \\ &\quad + (a_5+a_6+a_7+a_8+a_9)\, max\, \{d(SAx_n, Az), d(TAx_n, Az), d(A^2x_n, Az)\}. \end{aligned}$$

Taking the *lim Sup*, we have

$$d(Az,Sz) \leq \underset{x,y\,\varepsilon X}{Sup}\, (a_3 + a_4 + a_7 + a_8)\ max\, \{d(Az,Sz), d(Az, Tz)\}$$

Similarly

$$d(Az,Tz) \leq \underset{x,y\,\varepsilon X}{Sup}\, (a_1 + a_2 + a_5 + a_6)\ max\, \{d(Az,Sz), d(Az, Tz)\}$$

Then from (1.2) it follows $Az = Sz = Tz$.

Again using (1.1) with $a_i = a_i(x_n, Ax_n)$,

$$\begin{aligned} d(Sx_n, TAx_n) \leq{}& a_1 d(Sx_n, Ax_n) + a_2\, d(Tx_n, Ax_n) + a_3 d(SAx_n, A^2x_n) \\ &+ a_4 d(TAx_n, A^2x_n) + (a_5+a_6+a_7+a_8+a_9)\, max\, \{d(SAx_n, A^2x_n), \\ &d(Tx_n, A^2x_n), d(SAx_n, Ax_n), d(TAx_n, Ax_n), d(Ax_n, A^2x_n)\}. \end{aligned}$$

Taking *Lim Sup* of both sides, yields

$$d(z,Az) \leq \underset{x,y\,\varepsilon X}{Sup}\, (a_5 + a_6 + a_7 + a_8 + a_9)\ d(z, Az) < d(z, Az),$$

which gives $z = Az$ and hence z is a common fixed point of A, S and T. Uniqueness follows easily. This completes the proof.

Replacing the asymptotic continuity of A with asymptotic continuity of S or T, we have the following theorem.

Theorem 3. Let A, S and T be three selfmaps of a metric space (X,d) satisfying condition (1.1), (1.4), $a_1 + a_2 + a_3 + a_4$ bounded on X and $\underset{x,y\,\varepsilon X}{Sup}(a_5 + a_6 + a_7 + a_8 + a_9) < 1$. If T is asymptotically continuous, weakly commuting with A and S; and X is A-asymptotically complete, then T has a fixed point.

An analogous theorem can be proved using asymptotic continuity of S instead of T. Note that in general, A, S and T not necessarily have a common fixed point (Example 3).

Theorem 4. Let $\{S_n\}$ be a sequence of selfmaps of a metrica space (X, d) and A, a selfmap satisfying with $i \neq j$,

$$d(S_i x, S_j y) \leq a_1 d(S_i x, Ax) + a_2 d(S_j x, Ax) + a_3 d(S_i y, Ay) + a_4 d(S_j y, Ay)$$
$$+ a_5 d(S_i x, Ay) + a_6 d(S_j x, Ay) + a_7 d(S_i y, Ax) + a_8 d(S_j y, Ax)$$
$$+ a_9 d(Ax, Ay)$$

for all x, y in X, where $a_k = a_k(x,y)$, $k = 1,2,...,9$, are nonnegative functions satisfying (1.2). If A weakly commutes with each S_n and there exists an asymptotically S_n-regular sequence with respect to A for every $n = 1,2,...$, such that A is asymptotically continuous and X is A-asymptotically complete. Then the family $\{A, S_1, S_2, ...\}$ has a uniqeue common fixed point.

Examples.

In support of theorem 2, we give following example:

(1) Let $X = [0,1)$ with usual metric d. Let $S = T$ and A, the selfmaps of X defined by

$$S(x) = x/(x+16) \text{ if } x \in [0, 1/2] \quad \text{and} \quad Ax = x/8 \text{ if } x \in [0, 1/2]$$
$$= 1/8 \quad \text{if } x \in (1/2, 1) \qquad\qquad = 1/4 \text{ if } x \in [1/2, 1]$$

Clearly A and S weakly commuting mappings. Let $x_n = 1/(n+2)$ (or any sequence $x_n \neq 0$, converging to 0), since $d(Sx_n, Ax_n) = x_n(x_n + 8)(x_n + 16)/8$, $\{x_n\}$is asymptotically S-regular w.r.t. A. Inequality (1.1) also holds for $a_9 = 1/2$, $a_i = 0$, for $i = 1,2,...,8$. X is asymptotically complete. Then all the assumptions of theorem 2 are satisfied of course assumptions of theorem 1 are not satisfied since A is discontinuous and X is not a complete metric space.

The idea of following example appears in [13],

(2) This example shows that conditions of asymptotic continuity of A, can not be dropped. Let $x = [0, 1]$ and $S = T$, $A: X \rightarrow X$ given by

$$Sx = 1/2 \text{ if } x = 0 \quad \text{and} \quad Ax = 1 \text{ if } x = 0$$
$$= x/4 \text{ if } x \neq 0 \qquad\qquad = x/2 \text{ if } x \neq 0$$

then except asymptotic continuity of A, all other assumptions of theorem 2 are satisfied with $a_9 = 1/2$, $a_i = 0$, for $i = 1,2,..., 8$. On the other hand A and S have no fixed point.

(3) Under the assumptions of theorem 3, A, S and T are not necessarily have a common fixed point.

Let $X = [0, 1]$, A, S and $T: X \rightarrow X$ given by

$Ax = 1$ if $x = 0$ and $Sx = 1/4$ if $x = 0$ $Tx = x/2$ for all x in X
$= x$ if $x \neq 0$ $= x/2$ if $x \neq 0$

Let $a_1 = a_9 = 1/2$ and $a_i = 0$ for $i = 2, 3, \ldots, 8$.
Then all the assumption of theorem 3 are satisfied, but 0 is not a fixed point of either S or A.

Remarks.

(1) Assuming continuity of A in place of asymptotic continuity and X is complete in place of asymptotic completeness, theroms 2,3 and 4 are the theorems 1,1.1 and 4.2 of [13].

(2) Theorem 2 may be regarded as an extension of the well known result of Hardy and Rogers [7], which considered the following condition :

$$d\{Tx,Ty\} \leq b_1 d(Tx, x) + b_2 d(Ty,y) + b_3 d(Tx,y) + b_4 d(Ty,x) + b_5 d(x,y)$$

for all x, y in X where $b_1 \geq 0$, $i = 1, 2, \ldots 5$, $b_1+b_2+b_3+b_4+b_5 < 1$

(3) Above contractive condition has been also used by Guay and Singh [6] for $max\ \{b_3+b_4+b_5+b_1+b_3\} < 1$.

(4) In Jungck [8], the continuity of the mapping $S = T$ is a consequence of his contractive condition and it is used in his proof. But in theorem 2, the continuity of his mappings S and T are neither assumed nor implied by the contractive condition (1.1).

(5) Das and Naik [3], generalize Jungck's theorem by considering the following condition for commuting mappings,

$$d(Sx, Sy) \leq c\ max\{d(Sx,Ax),\ d(Sy, Ay),\ d(Sx, Ay),\ d(Sy,Ax),\ d(Ax,Ay)\}$$

for all x, y in X where $0 \leq c \leq 1$. As indicated in Massa [9], this contractive condition is equivalent to the following condition,

$$d(Sx, Sy) \leq a_1 d(Sx,Ax)+a_2 d(Sy, Ay)+a_3 d(Sx, Ay)+a_4 d(Sy,Ax)+ a_5 d(Ax,Ay)\}$$

for all x, y in X, $a_i = a_i(x, y)$, $i = 1,2,\ldots,5$ and $\sup_{x,y\ \varepsilon X}(a_1+a_2+a_3+a_4+a_5) < 1$.
Clearly theorem 2 is a generalization of result of Das and Naik, which has been extended also by Fisher [5], Rhoades [11] and several others. We refer [13], for further details.

(6) Theorem 4 can be regarded as an improvement of theorem 1 of Singh and Tiwari [14].

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MULTIVALENT FUNCTIONS WITH NEGATIVE AND MISSING COEFFICIENTS

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ABSTRACT

Let $P_k(A, B, p, \lambda)$ be the class of functions $f(z) = z^p - \sum\limits_{n=k+1}^{\infty} |a_n| z^n$, $k \geq p \geq 1$ analytic and multivalent in the unit disc $U = \{z : |z| < 1\}$ and satisfying.

$$\left| \frac{(1-\lambda)\dfrac{f(z)}{z^p} + \lambda \dfrac{f'(z)}{pz^{p-1}} - 1}{B\left\{(1-\lambda)\dfrac{f(z)}{z^p} + \lambda \dfrac{f'(z)}{pz^{p-1}}\right\} - A} \right| < 1, \qquad z \in U,$$

where $-1 \leq B < A \leq 1$ and $0 < \lambda \leq 1$.

In this paper we obtain coefficient estimate, distortion, closure theorems and redius of convexity for the class $P_k(A, B, p, \lambda)$ under the assumptions $-1 \leq B < 0$. We also obtain class preserving integral operators of the form

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt, \qquad c > -p$$

for the class $P_k(A, B, p, \lambda)$. Conversely when $F(z) \in P_k(A, B, p, \lambda)$ radius of multivalence of $f(z)$ has been determined.

1. Introduction. Let S_p denote the class of functions of the form $f(z) = z^p + \sum\limits_{n=k+1}^{\infty} |a_n| z^n$, which are analytic and multivalent in the unit disc. $U = \{z : |z| < 1\}$ for $-1 \leq B < A \leq 1$ and $0 < \lambda \leq 1$ let $S_p(A, B, \lambda)$ be the class of those functions $f(z)$ of S_p satisfying

$$\left| \frac{(1-\lambda)\dfrac{f(z)}{z^p} + \lambda \dfrac{f'(z)}{pz^{p-1}} - 1}{B\left\{(1-\lambda)\dfrac{f(z)}{z^p} + \lambda \dfrac{f'(z)}{pz^{p-1}}\right\} - A} \right| < 1, \qquad z \in U,$$

Let T_p denote the subclass of S_p consisting of functions analytic and multivalent which can be expressed in the form

$$f(z) = z^p - \sum_{n=k+1}^{\infty} |a_n| z^n, \qquad (k \geq p \geq 1) \tag{1.1}$$

Gupta and Jain [2] obtained results for certain classes of univalent functions with negative coefficients. Vinod Kumar [3] studied an interesting class of univalent functions with negative as well as missing coefficients. Shukla and Dashrath [4] also obtained results on certain classes of multivalent functions with negative coefficients.

In fact, in this paper an attempt has been made to have a unified and detailed study of certain subclasses of multivalent functions. Our approach not only yields a generalisation of the various known results but also gives rise to new estimates.

For this, we define

$P_k(A, B, p, \lambda) = S_p(A, B, \lambda) \cap T_p$.

In this paper, under the assumption $-1 \leq B \leq 0$. , we obtain coefficient estimate, distortion and covering theorems and radius of convexity for the class $P_k(A, B, p, \lambda)$. We also determine the class preserving integral operators of the form

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt, \qquad c > -p \tag{1.2}$$

Conversely when $F(z) \in P_k(A, B, p, \lambda)$, radius of p-valence of $f(z)$ has been determined. Lastly we show that the class $P_k(A, B, p, \lambda)$ is closed under "arithmetic mean" and "convex linear combination". Throughout this paper we shall assume that $-1 \leq B \leq 0$ and $k \geq p \geq 1$.

2. Coeficient Estimate

Theorem 1. A function $f(z) = z^p - \sum_{n=k+1}^{\infty} |a_n| z^n$ is in $P_k(A, B, p, \lambda)$ if and only if

$$\sum_{n=k+1}^{\infty} (1-B)\left(1-\lambda+\frac{n\lambda}{p}\right)|a_n| \leq (A-B).$$

This result is sharp.

Proof. Let $|z| = 1$. Then

$$|_1(1-\lambda)\frac{f(z)}{z^p} + \lambda\frac{f'(z)}{pz^{p-1}} - 1|_1 - |_1 B\left\{(1-\lambda)\frac{f(z)}{z^p} + \lambda\frac{f'(z)}{pz^{p-1}}\right\} - A|_1$$

$$=\left| -\sum_{n=k+1}^{\infty} (1-\lambda+\frac{n\lambda}{p})\,|a_n|\,z^{n-p}\right| - \left|B\left\{1-\sum_{n=k+1}^{\infty} (1-\lambda+\frac{n\lambda}{p})\,|a_n|\,z^{n-p}\right\} -A\right|$$

$$\le \sum_{n=k+1}^{\infty}\left[(1-\lambda+\frac{n\lambda}{p}) - B\,(1-\lambda+\frac{n\lambda}{p})\right]|a_n| \quad -(A-B), \text{ (Since } -1\le B\le 0)$$

≤ 0, (by assumption) .

Hence, by maximum modulus principle, $f(z)\in P_k\,(A, B, p, \lambda)$. To prove converse, let

$$\left|\frac{(1-\lambda)\,\frac{f(z)}{z^p} + \lambda\,\frac{f'(z)}{pz^{p-1}} - 1}{\left\{B\quad (1-\lambda)\,\frac{f(z)}{z^p} + \lambda\,\frac{f'(z)}{pz^{p-1}}\right\} - A}\right|$$

$$=\left|\frac{-\sum_{n=k+1}^{\infty}(1-\lambda+\frac{n\lambda}{p})\,|a_n|\,z^{n-p}}{B - B\sum_{n=k+1}^{\infty}(1-\lambda+\frac{n\lambda}{p})\,|a_n|\,z^{n-p} - A}\right| < 1, \qquad z\in U,$$

Since $|Re(z)|\le |z|$ for all z, we have

$$Re\left\{\frac{-\sum_{n=k+1}^{\infty}(1-\lambda+\frac{n\lambda}{p})\,|a_n|\,z^{n-p}}{(A-B) + B\sum_{n=k+1}^{\infty}(1-\lambda+\frac{n\lambda}{p})\,|a_n|\,z^{n-p}}\right\} < 1. \tag{2.1}$$

Choose values of z on the real axis so that

$$\frac{\lambda f'(z)}{pz^{p-1}} + (1-\lambda)\,\frac{f(z)}{z^p} \text{ is real.}$$

Upon clearing the denominator in (2.1) and letting $z\to 1$ over real values, we obtain

$$\sum_{n=k+1}^{\infty}(1-\lambda+\frac{n\lambda}{p})\,|a_n| \le (A-B) + B\sum_{n=k+1}^{\infty}(1-\lambda+\frac{n\lambda}{p})\,|a_n|$$

This completes the proof of theorem.

Sharpness follows if we take

$$f(z) = z^p - \frac{(A-B)}{(1-B)\,(1-\lambda+\frac{n\lambda}{p})}\,z^n \;(n\ge k+1,\ k\ge p\ge 1).$$

3. Distortion Properties

Theorem 2. If $f(z) \in P_k(A, B, p, \lambda)$, then for $|z| = r$

$$r^p - \frac{(A-B)\,r^{k+1}}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})} \le |f(z)| \le r^p + \frac{(A-B)\,r^{k+1}}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})} \tag{3.1}$$

This inequality is sharp.

Proof. Let $f(z) = z^p - \sum\limits_{n=k+1}^{\infty} |a_n|\, z^n$

Then, it follows from Theorem 1 that

$$\sum_{n=k+1}^{\infty} |a_n| \le \frac{(A-B)}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})} \;.$$

Hence $|f(z)| \le r^p + \sum\limits_{n=k+1}^{\infty} |a_n|\, r^n < r^p + r^{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|$

$$\le r^p + \frac{(A-B)\,r^{k+1}}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})}$$

and $|f(z)| \ge r^p - \sum\limits_{n=k+1}^{\infty} |a_n|\, r^n \ge r^p - r^{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|$

$$\le r^p - \frac{(A-B)\,r^{k+1}}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})} \;.$$

Thus we get

$$r^p - \frac{(A-B)\,r^{k+1}}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})} \le |f(z)| \le r^p + \frac{(A-B)\,r^{k+1}}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})}$$

This completes the proof.

Sharpness follows if we take

$$f(z) = z^p - \frac{(A-B)}{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p})}\, z^{k+1} \tag{3.2}$$

Note **.** For the above function, equality on the left hand side of (3.1) is obtained at $z = r$ whereas on the right hand side equality is obtained at $z = -r$ when $k = p+1, p+3, p+5, \ldots$, $z = ir$ when $k = p+2, p+6, p+10, \ldots$ and $z = r\,e^{i\pi/(k-p)}$, when $k = p+4, p+8, p+12, \ldots$ and $0 \le \lambda < 1$.

Corollary 1. If $f(z) \in P_k(A, B, p, \lambda)$, then disk U is mapped by $f(z)$ onto a domain that contains the disc

$$|w| < \frac{(1-B)\,(1-\lambda+\frac{(k+1)\lambda}{p}) - (A-B)}{(1-B)\left\{1-\lambda+\frac{(k+1)\lambda}{p}\right\}}\,.$$

The result is sharp with extremal function $f(z)$ given by (3.2). The above corollary follows if we let $r \to 1$ in the left hand side inequality in (3.1). An interesting case appears when $A = 1$.

In this case

$$|w| < \frac{\frac{(k+1)\lambda}{p} - \lambda}{(1-\lambda+\frac{(k+1)\lambda}{p})}\,.$$

4. Integral Operators

Theorem 3. Let c be a real number such that $c > -p$. If $f(z) \in P_k\,(A, B, p, \lambda)$, then the funciton $F(z)$ defined by

$$F(z) = \frac{(c+p)}{z^c} \int_0^z t^{c-1} f(t)\, dt, \tag{4.1}$$

also belongs to $P_k\,(A, B, p, \lambda)$.

Proof. Let $f(z) = z^p - \sum\limits_{n=k+1}^{\infty} |a_n|\, z^n$. Then from the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=k+1}^{\infty} |b_n|\, z^n,$$

where $\quad |b_n| = \left(\frac{c+p}{c+n}\right) |a_n|\,.$

Therefore,

$$\begin{aligned}\sum_{n=k+1}^{\infty} (1-B)\,(1-\lambda+\frac{n\lambda}{p})\,|b_n| &= \sum_{n=k+1}^{\infty} (1-B)\,(1-\lambda+\frac{n\lambda}{p})\left(\frac{c+p}{c+n}\right)|a_n| \\ &\le \sum_{n=k+1}^{\infty} (1-B)\,(1-\lambda+\frac{n\lambda}{p})\;|a_n| \\ &\le (A-B),\end{aligned}$$

since, $f(z) \in P_k\,(A, B, p, \lambda)$.

Hence by Theorem 1, $F(z) \in P_k\,(A, B, p, \lambda)$.

Theorem 4. Let c be a real number such that $c > -p$. If $F(z) \in P_k\,(A, B, p, \lambda)$, then the function $f(z)$ defined by (4.1) is p-valent in $|z| < R^*$, where

$$R^* = \inf_{n \ge k+1,\, k \ge p \ge 1} \left[\frac{(1-B)\,(1-\lambda+\frac{n\lambda}{p})}{n\,(A-B)}\left(\frac{c+p}{c+n}\right)\right]^{1/(n-p)}$$

This result is sharp.

Proof. Let $F(z) = z^p - \sum\limits_{n=k+1}^{\infty} |a_n| z^n$. It follows then from (4.1) that

$$f(z) = \frac{z^{1-c}}{c+p}\ [z^c F(z)]' = z^p - \sum_{n=k+1}^{\infty}\left(\frac{c+n}{c+p}\right) |a_n| z^n.$$

In order to obtain the required result it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p \text{ in } |z| < R^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = - \sum_{n=k+1}^{\infty}\left(\frac{c+n}{c+p}\right) |a_n| z^{n-p}\Bigg| \le - \sum_{n=k+1}^{\infty}\left(\frac{c+n}{c+p}\right) |a_n|\ |z|^{n-p}$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p, \text{ if}$$

$$\sum_{n=k+1}^{\infty} n\left(\frac{c+n}{c+p}\right) |a_n| z^{n-p} \le p. \tag{4.2}$$

But Theorem 1 confirms that

$$\sum_{n=k+1}^{\infty} \frac{(1-B)\,(1-\lambda+\frac{n\lambda}{p})\,|a_n|}{(A-B)} \le p.$$

Hence (4.2) will be satisfied if

$$n\left(\frac{c+n}{c+p}\right)|a_n|\ |z|^{n-p} < \frac{p\,(1-B)\,(1-\lambda+\frac{n\lambda}{p})\,|a_n|}{(A-B)}\ ,$$

$$n = k+1,\ k+2,\ \ldots;\ k \ge p \ge 1$$

or if

$$|z| < \left[\frac{(1-B)\,(1-\lambda+\frac{n\lambda}{p})}{n\,(A-B)}\left(\frac{c+p}{c+n}\right).\right]^{1/(n-p)}$$

$$n = k+1,\ k+2,\ \ldots;\ k \ge p \ge 1$$

Therefore $f(z)$ is p-valent in $|z| < R^*$. Sharpness follow if we take

$$F(z) = z^p - \frac{(A-B)\,z^n}{(1-B)\,(1-\lambda+\frac{n\lambda}{p})},\ n \ge k+1;\ k \ge p \ge 1.$$

5. Radius of Convexity

Theorem 5. if $f(z) \in P_k\,(A, B, p, \lambda)$, then $f(z)$ is p-valently convex in the disc $|z|<R$, where

$$R = \inf_{n \geq k+1,\ k \geq p \geq 1} \left[\frac{(1-B)\left(1-\lambda+\frac{n\lambda}{p}\right)p^2}{(A-B)\,n^2} \right]^{1/(n-p)}.$$

The result is sharp.

Proof. In order to establish the required result it suffices to show that

$$\left| 1+\frac{zf''(z)}{f'(z)} - p \right| \leq p \text{ in } |z|<R.$$

Let

$$f(z) = z^p - \sum_{n=k+1}^{\infty} |a_n|\, z^n,$$

Then we have

$$\left| 1+\frac{zf''(z)}{f'(z)} - p \right| = \left| \frac{-\sum_{n=k+1}^{\infty} n(n-p)|a_n| z^{n-p}}{p - \sum_{n=k+1}^{\infty} n|a_n| z^{n-p}} \right|$$

$$\leq \frac{\sum_{n=k+1}^{\infty} n(n-p)|a_n| z^{n-p}}{p - \sum_{n=k+1}^{\infty} n|a_n| z^{n-p}}.$$

Therefore

$$\left| 1+\frac{zf''(z)}{f'(z)} - p \right| < p,$$

if $\sum_{n=k+1}^{\infty} n(n-p)\ |a_n|\ |z|^{n-p} < p^2 - \sum_{n=k+1}^{\infty} p\,n\,|a_n|\ |z|^{n-p}$

or if $\sum_{n=k+1}^{\infty} (n/p)^2\ |a_n|\ |z|^{n-p} < 1.$ (5.1)

Also, by Theorem 1, we have

$$\sum_{n=k+1}^{\infty} \left(1-\lambda+\frac{n\lambda}{p}\right)\frac{(1-B)\ |a_n|}{(A-B)} \leq 1.$$

Hence (5.1) will be satisfied if

$$(n/p)^2\ |a_n|\ |z|^{n-p} < \frac{(1-B)\left(1-\lambda+\frac{n\lambda}{p}\right)|a_n|}{(A-B)},$$

$$n = k+1,\ k+2,\ \ldots,\ k \geq p \geq 1$$

or if

$$|z| < \left[\frac{(1-B)(1-\lambda+\frac{n\lambda}{p})\ p^2}{n^2 (A-B)} . \right]^{1/(n-p)}$$

$$n = k+1,\ k+2,\ \ldots, k \geq p \geq 1$$

Therefore $f(z)$ is p-valently convex in $|z| < R$. Sharpness follows if we take

$$F(z) = z^p - \frac{(A-B)\, z^n}{(1-B)(1-\lambda+\frac{n\lambda}{p})} . \qquad n \geq k+1,\ k \geq p \geq 1.$$

6. Closure Properties

In this section we show that the class $P_k\,(A, B, p, \lambda)$ is closed under "arithmetic mean" and "convex linear combinations".

Theorem 6. Let $f_j(z) = z^p - \sum_{n=k+1}^{\infty} |a_{nj}| z^n\ j = 1, 2, \ldots m$.

If $f_j(z) \in P_k\,(A, B, p, \lambda)$ for each $j = 1, 2, \ldots, m$, then function

$$h(z) = z^p - \sum_{n=k+1}^{\infty} |b_n|\, z^n$$

also belongs to $P_k\,(A, B, p, \lambda)$, where $b_n = \frac{1}{m} \sum_{j=1}^{m} a_{nj}$.

Proof. Since $f_j(z) \in P_k\,(A, B, p, \lambda)$, it follows from Theorem 1 that

$$\sum_{n=k+1}^{\infty} (1-B)(1-\lambda+\frac{n\lambda}{p}) \quad |a_{nj}| \leq (A-B),\ j = 1, 2, \ldots m.$$

Therefore

$$\sum_{n=k+1}^{\infty} (1-B)(1-\lambda+\frac{n\lambda}{p})\, |b_n| \leq \sum_{n=k+1}^{\infty} \left[(1-B)(1-\lambda+\frac{n\lambda}{p})(\frac{1}{m} \sum_{j=1}^{m} |a_{nj}|) \right]$$

$$\leq (A-B).$$

Hence, by Theorem 1, $h(z) \in P_k\,(A, B, p, \lambda)$.

Theorem 7. Let $f_p(z) = z^p,\ f_n(z) = z^p - \frac{(A-B)\, z^n}{(1-B)(1-\lambda+\frac{n\lambda}{p})}$

$$(n \geq k+1,\ k \geq p \geq 1$$

Then $f(z) \in P_k\,(A, B, p, \lambda)$, if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{n=k+1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geq 0$ and $\lambda_p + \sum_{n=k+1}^{\infty} \lambda_n = 1$.

Proof. Let us assume that

$$f(z) = \lambda_p f_p(z) + \sum_{n=k+1}^{\infty} \lambda_n f_n(z) = z^p - \sum_{n=k+1}^{\infty} \frac{(A-B)}{(1-B)(1-\lambda+\frac{n\lambda}{p})} \lambda_n z^n$$

Then

$$\sum_{n=k+1}^{\infty}\left[(1-\lambda+\frac{n\lambda}{p})(1-B)\left\{\frac{(A-B)}{(1-B)(1-\lambda+\frac{n\lambda}{p})}\right\}\lambda_n\right] \le (A-B) \sum_{n=k+1}^{\infty}\lambda_n$$

$$\le (A-B)$$

Hence, by Theorem 1, $f(z) \in P_k(A, B, p, \lambda)$, It follows then from Theorem 1 that

$$|a_n| \le \frac{(A-B)}{(1-\lambda)(1-\lambda+\frac{n\lambda}{p})}, \quad n = k+1, k+2, \ldots, k \ge p \ge 1 .$$

Setting $\lambda_n = \dfrac{(1-B)(1-\lambda+\frac{n\lambda}{p})\ |a_n|}{(A-B)}, \quad n = k+1, k+2, \ldots, k \ge p \ge 1 .$

and $\lambda_p = 1 - \sum_{n=k+1}^{\infty} \lambda_n$

we have

$$f(z) = \lambda_p f_p(z) + \sum_{n=k+1}^{\infty} \lambda_n f_n(z)$$

This completes the proof of theorem.

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EXISTENCE OF MAXIMAL ELEMENTS AND APPLICATIONS

By

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Recall that a binary relation F on a set C is a subset of $C \times C$ or a mapping of C into itself. It is written yFx or $y \in Fx$ to mean that y stands in relation F to x.

A maximal element of F is a point x such that no point y satisfies $y \in Fx$, i.e., $Fx = \phi$. Thus the set of maximal elements is

$$\{x \in C : Fx = \phi\} = \bigcap_{x \in C} (C \setminus F^{-1}x),$$

where

$$F^{-1}x = \{y \in C : x \in Fy\}.$$

The following well known theorem is due to Sonnenschein [13].

Theorem *1 Let C be a compact convex subset of R^n and F a binary relation on C satisfying*

(i) $x \notin coFx$ for all $x \in C$ (co stands for convex hull),

(ii) if $y \in F^{-1}x$, then there exists some $x_1 \in C$ (possibly $x_1 = x$) such that $y \in intF^{-1}x_1$.

Then F has a maximal element.

Bergstrom proved the following [1].

Theorem *2 Let C be a nonempty, compact convex subset of R^n and $F: C \to 2^C$ a preference map satisfying*

(i) $x \notin coFx$ for each $x \in C$,

(ii) F is lower semicontinuous on C.

Then F has a maximal element.

We note that the results on maximal elements are useful in fixed point theory, variational inequalities, complementarity problems and best approximation (see [2] and [12]).

Ky Fan's Lemma [6]. *Let C be a nonempty compact convex subset of R^n and $F : C \to 2^C$ a multifunction such that*

(i) *Fx is convex for each $x \in C$,*

(ii) *F has open graph,*

(iii) *$x \notin Fx$ for each $x \in C$.*

Then F has a maximal element.

We prove the following for maximal elements :

Theorem 3. *Let C be a nonempty closed convex subset of a Hausdorff topological vector space X and $F{:}C \to 2^C$ satisfy*

(i) *$x \notin Fx$ for each $x \in C$,*

(ii) *Fx is closed for each $x \in C$,*

(iii) *$F^{-1}(y) = \{x \in C : y \in Fx\}$ is convex for each $y \in C$,*

(iv) *C can be covered by some finite number of closed sets $Fx_1, Fx_2, \dots, Fx_n$.*

Then F has a maximal element.

Proof.

Let $Fx \neq \phi$ for each $x \in C$. Define $G(x) = C \setminus Fx$. Then $G(x)$ is open for each $x \in C$ since Fx is closed. By (iv) $C = \cup_{i=1}^{n} Fx_i$, so $\cap_{i=1}^{n} Gx_i = \cap_{i=1}^{n} (C/Fx_i) = (\cup_{i=1}^{n} Fx_i)^c = \phi$. Therefore G is not a KKM-map.

Hence, there exists a finite set $\{x_1, x_2, \dots, x_k\}$ of C such that

$$z = \sum_{i=0}^{k} \lambda_i x_i \not\subset \bigcup_{1}^{k} Gx_i,$$

where $\lambda_i = 0$ and $\sum \lambda_i = 1$.

so $z \in \cap_{i=1}^{k} Fx_i$ and $x_i \in F^{-1}z$, $i = 1, 2, \dots, k$. Since $F^{-1}z$ is convex so

$$z = \sum_{i=0}^{k} \lambda_i x_i \in F^{-1}z$$

implying that $z \in Fz$, a contradiction to hypothesis (i). So $Fx = \phi$.

The following result is a corollary :

Corollary 1. *Let C be a nonempty closed convex subset of a topological vector space X and $F{:}C \to 2^C$ an upper semicontinuous convex-valued map. Further, assume that there is some finite subset B of C such that $Fx \cap B \neq \phi$ for every $x \in C$, and $x \notin Fx$ for each $x \in C$. Then F has a maximal element.*

Proof.

Let $G : C \to 2^C$ be defined by

$$Gx = F^{-1}x = \{y \in C : x \in Fy\}.$$

Since F is upper semicontinuous, each $G(x)$ is closed. Now, $G^{-1}(y) = (F^{-1}y)^{-1} = Fy$ is convex.

Also C can be covered by $\{G(x) : x \in B\}$, finitely many closed sets.

By hypothesis $x \notin Fx$ for $x \in C$. So by Theorem 3, F has a maximal element.

In the following, the KKM-map principle is applied :

Theorem 4. *Let C be a nonempty convex subset of a topological vector space X, and $F : C \to 2^C$ satisfy*

(i) $x \notin coFx$ *for each* $x \in C$,

(ii) *if* $y \in F^{-1}x$, *then there exists some* $x_1 \in C$ *(possibly* $x_1 = x$*) such that* $y \in intF^{-1}x_1$,

(iii) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C : y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

Proof. Define $G(x) = C \setminus int\, F^{-1}(x)$ for each $x \in C$. Then $G(x)$ is closed for each $x \in C$. We claim that G is a KKM-map. Let $z \in co(x_1, x_2, \ldots, x_n)$. *If* $z \notin \cup_{i=1}^{n} Gx_i$ then $z \notin Gx_i$, $i = 1, 2, \ldots n$, that is, $z \in F^{-1}x_i$, $i = 1, 2, \ldots, n$. Thus $x_i \in Fz$ and $z \in coFz$, contradiction to (i). Hence G is a KKM-map.

Now,

$$\begin{aligned} B &= \{x \in C : y \notin Fx \text{ for all } y \in D\} \\ &= \{x \in C : x \notin F^{-1}y \text{ for all } y \in D\} \\ &= \{x \in C : x \in Gy \text{ for all } y \in D\} \\ &= \cap_{y \in D} Gy \neq \phi, \end{aligned}$$

that is, $x_0 \in \cap_{y \in D} Gy$.

Thus $x_0 \notin F^{-1}y$ for all $y \in C$, that is, $y \notin Fx_0$ for all $y \in C$ and $Fx_0 = \phi$.

The following results are derived as corollaries :

Corollary 2. *Let C be a nonempty convex subset of a topological vector space E and $F : C \to 2^C$ satisfy:*

(i) $x \notin Fx$,

(ii) *Fx is convex for each* $x \in C$,

(iii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ *is open in C for each* $y \in C$,

(iv) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C : y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

The following is due to Lin [9,10]:

Corollary 3. *Let C be a nonempty subset of a topological vector space E and $F : C \to 2^C$ satisfy*

(i) *$x \notin co\, F(x)$ is convex for each $x \in C$,*

(ii) *$F^{-1}(y) = \{x \in C : y \in Fx\}$ is open for each $y \in C$,*

(iii) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C : y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

In case C is a nonempty compact convex subset of a topological vector space X and $F : C \to 2^C$ satisfy

(i) $x \notin coF(x)$ for each $x \in C$,

(ii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ is open for each $y \in C$,

then F has a maximal element.

Corollary 4. *Let C be a nonempty compact convex subset of a normed linear space X and $f : C \to X$ a continuous map. Then there is a $y_0 \in C$ such that*

$$\|y_0 - fy_0\| = d(fy_0, C),$$

where $d(x,C) = inf\,\{\|x - y\| : y \in C\}$.

The following variant of Hartman and Stampacchia [7] theorem is very useful in variational problems.

Theorem 5. *Let C be a closed bounded convex subset of a Hilbert space H. If $f : C \to H$ is a continuous and monotone map, then there exists a $y_0 \in C$ such that $\langle fy_0, y_0 - x\rangle \leq 0$ for all $x \in C$.*

Recall that f is monotone on C if $\langle fx - fy, x - y\rangle \geq 0$ for all $x, y \in C$.

The fixed point theorem due to Browder [3] follows as a corollary.

Corollary 5. *If C is a closed bounded convex subset of a Hilbert space H and $f : C \to C$ a nonexpansive map then f has a fixed point.*

Put $F = 1 - f$. Then $F : C \to H$ is continuous and monotone; hence there is a $y_0 \in C$ such that $(Fy_0, y_0 - x) \leq 0$ for all $x \in C$, i.e., $< (1-f)\, y_0,$

$$\begin{aligned} ||fx_0-y|| &= ||fx_0-\lambda x_0-(1-\lambda)fx_0|| \\ &= ||\lambda||\,||x_0-fx_0||<||x_0-fx_0||, \end{aligned}$$

a contradiction, so $x_0 = fx_0$.

Using Fan's best approximation theorem, one could prove the following variational inequality due to Hartman and Stampacchia [7].

Theorem 10. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function.*

Then there exists an $x_0 \in C$ such that

$$<fx_0, y-x_0> \geq 0 \text{ for all } y \in C.$$

Proof. Let $g = I-f: C \to R^n$. Then g is a continuous function. By Theorem 9, there exists a $y_0 \in C$ such that

$$||gy_0 - y_0|| \leq ||gy_0-x|| \text{ for all } x \in C,$$

that is,

$$< gy_0 - y_0, y_0-x> \geq 0 \text{ for all } x \in C.$$

Thus

$$< y_0 - fy_0 - y_0, y_0-x> \geq 0 \text{ for all } x \in C,$$

that is

$$<-fy_0, \ y_0 - x > \geq 0 \text{ for all } x \in C,$$

Hence

$$<fy_0, \ x - y_0 > \geq 0 \text{ for all } x \in C.$$

Note. If $f = I-g$, then

$<y_0- gy_0, \ x - y_0 > \geq 0$ for all $x \in C$,

gives that g has a fixed point by taking $x = gy_0$.

Theorem 9 is useful in proving the existence of zeros of a given function.

Theorem 11. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Let $x - \lambda fx \in C$ for all $x \in C$ and for some $\lambda > 0$.*

Then there is an $x_0 \in C$ such that $fx_0 = 0$ [14].

Proof. Let $gx = x - \lambda fx$. Then by Theorem 9 there is an $x_0 \in C$ such that

$$||gx_0-x_0|| = d(gx_0,C).$$

In the case $x - \lambda fx \in C$ for all $x \in C$, then $gx_0 \in C$ and $gx_0 = x_0$, that is, $x_0 - \lambda fx_0 = x_0$, so $fx_0 = 0$.

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$z \in Pz$. This contradicts assumption (i) and hence P has a maximal element.

The following result due to Ky Fan [5] is proved by using the existence of maximal elements.

Theorem 8. *Let C a compact convex subset of R^n and let $A \subset C \times C$ be a closed set such that*

(i) for each $x \in C$, $(x,x) \in A$,

(ii) for all $y \in C$, $\{x \in C : (x,y) \notin A\}$ is convex or ϕ.

Then there is a $\bar{y} \in C$ such that $C \times \{\bar{y}\} \subset A$.

Proof. Define F on C by

$$Fy = \{x \in C : (x,y) \notin A\}$$

Then $y \notin Fy$, and Fy is convex for each y.

F has open graph since $(x,y) \notin A$ so $(x,y) \in A^c$ is an open set. So by Ky Fan's Lemma F has a maximal element.

Hence, there exists a $\bar{y} \in C$ such that $F\bar{y} = \phi$, that is, $C \times \{\bar{y}\} \subset A$.

We prove the best approximation theorem by making use of maximal elements and then derive a few fixed point theorems.

The following is Fan's best approximation theorem [6]:

Theorem 9. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Then there exists an $x_0 \in C$ such that*

$$||x_0 - fx_0|| = d(fx_0, C) \le ||fx_0 - y|| \text{ for all } y \in C.$$

Proof. Define F on C by

$$y \in Fx \text{ if and only if } ||y - fx|| < ||x - fx||.$$

Therefore, by Ky Fan's Lemma we get that F has a maximal element. Since Fx is convex, $x \notin Fx$ and F has open graph; that is, $F\bar{x} = \phi$ for $\bar{x} \in C$. Hence

$$||\bar{x} - f\bar{x}|| \le ||f\bar{x} - y|| \text{ for all } y \in C.$$

Note. In case $f : C \to C$ in Theorem 9, then f has a fixed point.

In Theorem 9, if $\bar{x} \ne f\bar{x}$, then the following condition will guarantee that f has a fixed point.

If $\bar{x} \ne f\bar{x}$, then the line segment $[\bar{x}, f\bar{x}]$ has at least two points of C.

With this additional hypothesis f has a fixed point.

For instance, by Theorem 9 there is an $x_0 \in C$ such that $||x_0 - fx_0|| = d(fx_0, C)$.

Let $x_0 \ne fx_0$. Set $y = \lambda x_0 + (1-\lambda) fx_0$, $0 < \lambda < 1$. Then

$y_0 - x> \leq 0$ for all $x \in C$. If $x = fy_0$ then we get that $y_0 = fy_0$.

If C is not a weakly compact set, then the following statement gives a result similar to [7].

Theorem 6. *Let C be a nonempty convex subset of a Hilbert space H and $f : C \to H$ continuous monotone map. If C has a weakly compact, convex subset C_0 such that the set*

$$B = \{y \in C : < fy, y - x> \leq 0 \text{ for all } x \in C_0\}$$

is weaklky compact, then there is a $y \in C$ such that

$$<fy_0, y_0 - x> \leq 0 \text{ for all } x \in C.$$

For the following terminologies, see Mehta [11].

As regards the preference correspondence P one assigns for each bundle x in the consumption set, $P(x)$ is regarded as the set of all bundles that are strictly preferred to x, P is irreflexive binary relation. P has a maximal element x_0 if Px_0 is empty, i.e. if there is no bundle strictly preferred to x_0.

In the end, a result for maximal elements in Mathematical Economic is given. Mehta [11] has discussed results of this nature and has shown the existence of maximal elements for an economics agent.

Theorem 7. *Let $P : C \to 2^C$ be a preference correspondence where C is a nonempty closed convex subset of a Hausdorff topological vector space E. Suppose that*

(i) $x \notin Px$ for each $x \in C$,

(ii) Px is closed for each $x \in C$,

(iii) $P^{-1} y = \{x \in C : y \in Px\}$ is convex for each $y \in C$,

(iv) assume that C is covered by finitely many closed sets.

Then there is a maximal element for P.

Proof. Suppose there is no maximal element so

$$P(x) \neq \phi \text{ for all } x \in C.$$

Define $G : C \to 2^C$ by $Gx = C \setminus Px$. Then G is an open valued map for each $x \in C$.

Since $C = \bigcup_1^n Px_i$, so $\bigcap_1^n Gx_i = \bigcap_1^n (C/Px_i) = \cup_1^n Px^c_i = \phi$.

So G is not a *KKM*-map. Recall that if G is an open valued *KKM*-map then the family $\{Gx : x \in C\}$ has the finite intersection property.

Therefore $z = \sum_1^n \alpha_i x_i \notin \bigcup_1^n Gx_i$, and $z \in \bigcap_1^n Px_i$.

Thus $x_i P^{-1}z$. But $P^{-1}z$ is convex so $z = \sum_1^n \alpha_i x_i \in P^{-1}z$, *i.e.*,

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SOME EXPECTATIONS ASSOCIATED WITH MULTIVARIATE GAMMA AND BETA DISTRIBUTIONS INVOLVING THE MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

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ABSTRACT

In the present paper, we obtain some density functions associated with the multivariate gamma and beta distributions and make their applications to obtain the expectations involving multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha [51], p.64). Finally. we also derive the moments for these multivariate beta and gamma distributions and discuss their special cases.

1. Introduction. Different distributions have discussed by various authors Block and Rao [1], Carlson [2], Daley [5] Datt [6]. Kabe [8], Kaufman, Mathai and Saxena [9], Kendall [10], Khatri and Pillai [11,12]. Khatri and Srivastava [13], Littler and Fackerell [15], Lukacs and Naha [16], Lukacs [17], Mathai ([18] to [29]), Mathai and Rathie ([30] to [35]), Mathai and Saxena ([36] to [42]), Miller [43], Pillai, A1-Ani and Jouris [44], Pillai and Jouris [45], Pillai and Nagarsenker [46], Robbins and Pitman [47], Strawderman [52], Thaung [53], and Wilks [54]. Srivastava and Singhal [50], studied many of the classical statistical distributions, which were associated with the beta and gamma distributions. Further Exton [7], discussed generalized beta and gamma distributions with other special multivariate distributions,like Dirichlet distributions and multivariate normal distributions. He also discussed the expectations of some functions invoiving Lauricalla's multiple hypergeometric functions [14]. Recently the authors Chandel and Vishwakarma [4] discussed some multivariate beta and gamma distributions and made their applications to derive their expectations in terms of different multiple hypergermetric functions of several veriables.

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In the present paper, we extend the above work and establish some probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain some expectations involving the most generalized multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha ([51],p.64). Finally,we also drive the moments for these multivariate beta and gamma distributions and discuss their special cases.

2. Formulae Required. For ready stock, in this section we write the following results which will be used in our investigations:

The Liouville's Theorem (Also see Chandel [3,p.83 (3.1)]

$$(2.1)\quad \int_0^\infty \ldots \int_0^\infty f(x_1+\ldots+x_n) \quad x_1^{\mu_1-1}\ldots x_n^{\mu_n-1}\, dx_1\ldots dx_n$$

$$= \frac{\Gamma(\mu_1)\ldots\Gamma(\mu_n)}{\Gamma(\mu_1+\ldots+\mu_n)} \int_0^\infty f(t)\ t^{\mu_1+\ldots+\mu_{n-1}}\, dt,$$

provided that $Re\,(\mu_i) > 0,\ i=1,\ldots,n$.

Euler's definition for gamma function

$$(2.2)\quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad Re\,(z) > 0.$$

The definition of beta function (see, Srivastava and Manocha [51,p.26 eq.(4.6)])

$$(2.3)\quad B(\alpha,\beta) = \int_0^\infty \frac{\mu^{\alpha-1}}{(1+\mu)^{\alpha+\beta}}\, d\mu,\ Re\,(\alpha) > 0,\ Re\,(\beta) > 0.$$

3. Multivariate Gamma Distribution. Consider the function

$$(3.1)\quad f(x_1,\ldots,x_n) = \frac{\Gamma(\mu_1+\ldots+\mu_n)\,\lambda^{\mu+\mu_1+\ldots+\mu_n}}{\Gamma(\mu_1)\ldots\Gamma(\mu_n)\ \Gamma(\mu+\mu_1+\ldots+\mu_n)} \quad exp\{-(x_1+\ldots+x_n)\lambda\}\ (x_1+\ldots+x_n)^{\mu}$$

provided that $Re(\lambda) > 0,\ x_i \geq 0,\ Re\,(\mu_i) > 0,\ i=1,\ldots,n$ and $f(x_1,\ldots,x_n) = 0$ elsewhere.

Making an appeal to (2.1) and (2.2) the value of multiple integral of $f(x_1,\ldots,x_n)$ over the region defined above in (3.1) becomes unity. Hence $f(x_1,\ldots,x_n)$ is a probability density function for multivariate gamma distribution.

4. Expectation Associated with Multivariate Gamma Distribution.

The expectation value of the function $g(x_1,\ldots,x_n)$ is defined as

$$(4.1)\quad < g(x_1,\ldots,x_n) > \ = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f(x_1,\ldots,x_n)\, g(x_1,\ldots,x_n)\, dx_1\ldots dx_n$$

corresponding to density function $f(x_1,\ldots,x_n)$ defined by (3.1).

Consider the function

$$(4.2)\ g_1(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{array}{l}[(a):\theta',\dots,\theta^{(n)}];[(b'),\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}];[(d'),\delta'];\dots;\end{array}\right.$$

$$\left.\begin{array}{l}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{array}\ z_1(x_1+\dots+x_n)^{\nu_1},\dots,z_n(x_1+\dots+x_n)^{\nu_n}\right]$$

where $F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}$ is most generalized multiple hypergeometric function of Srivastava and Daoust [48] (also see Srivastava and Manocha [51, (18),(19),(20),p.64)]).

Now making an appeal to (2.1) and (2.2)the expectation of $g_1(x_1,\dots,x_n)$having density function $f(x_1,\dots,x_n)$ is given by

$$(4.3)\ <g_1(x_1,\dots,x_n)> = F^{A+1:B';\dots;B^{(n)}}_{C\ :D';\dots;D^{(n)}}\left[\begin{array}{l}[(a):\theta',\dots,\theta^{(n)}];\\ [(c):\Psi',\dots,\Psi^{(n)}];\end{array}\right.$$

$$\left.\begin{array}{r}[\mu+\mu_1+\dots+\mu_n:\nu_1,\dots,\nu_n]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{array}\ \frac{z_1}{\lambda_1^{\nu_1}},\dots,\frac{z_n}{\lambda_n^{\nu_n}}\right]$$

provided that

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\nu_i>0,\ i=1,\dots,n.$$

Corresponding to density fuction $f(x_1,\dots,x_n)$ defined by (3.1), if we consider the function

$$(4.4)\ <g_2(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{array}{l}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\\ [(c):\Psi',\dots,\Psi^{(n)}]:[(d'):\delta'];\end{array}\right.$$

$$\left.\begin{array}{l}\dots;[(b^n):\phi^n];\\ \dots;[(d^n):\delta^n];\end{array}\ z_1x_1^{\alpha_1}(x_1+\dots+x_n)^{\nu_1},\dots,z_nx_n^{\alpha_n}(x_1+\dots+x_n)^{\nu_n}\right]$$

Then expectation of g_2 is given by

$$(4.5)\ g_2(x_1,\dots,x_n)> = F^{A+1:B';\dots;B^{(n)}+1}_{C+1:D';\dots;D^{(n)}}\left[\begin{array}{l}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\Psi',\dots,\Psi^{(n)}],\end{array}\right.$$

$$\begin{array}{l}[\mu+\mu_1+\dots+\mu_n:\alpha_1+\nu_1,\dots,\alpha_n+\nu_n]:[(b'):\phi'],[\mu_1:\alpha_1];\dots;[(b^n):\phi^{(n)}],[\mu_n:\alpha_n];\\ [\mu+\mu_1+\dots+\mu_n:\alpha_1,\dots,\alpha_n]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{array}$$

$$\left.z_1/\lambda^{\alpha_1+\nu_1},\dots,z_n/\lambda^{\alpha_n+\nu_n}\right],$$

valid if

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\alpha_i-\nu_i>0,\ i=1,\dots,n.$$

5. The Multivariate Beta Distribution.

Consider the function $F(x_1,\dots,x_n)$defined by

$$(5.1)\ F(x_1,\dots,x_n)=\frac{\Gamma(\mu_1+\dots+\mu_n)\,\Gamma\alpha+\lambda+\mu_1+\dots+\mu_n)\,(x_1+\dots+x_n)^{\alpha}}{\Gamma(\mu_1)\dots\Gamma(\mu_n)\,\Gamma(\lambda)\,\Gamma(\alpha+\mu_1+\dots+\mu_n)\,(1+x_1+\dots+x_n)^{\alpha+\lambda+\mu_1+\dots+\mu_n}}$$

$Re(\alpha)>0, Re(\mu_i)>0$, $x_i>0$ $i=1,\dots,n$ and $F(x_1,\dots,x_n)=0$ elsewhere. Now making an appeal to (2.1) and (2.3) the value of multiple integral of $F(x_1,\dots,x_n)$ over the region defined above in (5.1), becomes unity. Hence $F(x_1,\dots,x_n)$ is probability density function for multivariate beta distribution.

6. Expectation Associated with Multivariate Beta Distribution.

Correspondiing to density function $f(x_1,\dots,x_n)$ defined by (5.1), consider the function

$$(6.1)\ G(x_1,\dots,x_n)=F^{A':B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;\end{matrix}\ \begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{z_1x_1^{\xi_1}(x_1+\dots+x_n)^{\eta_1}}{(1+x_1+\dots+x_n)^{\zeta_1+n_1}},\dots,\frac{z_nx_n^{\xi_n}(x_1+\dots+x_n)^{\eta_n}}{(1+x_1+\dots+x_n)^{\zeta_n+\eta_n}}$$

Then expectation of $G(x_1,..,x_n)$ is given by

$$(6.2)\ \langle G(x_1,\dots,x_n)\rangle$$

$$=F^{A+1:B'+;\dots;B^{(n)}+1}_{C+2:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],[\alpha+\mu_1+\dots+\mu_n:\eta_1+\xi_1,\dots,\eta_n+\xi_n]:\\ [(c):\psi',\dots,\psi^{(n)}],[\mu_1+\dots+\mu_n:\xi_1,\dots,\xi_n],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'],[\mu_1:\xi_1],\dots,[(b^{(n)}):\phi^{(n)}],[\mu_n:\xi_n];\\ [\alpha+\lambda+\mu_1+\dots+\mu_n:\eta_1+\zeta_1,\dots,\eta_n+\zeta_n]:[(d'):\delta'];\dots;[(d^{n}):\delta^{n}];\end{matrix}\ z_1,\dots,z_n\right]$$

where

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}+\zeta_i-\xi_i>0,$$

$i=1,\dots,n.$

7. Moment Generating Function (*m.g.f.*)for Gamma Distribution.

The *m.g.f.* is defined as

$$(7.1)\quad M(t_1,\dots,t_n)\quad\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}e^{x_1t_1+\dots+x_nt_n}f(x_1,\dots,x_n)\,dx_1\dots dx_n,$$

provided that the integral is a function of the parameters $t_1,\dots,t_n$ only. Thus *m.g.f.* for multivariate gamma distribution (3.1) is given by

$$(7.2)\ M(t_1,\dots,t_n)\quad=\quad\frac{\Gamma(\mu_1+\dots+\mu_n)\ \lambda^{\mu+\mu_1+\dots+\mu_n}}{\Gamma(\mu_1)\dots\Gamma(\mu_n)\ \Gamma(\mu+\mu_1+\dots+\mu_n)}$$

$$\int_0^{\infty}\dots\int_0^{\infty}e^{x_1t_1+\dots+x_nt_n}\,e^{-(x_1+\dots+x_n)\lambda}\,(x_1+\dots+x_n)^{\mu}\,x_1^{\mu_1-1}\dots x_n^{\mu_n-1}\,dx_1\dots dx_n$$

Now making an appeal to (2.1) and the result due to Srivastava[49,p,4.(12).]

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}=\sum_{M=0}^{\infty} f(M)\ (x_1+\ldots+x_n)^M, n\geq 1,$$

we finally derive

(7.3) $$M(t_1,\ldots,t_n)=F_D^{(n)}(\mu+\mu_1+\ldots+\mu_n\,,\mu_1,\ldots,\mu_n\,;\mu_1+\ldots+\mu_n\,;\ \frac{t_1}{\lambda},\ldots,\frac{t_n}{\lambda}),$$

where $F_D^{(n)}$ is Lauricella's fourth multiple hypergeometric function of several variables [14].

As a special case for $\mu = 0$, (7.3) gives

(7.4) $$M(t_1,\ldots,t_n) = \prod_{i=1}^{n} (1-t_i/\lambda)^{-\mu_i}.$$

8. Moments for Gamma Distribution. The moment $\mu'_{r_1,\ldots,r_n}$ for gamma distribution about $(0,\ldots,0)$ of order $r_1,\ldots,r_n$, is defined as the coefficient of $\frac{t_1^{r_1}}{r_1!}\cdots\frac{t_n^{r_n}}{r_n!}$ in $M(t_1,\ldots,t_n)$ when it is expended in powers of $t_1,\ldots,t_n$. Thus an appeal to (2.1) gives

(8.1) $$\mu'_{r_1,\ldots,r_n}\ \frac{(\mu_1)_{r_1}\cdots(\mu_n)_{r_n}\ \ (\mu+\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}}{(\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}\ \ \lambda^{r_1+\ldots+r_n}}.$$

9. Moment for Beta Distribution. The moment $\mu'_{r_1,\ldots,r_n}$ of density function $F(x_1,\ldots,x_n)$ about $(0,\ldots,0)$ for beta distribution is defined as

(9.1) $$\mu'_{r_1,\ldots,r_n}=\int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty} x_1^{r_1}\ldots x_n^{r_n}F(x_1,\ldots,x_n)\ \ dx_1\ldots dx_n.$$

Now substituting the value of $F(x_1,\ldots,x_n)$ from (5.1) in (9.1) and making an appeal to (2.1) and (2.3), we finally derive

(9.2) $$\mu'_{r_1,\ldots,r_n}\ =\ \frac{\Gamma(\alpha+\lambda-(r_1+,\ldots,+r_n)\ (\mu_1)_{r_1}\cdots(\mu_n)_{r_n}}{\Gamma(\lambda)\ \Gamma(\alpha+\mu_1+\ldots+\mu_n)\,(\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}}.$$

10. Special Cases. For $n=1$, from (7.3), we derive the following *m.g.f.* for gamma distribution:

(10.1) $$M(t_1)=(1-t_1/\lambda)^{-\mu+\mu_1}.$$

For $n=1$, from (8.1), we obtain the following moment of r_1th order about origin for gamma distribution:

(10.2) $$\mu'_{r_1}=\frac{(\mu+\mu_1)_{r_1}}{\lambda^{r_1}}.$$

Also for $n=1$, (9.2) gives the following moment of r_1th order for beta distribution :

(10.3) $$\mu'_{r_1}=\frac{\Gamma(\alpha+\lambda-r_1)}{\Gamma(\lambda)\,\Gamma(\alpha+\mu_1)}.$$

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MATHEMATICAL MODELS ON EPIDEMIOLOGICAL PROBLEMS WITH SPECIAL REFERENCE TO AIDS AND ITS REDUCTION STRATEGIES - USE OF COMPOUND BINOMIAL DISTRIBUTION.

By

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ABSTRACT

In this paper, an attempt has been made to make use of compound binomial distribution in dealing with problems of AIDS. Among methods include interval estimation, appoximation through Gaussian distribution and M.L.E. method etc. Finally, graphical approach has been used to demonstrate the comparative study of incidence infection rate corresponding to various sources of infection.

1. Introduction. When an outcome of the phenomenon is surrounded by uncertainties, it seems to be much difficult to quantify the amount of an outcome of that phenomenon. In order to effectively deal with probalilistic uncertainty, probability distribution theory has been developped to solve the above problem. Broadly categorised in two types-discret distribution and continuous distribution theories have tremendous potential for application in various wide ranging areas such as physical sciences, life sciences, epidemiology, management business, commerce, psychology and sociology etc. In 1974, Bachev and Petkova [1] have presented a mathematico-statistical analysis of the cycle recurrence in the droplet infections of communicable diseases. They have used dispersion analysis in the study of cycle recurrence or cyclicity.

In 1978, similarly, Bender [2] has conceptualised mathematical modelling in the areas of sex preference and sex ratio and problem of choices etc. Nevertheless, he has also studied Monte Carlo simulation technique which are applicable to a doctor's waiting room and sediment volume etc. Probabilistic model has also been used by him in the study of radio active decay, optimal facility location and distribution of particle size etc. Now a days, epidemiological problems have seriously attracted the attention of scientists involved therein. Different types of problems connected with epidemiology have been modelled mathematically and statistically.

In 1989, Edwards and Hamson [3] have also presented mathematical (statistical) modelling based on random variables. They have suggested various discrete and continuous random variables and their corresponding distributions to model different scientific and real life situations. They have especially discussed binomial, Poisson, exponential and normal distributions and their applications to various real life situations. Their approach to statistical modelling based on random varibles has undoubtedly added a new dimension to its study. Further more, in 1994 Kapur [5] has also described various aspects of mathematical modelling and its application in differnt areas such as physical and life sciences including epidemiology.

Under the category of epidemiological problems, recently, it is none but a problem of AIDS which has shaken the root of developed and developing societies and consequently gripped so badly that it has posed a great challenge before the epidemiologists. Unless, we effectively quantify the seriousness of disease, we cannot adopt any efficient reduction and prevention strategies to contain the HIV infection. Various mathematical modellings carried out previously could not have been able to precisely revealed such a fact which could be instrumental to adopting any effective reduction strategy. Recently Kotia and Srivastava [4] have made an effort to analyse impact of various sources of infection in order to seriously understand the problems of AIDS, so that infection rate could be contained effectively. Their study has made a hallmark in the investigations of problems of AIDS based on various sources of infection in the Indian perspective. Of late, Flanders, W.D. and Kleinbaum, D.G. [6] have presented a study on basic models for disease occurrence in Epidemiology. They have analysed disease occurrence with special reference to smoking deaths due to lung cancer in UK. More recently, Misra [7] has made an attempt in analysing the seriousness of AIDS infection by making use of Poisson distribution. However, his analytical approach could remain quite theoretical and couldnot reveal the process of data-fit without which purpose of investigation could not be fulfilled completely.

In the present paper an effort has been made by us to make use of compound binomial distribution in dealing with problems of AIDS. Here, it is a natural inquisitiveness that if any infected person of HIV transmits his or her HIV in several other persons with certain amount of probability along with the probability with which an individual infection is developing then what will be the probability or probabilitic rate that how many persons with how much probability would be infected in future out of total number of infected persons due to an individual HIV positive patient. Here we have been contemplating this situation of AIDS infection by making use of compound binomial distribution. Here the present paper is exclusively devoted to exhibit various analytical aspects such as theoretical, numerical and graphical in order to reach some index as a probabilistic rate of occurrence of AIDS infection which may be subsequently found helpful in formulating reduction as well as prevention strategies. Among methods include interval estimation, approximation through Gaussian distribution and MLE method etc. Finally, graphical aproach has been used to demonstrate the comparative study of incidence infection rate corresponding to various sources of infection.

2. Assumptions and Preliminaries. Before we use the compound binomial distribution in statistically analysing the problems of AIDS we have some underlined assumptions under which bionomial distribution can be applied in the afore said analysis :-

(i) AIDS disease can Independently occur among the group of people.

(ii) Risk of desease is equal with the change of time.

(iii) The occurrence of disease is rare and random in its beginning but subsequently it can cover the globly for the given geographical region.

Here we define compound binomial distribution as follows :

Let us suppose that $X_1, X_2, X_3, \ldots$ are indentically and independently distributed Bernouli variates with $P[X=0]=p$ and $P[X=1]=q=1-p$. For a fixed n, the random variable $X = X_1 + X_2 + X_3 + \ldots + X_n$ is a binomial variate with parameters n and p and probability function :

$$p\,(x=r) = \binom{n}{r} p^r q^{n-r},\ r = 0,1,2,\ \ldots\ ,\ n.$$

which gives the probability of r successes in n independent trials with constant probability p of success for each trial.

Now suppose that n, instead of being regarded as a fixed constant, is viewed as a random variable following Poisson law with Parameter λ.Then,

$$P\,(n=k) = \frac{e^{-\lambda}\lambda^k}{k!}\ ;\ k = 0,\ 1,2,\ \ldots$$

In such a case x is said to have compound binomial distribution. The joint probability funciton of x and n is given by

$$P\,(X = r \cap n = k) = P\,(n=k)\,P\,(x = \frac{r}{n} = k) = \frac{e^{-\lambda}\lambda^k}{r} - \binom{k}{r} p^r q^{k-r}.$$

Since $P\,(X = r/n = k)$ is the probability of r successes in k trials. Obviously

$r < k \Rightarrow k > r.$

The marginal distribution of x is given by :

$$P(x = r) = \sum_{k=r}^{\infty} P(X = r \cap n = k)$$

$$= e^{-\lambda} p^r \sum_{k=r}^{\infty} \binom{k}{r} \frac{\lambda^k q^{k-r}}{k!}$$

$$= \frac{e^{-\lambda}(\lambda p)^r}{k!} \sum_{k=r}^{\infty} \frac{(\lambda p)^{k-r}}{(k-r)!}$$

$$= \frac{e^{-\lambda}(\lambda p)^r}{k!} \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{(k-r)!}\,; j = k-r.$$

$$= \frac{e^{-\lambda}(\lambda p)^r}{k!} e^{\lambda q},$$

which is a probability function of a Poission Variate with parameter λp. Hence $E(x) = \lambda p$ and variate $(x) = \lambda p$.

Now we estimate λp by making use of maximum likelihood estimation *(MLE)* method these after use interprete it in the term of disease incidence after relating it to the Person-time.

3. Statstical Analysis. Source Wise Appromimation of Probabilities by Gaussian Distribution for 95% Confidence Limit. If *MLE* of λp is C then incidence rate of the disease is represented by C/PT. Here it may be of interest to note that if group of cases is smaller the value of probability can be estimated accurately but when group of cases becomes larger then it can be approximated with the help of Gaussian distribution which is shortly demonstrated in the next discussion.

We know that there are several sources of infection through which *HIV* of *AIDS* are spreading in the society. Here we have been made available the statistics corresponding to each and every source of infection from different sero surviellance centres established in various corner of India. This is being displayed by table given as under.

3.1 Table

S.No.	*Sources of Infection*	*No. of Cases*
1.	Sexual Promiscuous	4483
2.	Blood Donors	1682
3.	Recipients of blood	209
4.	Drug users	1647
5.	Others	2247

Source : Ministry of Health, India, 1994

1. **Sexual Promiscuous**

   $4363 \leq x \leq 4603$

   $0.0194646 < r < 0.0205353$

2. **Blood Donors**

   $1602 \leq x \leq 1762$

   $0.0158739 \leq r \leq 0.0174593$

3. **Recipients of blood**

   $181 \leq x \leq 237$

   $0.0133235 \leq r \leq 0.0174457$

4. **Drug Users**

   $1567 \leq x \leq 1727$

   $0.0190285 \leq r \leq 0.0209969$

5. **Others**

   $2157 \leq x \leq 2337$

   $0.0159991 \leq r \leq 0.0173342$

3.2 Table

S.No.	Sources of Infection	Rate of Occurrence
1.	Sexual Promiscuous	$0.0194646 \leq r \leq 0.0205353$
2.	Blood Donors	$0.0158739 \leq r \leq 0.0174593$
3.	Recipients of blood	$0.0133235 \leq r \leq 0.0174457$
4.	Drug users	$0.0190285 \leq r \leq 0.0209969$
5.	Others	$0.0159991 \leq r \leq 0.0173342$

4. Graphical Representation.

Here we plot a graph corresponding to each and every source of infection and subsequently we conduct the comparative study of probabilistic occurrence of this disease. From graph we can approximate the following informations :

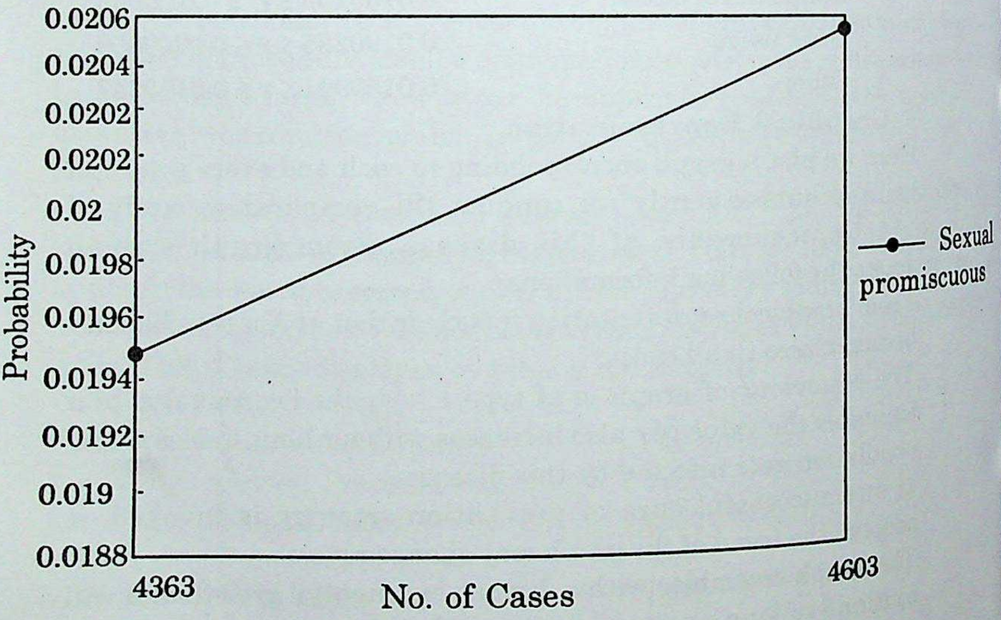
1. From graphical representation it is clear that at $X = 0$ probability becomes zero in all cases.
2. The behaviour of graph is of type $r = r_0\ ekx$ i.e. as value of x increases the value of r also increases without limit unless whole population gets infected by this disease.
3. If any successful cure of prevention srtategy is invoked or adopted the trend of the graph may change reverse.
4. This graph resembles with a famous exponential growth and will continue so long as any curve is not found.

5. Chances of occurrence of disease through recipients of blood source infected are being reflected higher than other sources of infection of a given number of cases.
6. 5% increase in the case will result in an increase of 0.0001 in case of sexual promiscuous, 0.0001 in case of blood donors, 0.0006 in case of recipients of blood, 0.0002 in case of drug users, 0.0001 in case of others.

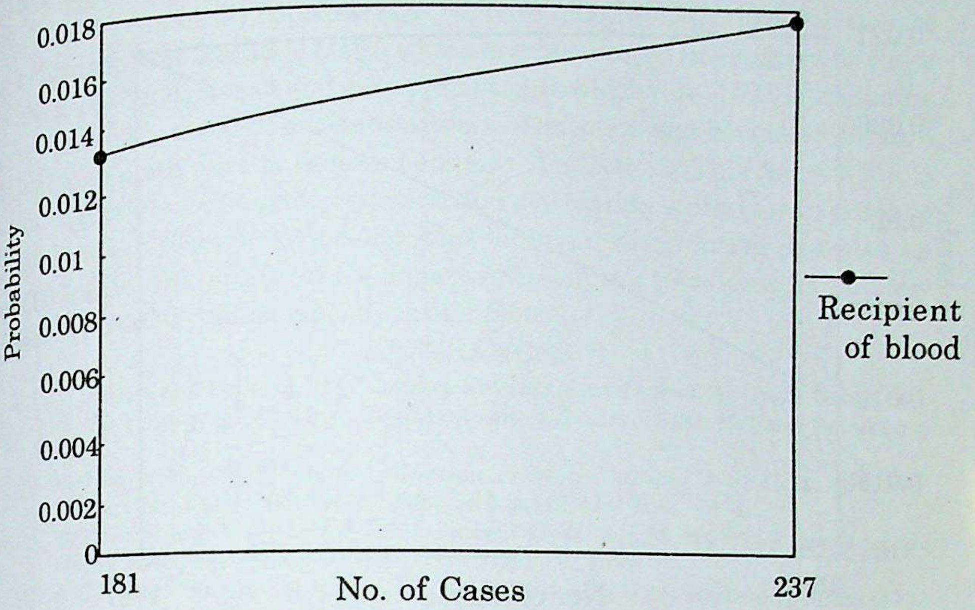
In view of above facts, reduction and prevention strategies may be chalked out efficiently and we may bring a better level of awareness among the people to reduce the infection rate. In the lack any vaccine or cure, we think preventive strategy is only sought cure which can be formulated effectively with the above set of informations.

As we have already mentioned that in case of larger group, probabilities are approximated through Gaussiam distribution with certain degree of confidence limit 95% and we describe the approximation corresponding to each and every source of infection as stated in the above table.

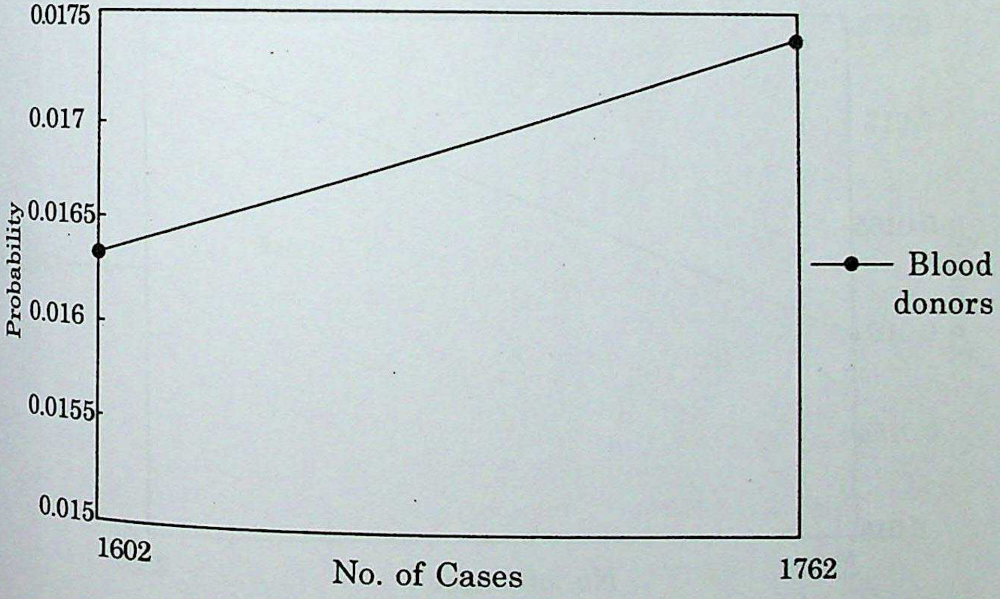
(4.1) Sexual Promiscuous



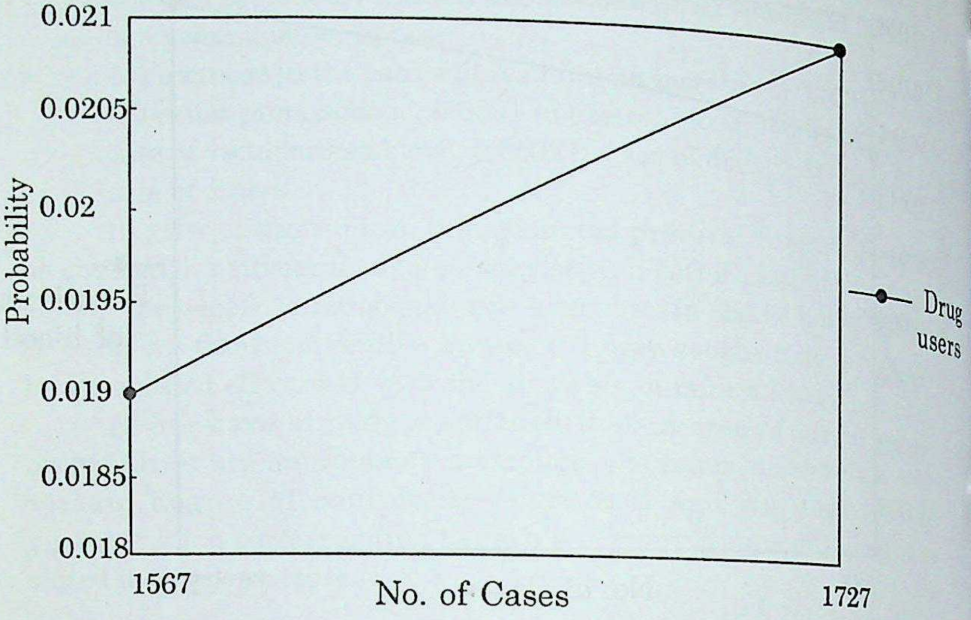
(4.2) Recipient of blood



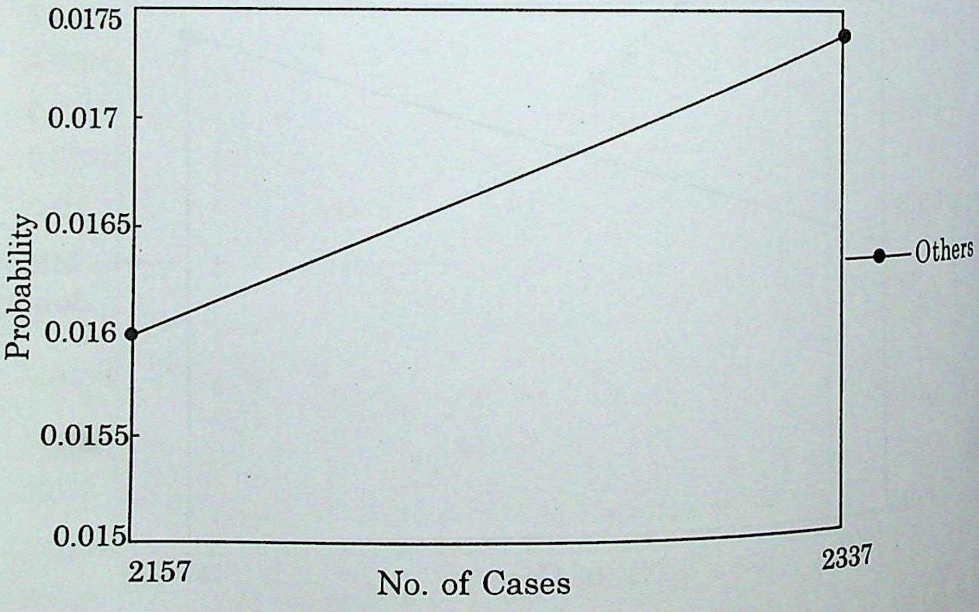
(4.3) Blood donors



(4.4) Drug users



(4.5) Others



5. Conclusion.

We can easily conclude that we have quantified the probabilitic rate of ocurrence of disease (*AIDS*) which is proven as an instrument to make the people understand and get acquainted at which rate *AIDS* is spreading in the society through different sources of infection and how intervariance is taking place dure to different sources of infections. This thing is a sole idea behind adopting the redution and prevention strategies in order to contain it. Nevertheless compound binomial distribution can also be applied in the study of various wide ranging problems as cancer, cardiovasenlar disease and Hepatitis *B* etc.

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CLASS OF CERTAIN UNIVALENT FUNCTIONS

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ABSTRACT

Let $D(A,B,\alpha)$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in unit disc $U = \{z : |z| < 1\}$, and satisfying

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w(z)}{1+B\, w(z)},$$

$$-1 \le B < A \le 1,\ z \in U,\ 0 \le \alpha < 1,$$

where $w(z)$ is regular in U and satisfies $w(0) = 0$, $|w(z)| < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U. In this paper we obtain coefficient estimates, radius of convexity for $f(z) \in D(A,B,\alpha)$ and some more interesting results for the subclasses of $D(A,B,\alpha)$.

1. Introduction. Let D denote the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in unit dist $U = \{z : |z| < 1\}$, and satisfying the condition

$$\text{(1.1)} \qquad \left| \frac{f'(z)}{g'(z)} - 1 \right| < 1, \qquad z \in U,$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

Let $D(\delta)$ denote the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in U and satisfying the condition

$$\text{(1.2)} \qquad \left| \frac{\{f'(z)/g'(z)\} - 1}{\{f'(z)/g'(z)\} + 1} \right| < \delta, \qquad z \in U,$$

where $0 < \delta \le 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

In this paper we introduce the class $D(A, B, \alpha)$ of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular in U and satisfying the condition

$$\text{(1.3)} \qquad \frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w(z)}{1+B\, w(z)},$$

$$-1 \leq B < A \leq 1,\ z \in U,\ 0 \leq \alpha < 1,$$

where $w(z)$ is regular in U and satisfies the conditions $w(0) = 0$, $|w(z)| < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

With appropriate choices of A and B and α, the class $D(A, B, \alpha)$ reduces to following known important subclasses :

(i) For $A = 1$ and $B = 0$ and $\alpha = 0$, the class $D(A,B,\alpha)$ reduces to the class D.

(ii) For $A = \delta$ and $B = -\delta$ and $\alpha = 0$, the class $D(A,B,\alpha)$ reduces to the class $D(\delta)$.

In this paper we obtain sharp coefficient estimates and sharp radius of convexity for $f(z) \in D(A,B,\alpha)$. Also, we obtain the sharp result concerning the radius of convexity for the subclasses of $D(A,B,\alpha)$ associated with each of the cases :

(i) $g(z)$ is starlike in U,

(ii) $g(z)$ is convex in U,

(iii) $Re\, g'(z) > 0$ in U.

Recently, Lakshminarashimhan [3] have studied the class $D(\delta)$ and Ratti [4] have studied the class D. Thus our results naturally generalize the corresponding results of Lakshminarasimhan [3] and Ratti [4].

In fact, an attempt has been made to have a unified and detailed study of these classes of univalent functions.

Throughout this paper we assume that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$ and R_1, R_2, K_1 and L_1 have the same values as in Lemma 2.1.

2. To establish the results of subsquent section we require the following lemmas:

Lemma 2.1. If $p(z) \in P(A,B,\alpha)$, then on $|z| = r < 1$,

$$Re \left\{\frac{z\,p'(z)}{p(z)}\right\} \geq \begin{cases} -\dfrac{(A-B)(1-\alpha)r}{[1+\{(A-B)(1-\alpha)+B\}r](1-Br)}, & R_1 \leq R_2 \\[2ex] \dfrac{(A-B)(1-\alpha)+2B}{(A-B)(1-\alpha)} + \dfrac{2[(L_1K_1)^{1/2} - [1-\{(A-B)(1-\alpha)+B\}Br^2]}{(A-B)(1-\alpha)(1-r^2)} & R_1 \leq R_2 \end{cases}$$

where $R_1 = (L_1/K_1)^{1/2}$, $R_2 = \dfrac{[1-\{(A-B)(1-\alpha)+B\}r]}{(1-Br)}$,

$$L_1 = [1-\{(A-B)(1-\alpha)+B\}]\,[1+\{(A-B)(1-\alpha)+B\}r^2]$$

and $K_1 = (1-B)(1+Br^2)$.

The result is sharp. The above lemma follows from Theorem 1 of Anh

$$Re\left[\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right]=R_1 \qquad \text{at } z=-r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 3) turn out to be special case by taking $A=1,\ B=0$ and $\alpha=0$.

Corollary. If $f(z)\in D(A,B,\alpha)$, $g(z)$ is regular and convex in U, then $f(z)$ is convex in

$$|z|<\left[\begin{array}{ll} r_3 & \text{for } R_1\le R_2 \\ r_4 & \text{for } R_2\le R_1 \end{array}\right.$$

where r_3 and r_4 respectively the smallest positive roots of the following equations:

$$(1-r)\,[1-\{(A-B)(1-\alpha)+B\}\,r]\,(1-Br)-(A-B)(1+r)\,r=0,$$

and

$$(1-r^2)\ (A-B)(1-\alpha)+\{(A-B)(1-\alpha)+2B\}(1-r^2)+2\,[\,(L_1K_1)^{1/2}-$$
$$\{1-((A-B)(1-\alpha)+B)\,Br^2\}]=0\,.$$

The resutl is sharp.

Theorem 3.5. If $f(z)\in D(A,B,\alpha)$, $Re\,g'(z)>0$ in U, then

$$Re\left\{1+\frac{z f''(z)}{f'(z)}\right\}\ \ge \left[\begin{array}{l} M_5(r) \text{ if } R_1\le R_2 \\ M_6(r) \text{ if } R_2\le R_1 \end{array}\right.$$

where

$$M_5(r)=\frac{1-2r+r^2}{1-r^2}-\frac{(A-B)(1-\alpha)\,r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)}\,,$$

and

$$M_6(r)=\frac{1-2r+r^2}{1-r^2}+\frac{(A-B)(1-\alpha)+2B}{(A-B)(1-\alpha)}+\frac{2[(L_1K_1)^{1/2}-\{1-(A-B)(1-\alpha)+B\}Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

The result is sharp.

Proof. Since $Re\,g'(z)>0$ in U, we have by Lemma 2.3

$$Re\left\{1+\frac{z\,g''(z)}{g'(z)}\right\}\ge\frac{1-2r+r^2}{1-r^2}\ .$$

Using this estimate in (3.4) we get the required result on the lines of proof of Theorem 3.2.

To see the bounds are sharp we consider the following two cases:

(i) $$\frac{f'(z)}{g'(z)}=\frac{1+\{(A-B)(1-\alpha)+B\}\,z}{1+B\,z}$$

with $g(z)=-z-2\log(1-z)$, when $R_1\le R_2$

and

(ii) $$\frac{f'(z)}{g'(z)}=\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}$$

and Tuan [1] by putting $\alpha = 0$ and $\beta = 1$.

Lemma 2.2. If $h(z) = 1+c_1z+ \ldots$ is regular in U and $Re\ h(z) > 0$ for $z \in U$, then

$$Re\ h(z) \geq \frac{1-r}{1+r},\ |z| = r .$$

This is a well known result due to Caratheodory.

Lemma 2.3. If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular $Re\ g'(z) > 0$ in U, then

$$Re\left\{1 + \frac{z\,g''(z)}{g'(z)}\right\} \geq \frac{1-2r+r^2}{1-r^2},\ |z| = r.$$

The result is sharp.

The above result is contained in the proof of Theorem 4 of Ratti [4].

3. In this section we obtain main results.

Theorem 3.1. If $f(z) \in D(A,B,\alpha)$, then

$$|a_n - b_n| \leq \frac{(A-B)(1-\alpha)}{n},\ n = 2, 3, \ldots \text{ and } 0 \leq \alpha < 1.$$

If $Re\ (a_k \overline{b_k}) \leq 0,\ k = 2, 3, \ldots$.

The result is sharp.

Proof. Since $f(z) \in D(A,B,\alpha)$, we have

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w(z)}{1+B\, w(z)},$$

or

$$f'(z) - g'(z) = [(A-B)(1-\alpha)\, g'(z) + B\, \{g'(z) - f'(z)\}]\, w(z).$$

That is

$$(3.1)\quad \sum_{n=2}^{\infty} n\,(a_n - b_n)\, z^{n-1} = \{(A-B)(1-\alpha)(1+ \sum_{n=2}^{\infty} n\, b_n z^{n-1}) + B \sum_{n=2}^{\infty} n\ (a_n - b_n)\, z^{n-1}\} \sum_{n=1}^{\infty} c_n z^n$$

For $n \geq 2$, we have from (3.1)

$$\sum_{k=2}^{n} k\,(a_k - b_k)\, z^{k-1} + \sum_{k=n+2}^{\infty} d_k z^{k-1} = [(A-B)(1-\alpha) + (A-B)(1-\alpha) \sum_{k=2}^{n-1} k\, b_k z^{k-1} + B \sum_{k=2}^{n-1} k\,(b_k - a_k)\, z^{k-1}]\, w(z)$$

which gives

$$\left| \sum_{k=2}^{n} k\,(a_k - b_k)\, z^{k-1} + \sum_{k=n+1}^{\infty} d_k z^{k-1} \right|^2 \leq \left| (A-B)(1-\alpha) + (A-B)(1-\alpha) \sum_{k=2}^{n-1} k\, b_k z^{k-1} + B \sum_{k=2}^{n-1} k\,(b_k - a_k)\, z^{k-1} \right|^2 .$$

On integrating over $|z| = r,\ 0 < r < 1$, we get

$$\sum_{k=2}^{n} k\,(a_k - b_k)\, z^{k-1} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k-2} \le (A-B)^2 (1-\alpha)^2 + (A-B)^2 (1-\alpha)^2$$

$$+ (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n-1} k^2\, |b_k|^2 r^{2k-2} + B^2 \sum_{k=2}^{n-1} k^2\, |b_k - a_k|^2 r^{2k-2}.$$

If we take the limit as $r \to 1$, then

$$\sum_{k=2}^{n} k^2\, |a_k - b_k|^2 \le (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n-1} k^2\, |b_k|^2 + B^2 \sum_{k=2}^{n-1} k^2\, |b_k - a_k|^2$$

or

$$n^2 |a_n - b_n|^2 \le (A-B)^2 (1-\alpha)^2 + (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n} k^2 |b_k|^2 + B^2 \sum_{k=2}^{n-1}$$

$$k^2\, [\,|b_k|^2 + |a_k|^2 - 2\,|a_k|\,|b_k|\,] - \sum_{k=2}^{n-1} k^2\, [\,|b_k|^2 + |a_k|^2 - 2\,|a_k|\,|b_k|\,]$$

$$= (A-B)^2 (1-\alpha)^2 + \sum_{k=2}^{n-1} k^2\, [\{ (A-B)^2 (1-\alpha)^2 + B^2 - 1\}\, |b_k|^2 + (B^2-1)\, |k|_2$$

$$+ 2 \sum_{k=2}^{n-1} k^2 (1-B^2)\, Re\,(a_k\, \overline{b_k})] \le (A-B)^2 (1-\alpha)^2,$$

$$\{A-1 \le B < A \le 1,\ 0 \le \alpha < 1,\ Re\,(a_k\, \overline{b_k}) \le 0\}$$

that is,

$$|a_n - b_n| \le \frac{(A-B)\,(1-\alpha)}{n},\ n = 2, 3, \ldots\ .$$

To establish the sharpness of the result we consider the following funcitons

$$f(z) = \int_0^z \frac{dt}{1 + \{(A-B)(1-\alpha) + B\}\, t^{n-1}},$$

and $$g(z) = \int_0^z \frac{(1 + B\, t^{n-1})\, dt}{[1 + \{(A-B)(1-\alpha) + B\}\, t^{n-1}]^2}.$$

It is obvious that $f(z) \in D(A,B,\alpha)$.

Now

$$f(z) = \int_0^z \frac{dt}{1 + \{(A-B)(1-\alpha) + B\}\, t^{n-1}},$$

$$= \int_0^z [1 + \{ (A-B)(1-\alpha) + B\} t^{n-1}]^{-1}\, dt$$

$$= z - \frac{\{ (A-B)(1-\alpha) + B\}}{n}\, z^n + \ldots$$

and

$$g(z) = \int_0^z \frac{(1 + B\, t^{n-1})\, dt}{[1 + \{(A-B)(1-\alpha) + B\}\, t^{n-1}]^2},$$

$$= \int_0^z [(1 + B\, t^{n-1})\, [1 + \{ (A-B)(1-\alpha) + B\} t^{n-1}]^{-2}\, dt$$

$$= z - \frac{[2\,\{(A-B)(1-\alpha)+B\} - B\,]}{n}\, z^n + \dots$$

Clearly, $a_n = -\dfrac{\{\,(A-B)(1-\alpha)+B\}}{n}$ and $b_n = -\dfrac{[2\{(A-B)(1-\alpha)+B\}-B]}{n}$.

Hence

$$|\,a_n - b_n\,| = \frac{(A-B)(1-\alpha)}{n}.$$

This shows the sharpness of the result.

Remark. On putting $A = \delta$ and $B = \delta$ and $\alpha = 0$ in Theorem 3.1 the result of Lakshminarasimhan([3], Theorem 5) follows.

Theorem 3.2. If $f(z) \in D(A,B,\alpha)$, then

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \le \left[\begin{array}{l} M_1(r) \text{ for } R_1 \le R_2 \\ M_2(r) \text{ for } R_2 \le R_1 \end{array}\right.$$

where

$$M_1(r) = \frac{1-4r+r^2}{1-r^2} - \frac{(A-B)(1-\alpha)\,r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)}.$$

$$M_2(r) = \frac{1-4r+r^2}{1-r^2} + \frac{\{(A-B)(1-\alpha)+2B\}}{(A-B)(1-\alpha)}.$$

$$+\frac{2\,[\,(L_1K_1)^{1/2} - \{1-(A-B)(1-\alpha)+B\}\,Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

Proof. Since $f(z) \in D(A,B,\alpha)$, we have

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)\,(1-\alpha) + B\}\,w(z)}{1+B\,w(z)},$$

(3.2) Let $\dfrac{f'(z)}{g'(z)} = p(z)$, where $p(z) = \dfrac{1 + \{(A-B)\,(1-\alpha) + B\}\,w(z)}{1+B\,w(z)}$.

On differentiating (3.2) logarithmically, we get

$$\frac{z f''(z)}{f'(z)} - \frac{z g''(z)}{g'(z)} = \frac{z p'(z)}{p(z)}.$$

or $Re\left\{1+\dfrac{z f''(z)}{f'(z)}\right\} - Re\left\{1+\dfrac{z g''(z)}{g'(z)}\right\} = Re\left(\dfrac{z p'(z)}{p(z)}\right)$.

Using Lemma 2.1 in (3.3), we have

(3.4) $$Re\left\{1+\frac{z f''(z)}{f'(z)}\right\} - Re\left\{1+\frac{z g''(z)}{g'(z)}\right\}$$

$$\ge \left[\begin{array}{ll} -\dfrac{(A-B)\,(1-\alpha)\,r}{[1-\{(A-B)\,(1-\alpha)+B\}r]\,(1-Br)}, & \text{if } R_1 \le R_2 \\ \dfrac{(A-B)\,(1-\alpha)+2B}{(A-B)\,(1-\alpha)} + \dfrac{2[(L_1K_1)^{1/2} - [1-\{(A-B)(1-\alpha)+B\}Br^2]}{(A-B)\,(1-\alpha)\,(1-r^2)} & \text{if } R_2 \le R_1 \end{array}\right.$$

Since $g(z)$ is univalent, we have [2]

(3.5) $Re\left\{1+\dfrac{z\,g''(z)}{g'(z)}\right\} \geq \dfrac{1-4r+r^2}{1-r^2},\ |z| = r.$

Combining (3.4) and (3.5) we get the required result.

Sharpness of the bounds follows if we choose

(i) $\dfrac{f'(z)}{g'(z)} = \dfrac{1+\{(A-B)(1-\alpha)+B\}z}{1+Bz}$ with $g(z) = \dfrac{z}{(1-z)^2}$, when $R_1 \leq R_2$

and

(ii) $\dfrac{f'(z)}{g'(z)} = \dfrac{1+\{(A-B)(1-\alpha)+B\}w_1(z)}{1+B\,w_1(z)}$ with $g(z) = \dfrac{z}{(1-z)^2}$,

when $R_2 \leq R_1$

where $w_1(z) = \dfrac{z\,(z-c)}{1-cz}$ with c defined by the condition

$$Re\left[\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right] = R_1 \qquad \text{at } z = -r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 1) turn out to be special case by taking $A = 1$, $B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, then $f(z)$ is convex in

$$|z| < \begin{cases} r_1 & \text{for } R_1 \leq R_2 \\ r_2 & \text{for } R_2 \leq R_1 \end{cases}$$

where r_1 and r_2 are respectively the smallest positive roots of the following equations :

(3.6) $(1-4r+r^2)\,[1-\{(A-B)(1-\alpha)+B\}\,r]\,(1-Br) - (A-B)(1-\alpha)\,(1-r^2)\,r = 0$

(3.7) $(1-4r+r^2)\,\{(A-B)(1-\alpha)\} + \{(A-B)(1-\alpha)+2B\}(1-r^2)$

$+2\,[\,(L_1K_1)^{1/2} - \{1-((A-B)(1-\alpha)+B)\,Br^2\}] = 0.$

The result is sharp.

Theorem 3.3. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and starlike in U, then

$$Re\left\{1+\frac{z\,f''(z)}{f'(z)}\right\} \geq \begin{cases} M_1(r) & \text{if } R_1 \leq R_2 \\ M_2(r) & \text{if } R_2 \leq R_1 \end{cases}$$

where $M_1(r)$ and $M_2(r)$ are same as in Theorem 3.2.

The result is sharp.

Proof. Since $g(z)$ is starlike in D implies $g(z)$ is univalent there, the proof of this theorem follows on the steps Theorem 3.2.

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 2) turn out to be special case by taking $A = 1$ and $B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and starlike in U, then $f(z)$ is convex in

$$|z| < \begin{bmatrix} r_1 & \text{for } R_1 \le R_2 \\ r_2 & \text{for } R_2 \le R_1 \end{bmatrix}$$

where r_1 and r_2 are respectively the smallest positve roots of the equations (3.6) and (3.7).

Theorem 3.4. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and convex in U, then

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \ge \begin{bmatrix} M_3(r) \text{ if } R_1 \le R_2 \\ M_4(r) \text{ if } R_2 \le R_1 \end{bmatrix}$$

where

$$M_3(r) = \frac{1-r}{1+r} - \frac{(A-B)(1-\alpha)\, r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)},$$

and

$$M_4(r) = \frac{1-r}{1+r} - \frac{\{(A-B)(1-\alpha)+2B\}}{(A-B)(1-\alpha)} + \frac{2[(L_1K_1)^{1/2} - \{1-(A-B)(1-\alpha)+B\}Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

The result is sharp.

Proof. Since $g(z)$ is convex in U, therefore $g'(z) \neq 0$ in U and

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} > 0 \text{ in } U.$$

The function

$$1 + \frac{z g''(z)}{g'(z)} = 1 + c_1 z + \dots$$

is regular in U and has positive real part, therefore by Lemma 2.2

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} \ge \frac{1-r}{1+r},\ |z| = r.$$

Using this estimate in Theorem 3.4, we get the required result on the lines of proof of Theorem 3.2.

To see that the bounds are sharp, we consider the following two cases.

(i) $\dfrac{f'(z)}{g'(z)} = \dfrac{1 + \{(A-B)(1-\alpha) + B\} z}{1+Bz}$ with $g(z) = \dfrac{z}{1-z}$, when $R_1 \le R_2$

and

(ii) $\dfrac{f'(z)}{g'(z)} = \dfrac{1 + \{(A-B)(1-\alpha) + B\} w_1(z)}{1+B w_1(z)}$ with $g(z) = \dfrac{z}{1-z}$, when $R_2 \le R_1$

where $w_1(z) = \dfrac{z(z-c)}{1-cz}$ with c defined by the condition where $w_1(z) = \dfrac{z\,(z-c)}{1-cz}$ with c defined by the condition

$$Re\left[\frac{1+\{(A-B)\,(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right] = R_1 \qquad \text{at } z=-r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 4) turn out to be special case by taking $A=1,\ B=0$ and $\alpha=0$.

Corollary. If $f(z)\in D(A,B,\alpha)$, $Re\, g'(z) > 0$ in U, then $f(z)$ is convex in

$$|z| < \begin{cases} r_5 & \text{for } R_1 \le R_2 \\ r_6 & \text{for } R_2 \le R_1 \end{cases}$$

where r_5 and r_6 respectively the smallest positive roots of the following equations:

$$(1-2r-r^2)\,[1-\{(A-B)\,(1-\alpha)+B\}\,r]\,(1-Br)-(A-B)(1-\alpha)\,(1-r^2)\,r=0$$

and

$$(1-2r-r^2)\,\{(A-B)\,(1-\alpha)\}+\{(A-B)(1-\alpha)+2B\}(1-r^2)+2\,[\,(L_1K_1)^{1/2}-\{1-((A-B)\,(1-\alpha)+B)\,Br^2\}]=0.$$

The result is sharp.

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ON FLEXIBILITY OF PRIME ASSOSYMMETRIC RINGS

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ABSTRACT

In this paper we show that a non-associative 2-and 3-divisible prime assosymmetric ring is flexible.

1. Introduction. E. Kleinfeld [1] introduced a class of non-associative rings called as assosymmetric rings in which the associator $(x,y,z) = (xy)z - x(yz)$ has the property $(x,y,z) = (p(x), p(y), p(z))$ for each permutation p of x,y and z. These rings are neither flexible nor power-associative. In [1] it is proved that the commutator and the associator are in the nucleus of this ring. By using these properties we show that a non-associative 2-and-3-divisible prime assosymmetric ring is flexible.

2. Preliminaries. Throughout this papr R will denote a non-associative 2-and 3-divisible assosymmetric ring. The commutator (x,y) of two elements x and y in a ring is defined by $(x,y) = xy - yx$. The nucleus N in R is the set of elements $n \in R$ such that $(n,x,y) = (x,n,y) = (x,y,n) = 0$ for all x,y in R. The center C of R is the set of elements $c \in N$ such that $(c,x) = 0$ for all x in R. A non-associative ring R is called flexible if $(x,y,x) = 0$ for all x,y in R. A ring is said to be power–associative if every subring of it generated by a single element is associative Let I be the associator ideal of R. I consists of the smallest ideal which contains all associators. R is called k-divisible if $kx = 0$ implies $x = 0$, $x \in R$ and k is a natural number.

In an arbitrary ring the following identities hold:

(1) $(wx,y,z) - (w, xy, z) + (w,x,yz) = w\,(x,y,z) + (w,x,y)z$

$f(w,x,y,z) = (wx,y,z) - x(w,y,z) - (x,y,z)\,w$

and

(2) $(xy,z) - x(y,z) - (x,z)y = (x,y,z) - (x,z,y) + (z,x,y).$

In any assosymetric ring (2) becomes

(3) $(xy,z) - x(y,z) - (x,z)\,y = (x,y,z)$

It is proved in [1] that in a 2-and 3-divisible assosymetric ring R the following identities hold for all w,x,y,z,t in R.

(4) $f(w,x,y,z) = 0$, that is, $(wx,y,z) = x(w,y,z) + (x,y,z)\,w$,

(5) $((w,x), y,z) = 0$

and

(6) $((w,x,y), z,t) = 0.$

That is, every commutator and associator is in the nucleus N. From

(3), (5) and (6), we obtain

(7) $x(y,z) + (x,z)y \subset N$.

Suppose that $n \in N$. Then with $w = n$ in (1) we get $(nx,yz) = n(x,y,z)$. Combining this with (5) yields.

(8) $(nx,y,z) = n(x,y,z) = (xn,y,z)$

From (7) and (8), we obtain

(9) $(y,z)(x,r,s) = -(x,z)(y,r,s)$.

3. Main Results

Lemma 1. Let $S = \{s \in N \mid s(R,R,R) = 0\}$. Then S is an ideal of R and $S(R,R,R) = 0$

Proof. By subsituting s for n in (8), we have $(sx,y,z) = s(x,y,z) = (xs,y,z) = 0$. Thus $sR \subset N \text{ and } Rs \subset N$. From (6), $sw(x,y,z) = s.w(x,yz)$. But (1) multiplied on the left by s yields $s.w(x,yz) = -s(w,x,y)z = -s(w,x,y).z = 0$. Thus $sw.(x,y,z) = 0$. From (9), we have $(s,w)(x,y,z) = -(x,w)(s,y,z) = 0$. Combining this with $sw.(x,y,z) = 0$, we obtain $ws.(x,y,z) = 0$. Thus S is an ideal of R. The rest is obvious. This completes the proof of the lemma.

Lemma 2. $(x,y,x) \in S$.

Proof. By forming the associatiors of both sides of (1) with u and v, and using (6), we obtain

(10) $(w(x,y,z), u,v) + ((w,x,y)z,u,v) = 0$.

Interchanging y and z in (10) and subtracting the result from (10), we get

(11) $((w,x,y)z,u,v) = ((w,x,z)y,u,v)$.

But $((w,x,z)y,u,v) = (y(w,x,z), u,v)$, because of (5). So that

(12) $((w,x,y)z,u,v) = (y(w,x,z), u,v)$, as a result of (11).

Also by permuting w and y in (10), we obtain $(y(w,x,z),u,v) + ((w,x,y)z,u,v) = 0$. This identity with (12) yields $2((w,x,y)z,u,v) = 0$. Thus

(13) $((w,x,y)z,u,v) = 0$.

From (6) we have $(x,y,x) \subset N$. Using (13) and (8), we get $0 = ((x,y,x)z,u,v) = (x,y,x)(z,u,v)$ for all x,y,z,u,v in R. Hence $(x,y,x) \in S$. This completes the proof of the lemma

Theorem. If R is a non-associative 2-and 3-divisible prime assosymmetric ring, then R is flexible.

Proof. Using lemma 1 and (1) we establish that $S.\,I = 0$. Since R is prime, either $S = 0$ or $I = 0$. If $I = 0$, R is associative. But we have assumed that R is not associative. Therefore $I \neq 0$. Hence $S = 0$. From lemma 2, $(x,y,x) \in S$. Thus $(x,y,x) = 0$. That is, R is flexible.

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NUMERICAL SOLUTION OF THE STEADY STATE NAVIER-STOKES EQUATIONS USING ELLIPTIC SOLVER

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ABSTRACT

The motion of a liquid under action of viscous forces is governed by Navier-Stokes equations. Exact solutions of *N-S* equations are available only under some ideal assumptions. Tremendous advances in the development and use of numerical methods, mainly due to availability of fast computing devices, have given rise to various types of numerical solutions of these equations. Driven cavity problem has worked as an ideal prototype, therefore present numerical scheme, which uses a fast direct elliptic equation solver is also tried on this problem. The results obtained compare well with those of previous authors.

1. Introduction. The Navier-Stokes equations for two dimensional steady flow of an incompressible fluid may be written in the vorticity (w) and streem function (ψ) formulation as :

$$\nabla^2\omega - Re\,(\psi_y\,\omega_x - \psi_x\omega_y) = 0 \qquad \text{... (1)}$$

$$\nabla^2\psi = -\,\omega \qquad \text{... (2)}$$

satisfying

$$u = \frac{\partial\psi}{\partial y},\ v = -\,\frac{\partial\psi}{\partial x} \qquad \text{... (3)}$$

and

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \text{... (4)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

The equation (1) and (2) can be formulated as follows.

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - Re \left\{ \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} \right\} = 0 \quad \text{... (5)}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = - w \quad \text{... (6)}$$

The equations (5) - (6) together with boudary conditions constitutes a non-linear elliptic boundary value problem. The degree of non-linearity increases with Reynolds number.

The numerical solution procedure generally consists of discretizing the domain over which equations (5) -(6) are defined. This gives two systems of algebraic equations. One can obtain the solution in terms of stream function and vorticity.

Burggraf [3] was first to employ central difference scheme for the solution of equations (5)-(6). The resulting systems of eqations were solved by using iterative procedures. But Iterative procedures fail to converge even at moderate Reynolds number. This difficulty originated the idea of upwind schemes. However, it is well known that upwind differncing introduces false diffusion effects and this will lead to additional error into the numerical solution as shown by Strikwerda [21]

Roache [16] gave the idea of what he calls Laplacian Driver Method. It is worth emphasizing that the superiority of Laplacian Driver Method over other methods is its simplicity to apply on computer, because one has to solve two Poisson's equations iteratively. We have applied an elliptic equation solver to solve equations (5) and Poisson's equation solver for equation (6). The system of equations (5) and (6) is solved iteratively until a desired convergence-criterion is satisfied.

Roache [16] has reported the solution of the driven cavity flow problem for Reynolds number $Re = 20$ with a 11×11 mesh. By using our technique, we have obtained the solutions upto Reynolds number $Re =$ 500 with a 21×21 mesh. The results obtained by us compare well with those of previous authors.

2. Derivation of Boundary Conditions. Since the values of vorticity on the boundary are not explicitly given, We compute these with the help of derivative boundary conditions on ψ Assuming equation (5) is valid near the wall and using n to represent the coordinate normal to the wall.

$$w(1) = - \left[\frac{\partial^2 w}{\partial n^2} \right]_1 \quad \text{... (7)}$$

where the argument and subscript '1' denote the mesh point on the wall. Using a Taylor's series expansion for ψ with step length Δn about the mesh point on the wall.

$$\psi(2) = \psi(1) + \Delta n \left[\frac{\partial \psi}{\partial n}\right]_1 + \frac{\Delta n^2}{2}\left[\frac{\partial^2 \psi}{\partial n^2}\right]_1 + \frac{\Delta n^3}{6}\left[\frac{\partial^3 \psi}{\partial n^3}\right]_1 + O\,(\Delta n^4) \qquad \dots (8)$$

$$\psi(3) = \psi(1) + 2\Delta n \left[\frac{\partial \psi}{\partial n}\right]_1 + \frac{4\Delta n^2}{2}\left[\frac{\partial^2 \psi}{\partial n^2}\right]_1 + \frac{8\Delta n^3}{6}\left[\frac{\partial^3 \psi}{\partial n^3}\right]_1 + O\,(\Delta n^4) \qquad \dots (9)$$

From (8) and (9) and making use of (7), we get

$$w(1) = \left[\psi(3) - 8\psi(2) + 7\psi(1) + 6\Delta n \left\{\frac{\partial \psi}{\partial n}\right\}_1\right] / (2\Delta n^2).$$

Since ψ (1) is zero on all four boundaries, the above equation reduces to

$$w(1) = \left[\psi(3) - 8\psi(2) + 6\Delta n \left\{\frac{\partial \psi}{\partial n}\right\}_1\right] / (2\Delta n^2) \qquad \dots (10)$$

This is second order approximation for vorticity and was used by Bozeman and Dalton [2]. Roache [16] calls it Jensen's fourmula. It is also referred as Briley's formula. Gupta and Manohar [9] have investigated the effects of boundary approximations on the solution and shown that Briley's formula gives more accurate results than any of the other formula.

3. Methematical Model. From a computational viwepoint, the cavity flow is an ideal prototype non-linear problem which is readily posed fo numerical solution. Because of its geometrical simplicity and comparatively minor singularities, it has served as a model problem for last twenty years for testing new numerical schemes and as a benchmark solution for making comparisons among various schemes. Fortunately, a number of numerical methods have been tried on this problem and results have been published. We have also applied our technique for the solution of this classical problem and obtained results which are comparable with the previous results.

The cavity, as shown in fig. (A) is rectangular in cross-section and filled with a Newtonian, viscous and incompressible fluid. The fluid is forced to make by the motion of the upper surface which travels with constant linear velocity in its own plane. The cavity is assumed to be long in the longitudinal (*Z*) direction so that the fluid motion is essentially two dimensional.

The formation of the governing equation for steady motion has been described in detail by Mills [13] and Burggraf [3]. The steady flow in the square cavity (fig. (B)) is governed by the equations (5) and (6) together with the under mentioned boundary conditions.

The boundary conditions for ψ may be derived from the conditions that each wall is impermeable and that the viscous boundary conditions implies that the fluid adjacent to the wall moves with the same velocity as the wall. The first condition implies that $\psi = 0$ on each boundary and second that the derivative of ψ in direction normal to the wall is equal to the tangential velocity of the wall. The boundary conditions thus obtained for driven cavity problem are as under :

$\psi = 0$ along all boundaries

$\psi_x = 0$ along vertical walls OC and AB ... (11)

$\psi_y = 0$ along the bottom wall OA ... (12)

$\psi_y = -1$ along th sliding wall BC. ... (13)

... (14)

4. Different Approaches Tried for This Problem. Burggraf [3] was first to employ central difference schemes for this problem using iterative procedures. Since the diagonal dominance property is not nacessarily satisfied, the standard iterative procedures such as Gauss-Seidel or SOR fail to converge rapidly even at moderate high Reynolds number. Later Spalding [20] discovered that stable solution could be obtained if upwind differencing methods were used for the convective terms in the vorticity equation. Further, Finite difference calculations using similar ideas have been presented by Nallaswamy and Krishna Prasad [14]. However, recently Strikwerda [21] and others have brought out clearly the drawbacks of upwind schemes.

The second approach to overcome the convergence problem is that at each outer iteration, the algebraic equations corresponding (5) - (6) are solved by direct fast solvers. Gupta [8] has applied this approach and used direct solver *MA 28* from the Harwell Package [5]. But These solvers do not take the benefit of sparsity into consideration and thus are not economical in computer-time and storage.

5. Difference Equations. The rectangular region over which the equations (1) and (2) are defined is divided into a uniform mesh by choosing mesh widths h along x-direction and k along y-direction. Applying central difference formulae to (1) and (2), relative to a uniform mesh system, we obtain

$$A_1 w_{i,j-1} + A_2 w_{i-1,j} + A_3 w_{i,j} + A_4 w_{i+1,j} + A_5 w_{i,j+1} = A_6 \qquad \text{... (15)}$$

where

$$A_1 = \frac{1}{k^2} + \frac{Re}{4hk}\left(w_{i+1,j} - w_{i-1,j}\right)$$

$$A_2 = \frac{1}{h^2} + \frac{Re}{4hk}\left(w_{i,j+1} - w_{i,j-1}\right)$$

$$A_3 = -2\left[\frac{1}{h^2} + \frac{1}{k^2}\right]$$

$$A_4 = \frac{1}{h^2} + \frac{Re}{4hk}\,(w_{i\,,\,j+1} - w_{i\,,\,j-1})$$

$$A_5 = \frac{1}{k^2} + \frac{Re}{4hk}\,(w_{i+1\,,\,j} - w_{i-1\,,\,j})$$

$$A_6 = 0$$

and

$$B_1\psi_{i,\,j-1} + B_2\,\psi_{i-1\,,\,j} + B_3\,\psi_{i\,,\,j} + B_4\,\psi_{i+1\,,\,j} + B_5\psi_{i\,,\,j+1} = B_6 \qquad \ldots (16)$$

where

$$B_1 = \frac{1}{k^2},\quad B_2 = \frac{1}{h^2},\quad B_3 = -2\,\left[\frac{1}{h^2} + \frac{1}{k^2}\right],\ B_4 = \frac{1}{h^2},\ \ B_5 = \frac{1}{k^2}\ ,\ B_6 = -w_{i\ j}.$$

6. Computational Procedure. Following are the steps involved in the computational procedure.

(a) Assign initial approximation $\psi^{(m)}$ with $m = 0$. Here approximation is taken as $\psi^{(0)} = 0$.

(b) Compute boundary values for ω by boundary conditions as in (10).

(c) Solve for Vorticity $\varpi^{(m+1)}$ from

$$A_1 w^{m+1}_{i,\ j-1} + A_2\, w^{m+1}_{i-1,\,j} + A_3\, w^{m+1}_{i,\,j} \ \ + A_4\, w^{m+1}_{i+1,\,j} + A_5 w^{m+1}_{i,\,j+1} = A_6$$

(d) Damp values of Vorticity in interior of domain as flollows

$$\omega^{(m+1)} = \delta\varpi^{(m+1)} + (1\text{–}\delta)\,\omega^{(m)}, \qquad\qquad 0 < \delta < 1.$$

(e) Solve stream function equation to obtain $\psi^{(m+1)}$ as

$$B_1\psi^{m+1}_{i,\ j-1} + B_2\,\psi^{m+1}_{i-1,\,j} + B_3\,\psi^{m+1}_{i,j} \ \ + B_4\,\psi^{m+1}_{i+1,\,j} + B_5\psi^{m+1}_{i,\,j+1} = B_6 \ \ldots (16)$$

(f) Damp values of stream function as

$$\psi^{(m+1)} = \delta\psi^{(m+1)} + (1\text{–}\delta)\ \psi^{(m)}$$

(g) Repeat steps (b) - (f) until following convergence criterion is satisfied :

$$max\ |\ \omega^{(m+1)}_{i,\,j} - \omega^{(m+1)}_{i,\,j}\ | < \in .$$

(h) After above convergence criterion is satisfied, boundary values for ω are again computed as in (b) and solution is attained over whole domain.

7. Algorithm. The resulting structure of system of equation (15)-(16) can be rewritten as :

$$A\overline{U} = \overline{V} \qquad (17)$$

Coefficient matrix A is a block tri-diagonal system having n blocks where each block is of order (mxm). $\overline{U}$ is a column matrix of unknown and $\overline{V}$ is also a column matix containing right hand sides of equations. Since diagonal dominance property in such a system does not always hold, so iterative methods sometimes fail to converge. Recently some efficient direct methods have been employed to solve such a system. Linger [12] has applied a sem-direct method to solve Poisson's equation.

Elsner and Mehramann [6] have dealt in detail on the convergence condition of block iterative methods. The block tri-digonal system of linear equations is a sparse coefficient matrix system and it is possible to take the advantage of the sparsensess in order to reduce both computation time and storage requirements. Duff [4], Erisman and Reid [7] have dealt in detail with the direct methods for sparse matrices. Jennings [11] has also given methods like elimination using submatrices to deal with a sparse structure of such type.

Present algorithm also employs a direct elimination technique to solve block tri-diagonal system of linear equations. Imortant steps in brief are as follows. Detailed analysis of algorithm and storage etc. are given in Sharma and Agarwal [19].

The system (13) is written in matrix form as follows.

$$\begin{bmatrix} A_1 & B_1 & & & \\ C_2 & A_2 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n-1} & A_{n-1} & B_{n-1} \\ & & & C_n & A_n \end{bmatrix}_{mn \times mn} \begin{bmatrix} \overline{U}_1 \\ | \\ | \\ | \\ \overline{U}_{n-1} \\ \overline{U}_n \end{bmatrix} = \begin{bmatrix} V_1 \\ | \\ | \\ | \\ V_{n-1} \\ V_n \end{bmatrix}_{mn \times 1} \quad \dots (18)$$

where A_j' s, $j = 1\ (1)\ n$ are tri-diagonal matrices, B_j's and C_j's, are diagonal matrices given by:

$$\begin{bmatrix} \beta_{1j} & \gamma_{1j} & & & \\ \alpha_{2j} & \beta_{2j} & \gamma_{2j} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \beta_{m-1,j} & \gamma_{m-1,j} \\ & & & \alpha_{mj} & \beta_{mj} \end{bmatrix}_{m \times m} \quad \dots (19)$$

$$B_j = Diag\ [\delta_{1j}, \delta_{2j}, \dots \delta_{mj}]_{m\times m} \quad \dots (20)$$

$$C_j = Diag\ [\eta_{1j}, \eta_{2j}, \dots \eta_{mj}]_{m\times m} \quad \dots (21)$$

where $j = 1, 2, \dots, n$

$\overline{u}_j$, $j = 1\ (1)n$ is unkown column vector defined as

$\overline{u} = [u_{1j}, u_{2j}, \dots, u_{mj}]^T$

V_j, $j = 1\ (1)\ n$ is right hand side modified after necessary adjustment due to given boundary conditions.

$V = [V_{1j}, V_{2j}, \dots V_{mj}]^T$.

Let R_{ij}, $i = 1(1)m$, $j = 1(1)n$ denotes i^{th} row of j^{th} block. Elimination algorithm is as follows :

Step 1.

For $j = 1$

Do $R_{i,1} = R_{i,1} - (\alpha_{i,1}/\beta_{i-1,1})*(R_{i-1,1})$ for $i = 2$ to m.

Step 2.

For $j = 1$

(a) $i = 1$

(b) (i) $k = 1$

$R_{i,j} = R_{i,j} - (\eta_{i,j}/\beta_{k,j-1})*(R_{k,j-1})$.

By this operation, first entry of first row of second block becomes zero and second entry of first row of second block becomes non-zero, say $\eta_{i,j}$.

(ii) For $k = k+1$ up m do

$\overline{R}_{i,j} = \overline{R}_{i,j} - (\eta_{i,j}/\overline{\beta}_{k,j-1})*(\overline{R}_{k,j-1})$

Where "–" refers to modified entries.

(c) $i = i+1$

(i) $R_{i,j} = R_{i,j} - (\eta_{i,j}/\overline{\beta}_{k,j-1})*(\overline{R}_{k,j-1})$

(ii) for $k = i$ to m (just like 2(b))

$\overline{R}_{i,j} = \overline{R}_{i,j} - (\eta_{i,j}/\overline{\beta}_{k,j-1})*(\overline{R}_{k,j-1})$

(iii) $\overline{R}_{i,j} = \overline{R}_{i,j} - (\alpha_{i,j}/\overline{\beta}_{i-1,j})*(\overline{R}_{i-1,j})$.

(d) Again go to step 2(c) upto $i = m$

Step 3.

(i) Put $j = j+1$ and repeat steps 2(a) to 2 (d)

(ii) Continue this process upto $j = n$.

Step 4.

Now the system is reduced to upper triangular form and by back substitution process, we may obtain values of unknowns u_{ij}, $i = 1(1)m$, $j=1(1)n$.

8. Result and Discussions. Computations are made taking uniform mesh-size $h = 1/20$ for various Reyonlds number. The point at which the value of ψ attains it's absolute maximum is called the center of primary vortex (*vc*). We denote the values of ψ and at the vortex center by ψ_{max} and ω_{vc} respectively. We also give the value of drag-coefficient on the sliding wall defined by-

$$cd = \frac{2}{Re}\int_0^1 \omega\ (s,1)\ ds = \frac{2}{Re}\overline{F}$$

where $\overline{F}$ is the value of shear force on sliding wall. The integral here is obtained by using Simpson's one-third rule over mesh points on the sliding wall.

It is evident from the table (I), (II) and (III) that resuts obtained compare well with those of previos authors [[2],[9], [14], [8], [10], [17], [18], [1]]. Streamlines and eqivorticities curves for different Reynolds number have been analysed. It is clear that there is no secondary vortex at $Re = 1$ and 10, but there exists two secondary vorticities at the downstream corners for $Re = 100$ to $Re = 500$. Also the size of secondary vortices increases with the increase in Reynolds number as observed experimentally by Pan and Acrivos [15]. The equivorticity curves become more asymmetrical and recirculating eddies become more dominant with the increase in Reynolds number. The equivorticity curve at $Re = 500$ has a secondary eddy on the bottom wall at the level -1.0. The same nature of vorticity curve was observed by Ghia, Ghia and Shin [10] at $Re = 400$ using a much finer mesh.

TABLE -1

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for Re =1 and 10.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference
1	0.0978	3.3553	(0.35,0.75)	21.1438	$\psi_{max} = 0.0982$	h=1/20 (9)
					$= 0.0995$	h=1/20 (9)
					$= 0.0995$	h=1/20 (8)
					$= 0.1082$	h=1/20 (9)
					$\omega_{vc} = 2.95$	h=1/20 (9)
					$= 3.02$	h=1/20 (9)
					$= 3.02$	h=1/20 (8)
					$= 3.33$	h=1/20 (9)
10	0.1130	3.3496	(0.35,0.75)	2.1170	$\psi_{max} = 0.1043$	h=1/30 (17)
					$= 0.1095$	h=1/20 (9)
					$\omega_{vc} = 3.570$	h=1/20 (9)
					$= 3.155$	h=1/50 (14)

TABLE -II

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for *Re* =*100* and *400*.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference
100	0.113	3.3496	(0.35,0.75)	0.2321	$\psi_{max} = 0.1043$	$h=1/30$ (17)
					$= 0.1095$	$h=1/20$ (9)
					$\omega_{vc} = 3.570$	$h=1/20$ (9)
					$= 3.155$	$h=1/50$ (14)
400	0.1027	2.2363	(0.40,0.60)	0.0763	$\psi_{max} = 0.1129$	$h=1/40$ (18)
					$= 0.1139$	$h=1/30$ (10)
					$= 0.0970$	$h=1/20$ (1)
					$\omega_{vc} = 2.2810$	$h=1/40$ (18)
					$= 2.2947$	$h=1/30$ (10)
					$= 2.3600$	$h = 1/20$ (1)

TABLE -III

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for *Re* =500 and 1000.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference
500	0.1009	2.0132	(0.45,0.60)	0.0641	$\psi_{max} = 0.1024$	$h=1/20$ (8)
					$\omega_{vc} = 2.0504$	$h=1/20$ (8)
					$= 1.9048$	$h=1/20$ (8)
1000	0.07914	1.6399	(0.45,0.60)	0.0367	$\psi_{max} = 0.0812$	$h=1/50$ (2)
					$= 0.0972$	$h=1/20$ (8)
					$\omega_{vc} = 1.7452$	$h=1/20$ (8)

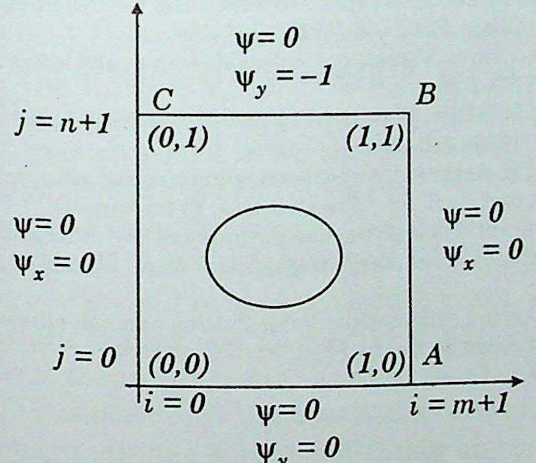

FIG. (A)- SQUARE DRIVEN CAVITY FLOW PROBLEM

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WAITING TIME DISTRIBUTION IN A REPAIR SYSTEM WITH MULTIDIMESIONAL STATE SPACE AND RANDOM BATCH ARRIVAL FOR EXCHANGABLE ITEMS

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ABSTRACT

In the present paper, we consider multidimensional state space with random batch arrival and compute the probability distributions of the waiting times under steady state for exchangable items in the system.

We finally discuss the working and non-working states of the system and present the transitional probability distributions of it under these two states by making use of Markov Process and Kolmogorov forward Equations.

1. Introduction. Stochastic networks are models for flexible manufacturing system (*FMS*) computer networks, communication systems and complex repair system etc. From a customer's point of view, perhaps the most important performance measure of a such system is the sojourn time (or moment of it) in the network or in parts of it. These sojourn times represent (individual) production or repair times for an items, the time for a data packet to be transmitted by the communication system etc. In all these cases, we observe the individual possibly random route of a typical (tagged) customer from arrival at the system until departure. For a review of the results concerning the distribution of these individual sojourn times in stochastic networks, see for example Schassberger [5] and Diney and keing [4].

Breg and Posner [1] observe that these are important problems with sojourn times (delay times) in repair system or *FMS* that can not be modeled by standard individual sojourn times. These problems arise (in the example of Breg and Posner [1], p. 287) when customers bring failed items to the repair system and the items are exchangable in the sense that a customer does not necessarily want to back the particular item be brought into the system but rather any good item of the same kind. They distinguish a "Customer queue" under a first come first served

(FCFS) discipline where the customers wait for any repaired item and "item queue" of ***M/M/C-FCFS*** system where the items wait for repair at one of the repair channels. The multichannel structure of the repair system implies the possibility of overtaking for items such that customer does not necessarily obtain his own item back.

While delay times usually are computed for individual items at the item queue (which is easy to do for the ***M/M/C*** system under consideration) Breg and Posner [1] investigated delay times at the customer queue, a problem which turned out to be much more complicated. In addition, they introduce spares at the service centre which, if at hand, may be handed immediately to customers who brought in failed items.

Clearly, such problems arise in general complex repair system or in *FMS* (for an example see Schroder [6]) which leads to the questions: what can be said about the delay distribution when customers are waiting in a queue (under some general discipline) for exchangable items which migrate through a general stochastic network.

A careful investigation of the proof of Berg and Posner [1] shows that it strongly depends on the fact that the state space of the system is one dimensioanl and it turns out that the necessity of more complicated state space to describe general (multiqueue) repair systems which lead to some apparently new problems. They considered the simplest system with two dimensional state space i.e. two different parallel sources each having its own queues under *FCFS* and computed the steady state probability distribution of the waiting times for exchangable items in that systems. They preferred two statespace case avoiding complexities of probabilty distribution, their generating functions and *LSTs* in the analysis of the multistate problem.

In the present work, we consider multidimensional state space with random batch arrival and compute the probability distribution of the waiting times under steady state for exchagable items in the system assuming that size of the batch of the customers be represented by random variable X such that $E(X) = a$.

We finally discuss the working and non-working states of the system and present the transitional probability distributions of it under these two states by making use of Markov Process and Kolmogorov Forward Equations.

2. The System. Customers arrive to the system (see figure) in Poisson stream of intensity a $\lambda > 0$ at a repair station deliver a failed item to the station and proceed to customer queue, which is organized by using ***FCFS*** rule. The items ae assumed to be exchangable that is customers accept any repaired item that is given to them. An item leaving the repair station is given to the customer at the head of queue which immediately departs from the queue.

The repair station consists of n-batch servers each having its own queue with queueing discipline ***FCFS***. An arriving customer joins the queue of server 1 with probability $p_1 \in (0,1)$, server 2 with probility $p_2 \in (0,1)$ ... server n-1 with probability $p_{n-1} \in (0,1)$ and the queue of server n with probability $p_n = 1 - \sum_{i=1}^{n=1} p_i$. The service times at server 1 is $exp\ (\mu_i)$, server 2 is $exp\ (\mu_2)$ ... server $n-1$ is $exp\ (\mu_{n-1})$, and at server n is $exp(\mu_n)$ distributed. We assume the family of inter arrival times, service times and routing decisions to be an independent family of random variables.

Because we are intrested in steady state results, we assume that $\lambda p_1 < \mu_1, \lambda p_2 < \mu_2, \ldots, \lambda p_{n-1} < \mu_{n-1}$ *and* $\lambda p_n < \mu_n$.

A Markovian description of the systems development over time is given by recording the joint queue length process at server $1, 2, \ldots, n-1$ and n. We assume that the associated Markov process is stationary with steady state distribution.

$$\Pi(m_1, m_2, \ldots, m_{n-1}, m_n) = \prod_{i=1}^{n} \left(1 - \frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} \prod_{i=1}^{n-1} \left(1 - \frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i}$$

$$(m_1, m_2, \ldots, m_{n-1}, m_n) \in N^n$$

Theorem.

$$f(s) = \prod_{i=1}^{n} \left[\frac{\mu_i - a\lambda p_i}{\mu_i - a\lambda p_i \left[\dfrac{\mu_i}{\mu_i + \mu_{i+1}}\right]} \cdot \frac{\mu_{i+1} - a\lambda p_{i+1}}{\mu_{i+1} - a\lambda p_{i+1} \left[\dfrac{\mu_{i+1}}{\mu_i + \mu_{i+1}}\right] s} \left(1 + \frac{s}{\mu_i + \mu_{i+1}}\right) \right]$$

$$- \prod_{i=1}^{n} \Bigg[- \frac{s}{\mu_i + \mu_{i+1}} \Bigg\{ p_{i+1} \frac{\mu_i - a\lambda p_i}{\mu_{i+1} - a\lambda p_{i+1} + \left[\dfrac{\mu_n}{\mu_i + \mu_{i+1}}\right] s} \left(1 - s \frac{a\lambda p_{i+1}}{\mu_{i+1} - a\lambda p_{i+1}}\right)$$

$$\left(a\lambda p_i + \mu_i + \mu_{i+1} - \frac{\mu_{i+1}}{\mu_{i+1} + a\lambda p_{i+1}} s - \frac{1}{\theta(\mu_{i+1} p_{i+1} a\lambda p_i \mu_i)}\right)^{-1}$$

$$+ \prod_{i=1}^{n} p_i \frac{\mu_i - a\lambda p_{i+1}}{\mu_i - a\lambda p_i + \left[\dfrac{\mu_i}{\mu_i + \mu_{i+1}}\right]} \left(1 - s \frac{a\lambda p_{i+1}}{\mu_i - a\lambda p_i} \cdot\right)$$

$$\left(a\lambda p_{i+1} + \mu_i + \mu_{i+1} - \frac{\mu_i}{\mu_i + a\lambda p_i} s - \frac{1}{\theta(\mu_i, p_i, \mu_i, s)}\right)^{-1} \Bigg\} \Bigg]$$

where $\theta(\eta, r, w, s) = \dfrac{A}{2a\lambda rw} \eta > 0,\ r \in [0,1],\ w \geq 0,\ s \geq 0$

where $A = \eta \dfrac{(\lambda\,(1-2r)+\mu_i+\mu_{i+1}+s)}{-\sqrt{\eta^2\,(\lambda\,(1-2r)+\mu_i+\mu_{i+1}+s)^2-4a\lambda p_i p_{i+1}\mu_i\mu_{i+1}}}$.

Proof. We introduce a sequence of conditional *LSTs* associated with the departure proce ss from the system.

Suppose that the system jumps any transition into the state $(m_1, m_2, \dots, m_{n-1}, m_n) \in N^2 - [0,0]$ and a clock is started at this jump instant. For some $k \in [1, 2, \dots m_1 + m_2, \dots, m_{n-1} + m_n]$. Let the clock be stopped at the k^{th} departure instant after the clock is started and denoted by

$$f\,(k\,;\, m_1, m_2, \dots, m_{n-1}, m_n)\,(s)\,,\ s \ge 0.$$

The *LST* of the distribution of the time recorded by that clock. Then

$$f(s)=\Sigma\pi(m_1,m_2,\dots,m_n)\{p_1 f(m_1+m_2+1,m_1+1,m_2)(s)+p_2\,f(m_1+m_2+1,m_2+1)(s)\},$$
$$\dots, + p_n\,f(m_1, m_2, \dots, m_{n-1}, m_n, m_{n+1} = m_1)(s)\},\ s \ge 0 \qquad \dots(1)$$

The key to our analysis is the following sequence of first entrance equation for the conditional *LSTs*. For

$1 \le k \le m_1+m_2+\dots+m_n$.

$$f(k;\, m_1,m_2, \dots, m_n)\,(s) = \frac{\pi(m_1,m_2, \dots, m_n)}{\pi(m_1,m_2, \dots, m_n)+s}\Big[\frac{a\lambda p_1}{\pi(m_1,m_2)}$$
$$f\,(k; m_1+1,\, m_2)(s) + \dots + \dots + \frac{a\lambda p_n}{\pi(m_1)}\,f(k;\, m_n+1,\, m_1)(s)+ I_{m_{1>0}}\frac{\mu_1}{\lambda(m_1,m_2)}$$
$$f\,(k-1,m_2-1,m_2)(s) + I_{m_{2>0}}\frac{\mu_2}{\pi\ m_2,m_3} f\,(k-1,\, m_2-1,\, m_3)(s) + \dots +$$
$$I_{m_{n>0}}\ \frac{\mu_n}{\pi\ (m_n,m_1)}\,f\,(k-1,\, m_n,\, m_1-1\,)(s)\Big]\,,\ s \ge 0 \qquad \dots(2a)$$

where $\lambda\,(m_1,m_2, \dots, m_n) = a\lambda+I_{m_{1>0}}\mu_1+ \dots + I_{(m_{n>0})}\mu_n$ and for

$$m_1+m_2+m_3+\dots+m_n \ge 0,\ f(0,\, m_1,m_2, \dots, m_n)(s) = 1,\ s \ge 0 \qquad \dots(2b)$$

We shall concentrate on cmputing in first step

$$F(0,s) = \Big(\sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty}\ \prod_{i=1}^{m}\ \Big(\frac{a\lambda p_i}{\mu_i}\Big)^{m_i} . f\,(m_1+m_2+\dots+m_n)(s),\ s \ge 0 \qquad \dots 3$$

which by a symmetry argument will provide (1). It turns out that we have to compute a recursion for the sequence

$F(h,s)\,;\ h = 0,1,\dots,\ s \ge 0$

to eventually obtain $F(0,s)$, *where*

$$F(h,s) = \sum_{m_1=h_1}^{\infty} \dots \sum_{m_n=h_n}^{\infty}\ \prod_{i=1}^{m}\ \Big(\frac{a\lambda p_i}{\mu_i}\Big)^{m_i} . f\,(\,m_1+m_2+\dots+m_n+1-h_i, m_i+1, \dots, m_n\,)s, \qquad \dots(4)$$
$$s \ge 0,\ h_i=0,1\dots$$

For the given $h_i \in n,\ m_i \in h_i, \dots, m_{n-1} \in h_i, m_n \ge 0$ and

$K = m_1 + m_2 +, \ldots, m_n - h_i$
we multiply (2a) (*with* $m_1 + 1, m_2 + 1, \ldots,$ *instead of*
$m_1, m_2, \ldots, +m_n$) *by*
$(\Lambda\ (m_1 + 1, m_2 + 1, \ldots, m_{n-1}, m_n) + s) \prod_{i=1}^{n} \left(\frac{\lambda p_i}{\mu_i}\right)^{m_1}$
and summing over $m = h_i, h_i + 1, \ldots, m_n = 0, 1, \ldots,$
we obtain, after some manipulations

$$
\begin{aligned}
F(h_1, s) &= F(h_1 + 1, s)\frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + s} + \frac{a\lambda p_i}{\mu_1}\ \frac{\mu_1}{(\mu_1 + \mu_2 + s)} \\
&+ (D(h_1, s) + D(h_1 + 1; s)\ \left(\frac{\mu_2}{a\lambda p_1}\right) \frac{\mu_1}{(\mu_1 + \mu_2 + s)} \\
&+ (c(h_1, s) + c(h_1 + 1; s)\ \frac{\mu_1}{\mu_1 + \mu_2 + s}, s \geq 0,\ h_1 \in \mathrm{N} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
F(h_n, s) &= F(h_n + 1; s)\frac{\mu_n + \mu_1}{\mu_n + \mu_1 + s} + \frac{a\lambda p_n}{\mu_n}\frac{\mu_n}{(\mu_1 + \mu_n + s)} \\
&+ (D(h_n, s) + D(h_n + 1; s)\ \left(\frac{\mu_n}{a\lambda p_n}\right)\frac{\mu_n}{(\mu_n + \mu_1 + s)} \\
&+ (c(h_n, s) + c(h_n + 1, s)\ \frac{\mu_n}{\mu_n + \mu_1 + s}, s \geq 0,\ h_1 \in \mathrm{N}.
\end{aligned}
$$

We can write the above expression in general terms

$$
\begin{aligned}
F(h_i; s) &= F(h_i + 1; s)\ \frac{\mu_i + \mu_{i+1}}{\mu_i + \mu_{i+1} + s}\ \frac{\alpha\lambda p_i}{\mu_i}\frac{\mu_i}{(\mu_i + \mu_{i+1} + s)} \\
&+ (D(h_i, s) + D(h_i + 1; s)\ \left(\frac{\mu_i}{a\lambda p_n}\right)\frac{\mu_i}{(\mu_i + \mu_{i+1} + s)} \\
&+ (c(h_i, s) - c(h_i + 1; s)\ , \frac{\mu_i}{\mu_i + \mu_{i+1} + s}, s \geq 0,\ h_1 \in \mathrm{N}
\end{aligned}
$$

where

$$
\begin{aligned}
C(h_1; s) &= \sum_{m_1 = h_1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1} f(m_1 + 1 - h_1; m_1 + 1, 0)(s), s \geq 0, h_1 \in \mathrm{N} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
C(h_n; s) &= \sum_{m_n = h_n}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n + 1 - h_n; m_n + 1, 0)(s), s \geq 0, h_n \in \mathrm{N} \\
C(h_i; s) &= \sum_{m_1 = h_1}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_n} f(m_i + 1 - h_i; m_i + 1, 0)(s), s \geq 0, h_i \in \mathrm{N} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
D(h_i; s) &= \sum_{m_1 = 1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{h_1}\left(\frac{a\lambda p_2}{\mu_2}\right)^{m_2} f(m_1, h_1; m_2)(s), s \geq 0, h_1 \in \mathrm{N} \\
D(h_i; s) &= \sum_{m_n = 1}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n, h_n; m_n)(s), s \geq 0, h_n \in \mathrm{N} \\
D(h_i; s) &= \sum_{m_i = 1}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i, h_i; m_i)(s), s \geq 0, h_i \in \mathrm{N}.
\end{aligned}
$$

Evaluating the generating functions $\sum_{h_n=1}^{\infty}(h_n,s)z^{h_n}, |z|<1$, for any $s\geq 0$, at $z=\left(\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s}\right)$ we eventually obtain

$$F(0,s)=\frac{\mu}{\mu_1(\mu_1+\mu_2+s)-a\lambda p_1(\mu_1+\mu_2)}$$

$$+c(0,s)\frac{\mu_2}{\mu_1+\mu_2}-D(0,s)\frac{\mu_1\mu_2}{a\lambda_1 p_1(\mu_1+\mu_2)}$$

$$-\sum_{h_1=0}^{\infty}\left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}\right)^{h_1}c(h,s)\frac{\mu_2}{(\mu_1+\mu_2)(\mu_1+\mu_2+s)}$$

$$+\sum_{h_1=0}^{\infty}\left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}\right)^{h_1}D(h_1,s)\frac{\mu_2(\mu_1+\mu_2+s)+a\lambda p_1(\mu_1+\mu_2)}{a\lambda p_1(\mu_1+\mu_2)(\mu_1+\mu_2+s)}, s>0.$$

$$\begin{matrix}\vdots & \vdots & \vdots\end{matrix}$$

$$F(0,s)=\frac{\mu_n^2}{\mu^2_n(\mu_n+\mu_1+s)-a\lambda p_1(\mu_n+\mu_1)}+C(0,s)\frac{\mu_n}{\mu_n+\mu_1}$$

$$-D(0,s)\frac{\mu_n\mu_1}{a\lambda p_n(\mu_n+\mu_1)}-\sum_{h_n=0}^{\infty}\frac{\mu_n\mu_1}{a\lambda p_n(\mu_1+\mu_2)}$$

$$-\sum_{h_n=0}^{\infty}\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s}-D(h_n,s)\frac{\mu_2(\mu_2+\mu_1+s)+a\lambda p_1(\mu_n+\mu_1)}{a\lambda p_n(\mu_1+\mu_n)(\mu_1+\mu_n+s)}.$$

Finally, general expression can be established as

$$F(0,s)=\frac{\mu_i^2}{\mu_i(\mu_i+\mu_{i+1}+s)-a\lambda p_1(\mu_i+\mu_{i+1})}-$$

$$+C(0,s)\frac{\mu_{i+1}}{\mu_i+\mu_{i+1}}-D(0,s)\frac{\mu_i\mu_{i+1}}{a\lambda p_i(\mu_1+\mu_{i+1})}$$

$$-\sum_{h_i=0}^{\infty}\left(\frac{\mu_i+\mu_{i+1}}{\mu_i+\mu_{i+1}+s}\right)^{h_i}-D(h_i,s)\frac{\mu_i(\mu_i+\mu_{i+1}+s)+a\lambda p_n(\mu_i+\mu_{i+1}+s)\mu_i}{a\lambda p_i(\mu_i+\mu_{i+1})(\mu_i+\mu_{i+1}+s)} \quad ...(5)$$

It remains to compute the boundary terms. This will be done using the observations that the following equations hold

$$f(k;k+m_1,m_2)(s)=f(k;k,m_2)(s),\ k,m_1,m_2\in N \quad ...(6a)$$

$$f(k;m_1,k+m_2)(s)=f(k;m_1,k)(s),\ k,m_1,m_2\in N \quad ...(6b)$$

$$\begin{matrix}\vdots & \vdots & \vdots\end{matrix}$$

$$f(k;k+m_{n-1},m_n)(s)=f(k;k,m_n)(s),\ k,m_{n-1},m_n\in N \quad ...(6c)$$

$$f(k;m_{n-1},k+m_n)(s)=f(k;m_{n-1},k)(s),\ k,m_{n-1},m_n\in N \quad ...(6d)$$

$$f(k;k+m_n,m)(s)=f(k;k,m_{n-1})(s),\ m_n,m_n\in N \quad ...(6e)$$

$$f(k;m_n,k+m_1)(s)=f(k;m_n,k)(s),\ k,m_n,m_1\in N \quad ...(6f)$$

$$\begin{matrix}\vdots & \vdots & \vdots\end{matrix}$$

$$f(k;m_i,k+m_{i+1})(s)=f(k;m_i,k)(s),\ k,m_i,m_{i+1}\in N \quad ...(6g)$$

These equations are easily interpreted. In the time for the next departures from system that to be computed, we can neglect queuing customers that set atleast k customers in front of themselves (in their own queue) because after the departure of these k customers the requested time has already passed.

We introduce,

$$B(j,s)=\sum_{m_1=0}^{\infty}\left(\frac{a\lambda p_1}{\mu_n}\right)^{m_1} f(m_1+1,m_1+1,j)(s);s\geq 0,\ j\in N$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$B(j_n,s)=\sum_{m_n=0}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1,m_n+1,j_n)(s);s\geq 0,\ j\in N$$

We can write the general expression of the above equations

$$B(j_i,s)=\sum_{m_i=0}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i+1,m_i+1,j_i)(s_i);s\geq 0,\ j_i\in N$$

Now,

$$C(h_1,s)=\sum_{m_1=h_1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1} f(m_1+1-h_1,m_1+1-h_1,0)(s);s\geq 0\ ,\ h_1\in N$$

$$=B(0,s)\left(\frac{a\lambda p_i}{\mu_i}\right),s\geq 0\ h_1\in N.$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$C(h_n,s)=\sum_{m_1=0}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1-h_n,m_n+1-h_n,0)(s);s\geq 0,\ h_n\in N.$$

Thus general expression from equations (6a) (6g) may be obtained as

$$C(h_i,s)=\sum_{m_i=h_i}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i+1-h_i,m_i+1-h_i,0)(s);s\geq 0,\ h_i\in N.$$

To obtain $B(0,s)$ note that for the conditional *LSTs* introduces (2a) reduces

$$\begin{aligned}&f(m_1+1,m_1+1,j_1)(s)\ [a\lambda p_2+\mu+I_{(j>0)}\mu_2+s]\\&=f(m_1+1,m_1+1,j_1+1)(s)a\lambda p_2+f(m_1,m_1,j)(s)(\mu_1)+f(m_1,m_1,j\text{-}1)(a\lambda)I_{(j_1>0)}\\&\qquad\qquad j\in N,\ m_1\in N,\ s\geq 0.\end{aligned} \qquad \ldots(7)$$

$$\begin{aligned}&f(m_n+1,m_n+1,j_n+1)\ (s)\ [a\lambda(p_1)+\mu_n+I_{(j_n>0)}\mu_1+s]\\&=f(m_n+1,m_n+1)\ (s)\ a\lambda(p_n+1)\\&+f(m_n,m_n,j_n)\ (s)\ (\mu_n)+f(m_n,m_n,j_n\text{-}1)(\lambda)((\mu_{n+1}z\mu_1)I_{(j_n>0)}\\&\qquad\qquad j_n\in N,\ m_n\in N,\ s\geq 0\end{aligned} \qquad \ldots(8)$$

Multiplication of (7) and (8) by $\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1}$ and $\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n}$ respectively and summation over m_1 and $m_n\in N$ yeilds the $B(j_n,s)$ respectively.

Evaluating the generating functions

$$\sum_{j_1=0} z^{j_1} B(j_1,s), |z| < 1,\ s \geq 0$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\sum_{j_n=0} z^{j_n} B(j_n,s), |z| < 1,\ s \geq 0$$

at $\theta(\mu_1, p_1, \mu_2, s)$ and $\theta_1(\mu_n, p_n, \mu_1, s)$ the smaller roots of the equation

$$a\lambda p_1 \mu_2 z \ldots \mu_1(\lambda(1\text{-}2p_1) + \mu_1 +$$

$$\mu_2 + s)z + a\lambda p_2 \mu_1 = 0$$

$$a\lambda p_{n\text{-}1} \mu_n z \ldots \mu_n(\lambda(1\text{-}2p_n) + \mu_n, \mu_1, s)z + a\lambda p_1 \mu_n = 0$$

We obtain general expression as under

$$B(0,s) = \frac{\theta(\mu_i + p_i + \mu_{i+1}, s)}{1 - \theta(\mu_i, p_i, \mu_{i+1}, s)} \; \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1} + s)}$$

and

$$C(0,s) \frac{\mu_{i+1}}{\mu_i + \mu_{i+1}} - \sum_{h_1=0}^{\infty} \left(\frac{\mu_i + \mu_{i+1}}{\mu_1 + \mu_{i+1} + s} \right)^h C(h,s) \frac{s\mu_{i+1}}{(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)}$$

$$= \frac{\theta(\mu_i, p_i, \mu_{i+1}, s)}{1 - \theta(\mu_i, \mu_{i+1}, s)} \cdot \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\lambda p_{i+1} - \mu_{i+1}\ \theta(\mu_i, p_i, \mu_{i+1}, s)}$$

$$\mu_{i+1} \; \frac{\mu_{i+1} + a\lambda p_i}{\mu_i(\mu_i + \mu_{i+1} + s) - a\lambda(\mu_i + \mu_{i+1})} \qquad \ldots(9)$$

To obtain $D(h,s)$, $h \in N$ note that for the conditinoal *LSTs* envolved, because of (6), (2a) reduces to

$$f(m_n, h, m_n)(s)\,[\lambda p_i + I_{(h>0)} \mu_i + \mu_i + s]$$

$$= f(m_i, h+1, m_i)(s)[a\lambda p_i + I_{(h>0)} \mu_i f(m_n - 1, h-1, m_n\text{-}1]$$

$$+ \mu_n f(m_n - 1, h, m_n - 1)\,(s)\ s \geq 0,\ h \in N,\ m_n \geq 1. \qquad \ldots(10)$$

Multiplication of (6) by $\left(\frac{a\lambda p_i}{\mu_i}\right)^h \left(\frac{a\lambda p_{i+1}}{\mu_{i+1}}\right)^{m_n}$ and summation over

$m_n = 1, 2 \ldots$*yeilds*

$$D(h,s)[a\lambda\,(1\text{-}2\,p_{i+1}) + 1_{(h>0)} \mu_i + \mu_{i+1} + s]$$

$$= \left(\frac{a\lambda p_i}{\mu_i}\right)^h a\lambda p_{i+1} (1 + I_{(h_1>0)} \frac{\mu_i}{\mu_{i+1}})$$

$$+ D\,(h_{i+1}, s)\,\mu_i$$

$$+ D(h\text{-}1,s) \frac{a\lambda^n p_i p_{i+1} I_{(h_1>0)}}{\mu_i}, s \geq 0,\ h \in N.$$

For generating function, we therfore have

$$\sum_{h=0}^{\infty} Z^h D(h,s) \left[(a\lambda(1 - 2p_{i+1}) + \mu_i + \mu_{i+1} + s) - \frac{\mu_i}{2} - \frac{a\lambda^2 p_i p_{i+1} z}{\mu_{i+1}} \right.$$

$$= D(0,s)\mu_i \left(\frac{z-1}{2}\right) + \frac{a\lambda^2 p_{i+1} \mu_i}{\mu_{i+1}} \cdot \frac{\mu_{i+1} + a\lambda p_i z}{\mu_i - a\lambda p_i z}, \qquad \ldots(11)$$

$$|z| < 1,\ s \geq 0$$

With $\theta(\mu_{i+1}, p_i, a\lambda p_i, s)$ the smaller root of the equation

$$p_i p_{i+1} - z^{\mu} - \mu_{i+1}(a\lambda(1-2p_{i+1}) + \mu_i + \mu_{i+1} + s)z + \mu_{i+1}\mu_i = 0,$$

we obtain from (11)

$$F(0,s) = \frac{a\lambda p_{i+1}\ \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_{i+1}\ 1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{\mu_{i+1} + a\lambda p_i}{\mu_i\ -}\ \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_i, p_{i+1}, a\lambda p_i, s)}$$

and

$$\sum_{h=0}^{\infty}\left(\frac{\mu_i + \mu_{i+1}}{\mu_i + \mu_{i+1} + s}\right)^h D(h,s)\ \frac{\mu_i(\mu_i + \mu_{i+1} + s) + a\lambda p_i(\mu_i + \mu_{i+1})}{a\lambda p_i(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)}$$

$$\mu_i - D(0,s)\frac{\mu_i + \mu_{i+1}}{a\lambda p_i, (\mu_i + \mu_{i+1})}$$

$$= -\frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\ \frac{\mu_i + a\lambda p_i\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{p_{i+1}\mu_i}{p_i(\mu_i + \mu_{i+1})}\left[1 + \frac{s\mu_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}\right] + \frac{p_{i+1}\mu^2}{p_i}$$

$$\frac{\mu_i + (\mu_i + \mu_{i+1})(\mu_i + a\lambda p_i)}{(\mu_i s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_i)(\mu_i s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})} \qquad \text{...(12)}$$

Inserting (7) and (12) into (5) we finally obtain

$$F(0,s) = \frac{\mu_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_i)}$$

$$\left(1 + \frac{p_{i+1}}{\mu_i} \cdot \frac{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} + a\lambda p_i)}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} + a\lambda p_i)}\right)$$

$$\frac{-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\ \frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{p_{i+1}\mu_i}{p_i}\ \frac{\mu_{i+1} - a\lambda p_{i+1} + s}{\mu_n s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}$$

$$+ \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)}\ \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)}{a\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)} \qquad \text{...(13)}$$

$$\mu_{i+1}\ \frac{\mu_i - a\lambda p_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - a\lambda p_i)}$$

The transformation

$$\begin{pmatrix}\mu_i\\ \mu_{i+1}\\ p_i\end{pmatrix} \xrightarrow{T} \begin{pmatrix}\mu_i\\ \mu_{i+1}\\ p_i\end{pmatrix}$$

$$\frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\;\frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\xrightarrow{T} \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)}\;\frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_{i+1}, a\lambda p_i, s)}$$

$$\xrightarrow{T} \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\;\frac{\mu_i + \alpha\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - \alpha\lambda p_i - \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

Therefore, applying *T* to *F(0,s)* and inserting the results and *F(0,s)* into (1) yeilds

$$F(0,s) = \frac{(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)(\mu_i - a\lambda p_i)(\mu_{i+1} - a\lambda p_{i+1})}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - a\lambda p_i)(\mu_{i+1} s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}$$

$$-s\left\{\frac{\theta(\mu_i, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\;\frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\right\}$$

$$\frac{1}{\mu_i}\;\frac{p_{i+1} \qquad (\mu_i - a\lambda p_i) \qquad (\mu_{i+1} - a\lambda p_{i+1})}{\mu_{i+1} s \quad + (\mu_i + \mu_{i+1}) \qquad (\mu_i - a\lambda p_{i+1})}$$

$$+\frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)}\;\frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)}$$

$$\frac{1}{\mu_i}\;\frac{p_i \;(\mu_i - \alpha\lambda p_i)(\mu_{i+1} - \alpha\lambda p_{i+1})}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - \alpha\lambda p_i)}.$$

After sum direct input this yield the theroem.

3. Probability Distribution of Working and Non Working States of the System. Assumptions

We analyse the probability distirbution of the system when it is in working and non-working state as under:

Following assumptions are laid down before we compute the transition probabilities from working state of the system to non-working, vice versa and when system remains at itself.

(i) Servers are parallelly arranged.

(ii) Servers are simulteneously busy and working as one single system.

(iii) System may go in non-working state at any epoch of the time and assumed to come back in normal state i.e.; working after certain time period. Reasons of non-working may be such as idleness, failure or any type of distribution etc.

(iv) Let working state of the system be represented by 1 and non working state be represented by *0*.

3.1 Transitional Probability distribution. Let the lengths of working and non-working periods be independent random variables having negative exponential distribution with means $1/b$ and $1/c$ respectively $(b,c > 0)$.

Now

$q_{01}(\Delta t) = Pr\ \{change\ of\ state\ from\ 0\ to\ 1\ in\ time\ \Delta t\}$
$= b\ \Delta t + o\ (\Delta t)$

$q_{10}(\Delta t) = Pr\ \{change\ of\ state\ from\ 1\ to\ 0\ in\ time\ \Delta t\}$
$= c\ \Delta t + o\ (\Delta t).$

Transition probabilities are

$$b_{01} = b, \quad b_{00} = -b \text{ and } b_{10} = c, \quad b_{11} = -c$$

Now the transition probabilities martix may be written as

$$A = \begin{bmatrix} -b & b \\ c & -c \end{bmatrix}$$

The Kolmogrov forward equations for $i = 0, 1$ are

$$q'_{i_0}(t) = -b\, q_{i_0}(t) + c q_{i_0}(t)$$
$$q'_{i_1}(t) = b\, q_{i_0}(t) - c q_{i_1}(t)$$

We use the following equations while finding the transition probabilities $q_{i_1}(t)$:

$$q_{00}(t) - q_{01}(t) = 1$$
$$q_{10}(t) + q_{11}(t) = 1.$$

We have

$$q'_{00}(t) + (b-c)\, q_{00}(t) = c$$

and

$$q'_{11}(t) + (b+c)\, q_{11}(t) = b$$

The solution of the frist of these differential equations is

$$q_{00}(t) = \frac{c}{b+c} + Ce^{-(b+c)t}$$

with $q_{00}(0) = 1$, we find $C = \frac{c}{b+c}$ so that

$$q_{00}(t) = \frac{c}{b+c} + \frac{c}{b+c} e^{-(b+c)t}$$

Hence

$$q_{01}(t) = 1 - q_{00} \frac{b}{b+c} + \frac{b}{b+c}\ ^{-(b+c)t}$$

Proceeding exactly in the same way, the solution of the second differential equation with the initial condition $q_{11}(t) = 1$, yelids

$$q_{11}(t) = \frac{b}{b+c} + \frac{b}{b+c} e^{-(b+c)t}$$

and therefore

$$q_{11}(t) = 1 - q_{11}(t)$$
$$= \frac{c}{b+c} - \frac{c}{b+c}\, e^{-(b+c)t}$$

We can finally conclude that the present work not only envisages the multistate space with working and non-working states of the system but also it respeakes more general and elegant unification of the results including Daduna [3] and others. The results are obtainable in turn, after specializing various parameters such as $i = 1,2$; $m_i = m_1$, $m_2 = n$, $E(x) = 1$ and non-working state is assumed not to occur with the system.

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A RESULT ON SIMPLE ACCESSIBLE RINGS

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ABSTRACT

In this paper we show that a simple accessible ring is either a (–1,1) ring or a commutative ring.

1. Introduction. Accessible rings were introduced by Kleinfeld [1]. A non-associative ring R is called accessible in case the following two identities hold :

(1) $(x,y,z) + (z,x,y) - (x,z,y) = 0,$

(2) $((w,x), y,z) = 0$

for all w,x,y,z in R, where the associator (x,y,z) is defined by $(x,y,z) = (xy)\,z - x(yz)$ and the commutator (x,y) is defined by $(x,y) = xy - yx$, An accessible ring is defined to be simple if it has no proper two sided ideals.

A (–1,1) ring is a non-associative ring in which the following identities hold :

(3) $(x,y,z) + (x,z,y) = 0$,

(4) $(x,y,z) + (y,z,x) + (z,x,y) = 0.$

In [4] it is proved that a simple non-associative ring of char $\neq 2$ satisfying the identity $((x,y,z),w) = 0$ is either commutative or associative, Kleinfeld [2] proved that the same result still valid when this identity is replaced by more general conditions. The identity $((x,y,z)\,v,w) = 0$ holds in accessible rings under the assumption that the rings are without nilpotent elements in the center [1]. Without this assumption we show that $((x,y,y)\,v,w) = 0$ holds in accessible rings. Using this identity we show that a simple accessible ring is either (–1,1) or commutative.

2. Preliminaries. By substituting $z = x$ in (1) we obtain the flexible law $(x,y,x) = 0$. The following identities hold in accessible rings

(5) $(x,y,z) = -\,(z,y,x)$,

(6) $(x,y,z) = x\,(y,z) + (x,z)\,y,$

(7) $(x,y,z) + (y,z,x) + (z,x,y) = 0,$

(8) $((w,x,y),z) = 0$.

The nucleus N of R is defined as the set of all elements n in R with the property $(n,R,R) = 0$, If n is an element of the nucleus N of R, then because of the flexible law $(R,R,n) = 0$, Finally because of (1) it follows that also $(R,n,R) = 0$.

The following identity holds in an arbitrary ring :

$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$.

If $w = n$ in the above equation, then it becomes

(9) $(nx,y,z) = n(x,y,z)$, n in N.

From (9) and from the fact that every commutator is in the nucleus, we get $(v,x)(x,y,z) = ((v,x)x,y,z)$.

It follows from (6) that $(v,x)x = (v,x,x)$. Consequently

$((v,x)x,y,z) = ((vx,x),y,z) = 0$, Thus

(10) $(v,x)(x,y,z) = 0$.

3. Main Result

Lemma 1. In an accessible ring R, $((x,y,y)v,w) = 0$.

Proof. Linearization of (10) becomes

(11) $(v,w)(x,y,z) = -(v,x)(w,y,z)$

By using (5), (11), (7) and (10),

$$\begin{aligned} &= (v,y)(w,y,x) \\ &= (v,y)[-(y,x,w) - (x,w,y)] \\ &= -(v,y)(y,x,w) + (v,y)(y,w,x) \\ &= 0. \end{aligned}$$

That is,

(12) $(v,w)(x,y,y) = 0$.

Now from(6),(12)and(8), we get $((x,y,y)v,w) = (x,y,y)(v,w) + ((x,y,y),w)v = 0$

Lemma 2. Let R be an accessible ring, then $U = \{u \in R / (u,r) = 0 = (uR,R)\}$ is an ideal of R.

Proof. If we put $w = u$ in (8), then $((u,x,y),z) = 0$. From this it follows that $(ux.y,z) - (u.xy,z) = 0$. Then $(ux,y,z) = 0$ by the definition of U. Thus $ux \in U$. So U is a right ideal of R. Since $(u,R) = 0$, $(u,x) = 0$. That is $ux = xu \in U$. So U is a left ideal of R. Hence U is an Ideal of R.

Theorem. If R is a simple accessible ring, then R is either a $(-1,1)$ ring or a commutative ring.

Proof. From (8) and lemma 1 (x,y,y) is in U. Since U is an ideal of R and R is simple, we have either $U=0$ or $U = R$. If $U = 0$, then R is right alternative, that is $(x,y,y) = 0$.

Linearization of this yields $(x,y,z) + (x,z,y) = 0$. From this and (7) it follows that R is a $(-1,1)$ ring. If $U = R$ then R is a commutative ring.

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SOME OPERATION- TRANSFORM FORMULAE FOR S_μ- TRANSFORM

By

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1. Introduction. In an earlier Paper [1], S_μ- transform has been defined by

$$(1.1)\quad S_\mu\,[f(t)] = F(s) = \int_0^\infty f(t)\,\frac{e^{-\mu(s/t)}}{s+t}\,dt,\ (\mu \geq 0),$$

where $f(t)$ is a suitably restricted conventional function defined on the real line $0 < t < \infty$ and $0 < Re(S) < \infty$. It has been generalised in the case of generalised functions as

$$(1.2)\quad S_\mu\,[f(t)] = F(s) = \left< f(t), \frac{e^{-\mu(s/t)}}{s+t} \right>,\ (\mu \geq 0).$$

Its inversion formula has been also derived. Here it is proposed to discuss some operation-transform formulae of the transform given by (1.1).

2. Operation- Transform Formulae for S_μ- Tranform

Differentation : If $\phi \in B_\mu$, where B_μ is the space of all complex valued smooth functions $\phi(t)$ such that for each $\phi(t) \in B_\mu$, we have

$$\rho_n(\phi) = \sup_{0<t<\infty} |D^n \phi(t)| \qquad (n = 0,1,2,\ldots)$$

bounded.

We shall prove that

$$(2.1)\quad \rho_n[-D\phi] = \rho_{n+1}[\phi].$$

Since,

$$\begin{aligned}\rho_n[-D\phi] &= \sup_{0<t<\infty} |D^n(-D\phi)| \\ &= \sup_{0<t<\infty} |D^{n+1}\phi| \\ &= \rho_{n+1}|\phi|.\end{aligned}$$

Therefore, we get

$$\rho_n[-D\phi] = \rho_{n+1}[\phi]$$

From (2.1) it follows that $\phi \to -D\phi$ is a continuous and linear mapping of B_μ on to itself. Therefore, from Theorem 1. 10–1 due to Zemanien [2.

p. 29], the adjoint mapping $f \to Df$ is also a continuous and linear mapping of β'_μ on to itself where B'_μ is the dual of B_μ and therefore we get

(2.2) $< Df(t), \phi(t) > = < f(t), -D\phi(t) >$

Now, we prove that the following operation-transform formula

(2.3) $S_\mu [D^n f] \leq K.S_\mu [|f(t)|]$

Proof. Using, the generalised definition of S_μ-transform and the relation (2.2), we get

$$S_\mu [D^n f] = < D^n f(t), \frac{e^{-\mu(s/t)}}{s+t} >$$

$$= < f(t), (-D)^n \frac{e^{-\mu(s/t)}}{s+t} >$$

$$= < f(t), \sum_{\upsilon=0}^{n} {}^n c_\upsilon (-D)^{n-\upsilon} e^{-\mu\, s/t} (-D)^\upsilon \frac{1}{s+t} >$$

Therefore, we get

(2.4) $S_\mu [D^n f(t)] = < f(t), \frac{e^{-\mu(s/t)}}{s+t} \cdot \frac{P_n(t)}{Q_n(t)} >$

where $P_n(t)$ and $Q_n(t)$ are the polynomials in t such that order of $Q_n(t) \geq$ order of $P_n(t)$.

Let us suppose that f is a regular generalised function of B'_μ. Therefore, for, $\phi \in \beta_\mu$, we have

$$< f, \phi > = \int_0^\infty f(t)\, \phi(t)\, dt$$

and

$$|< f, \phi >| \leq \int_0^\infty |f(t)|\ |\phi(t)|\ dt$$

Consequently, we get

(2.5) $|< f, \phi >| \leq < |f|, |\phi| >$.

An appeal to (2.4) and (2.5) gives

$$|S_\mu [D^n f]| \leq < |f(t), |-\frac{e^{-\mu s/t}}{s+t}|.|\frac{P_n(t)}{Q_n(t)}|>$$

$$\leq < |f(t), |\frac{e^{-\mu s/t}}{s+t}|.\, K >,$$

where $|\frac{P_n(t)}{Q_n(t)}| \leq K$ (const); $0 < t < \infty$; $0 < \mu.s < \infty$ and $\mu \geq 0$.

Therefore, we get

$$|S_\mu [D^n f]| \leq K < |f(t), |\frac{e^{-\mu s/t}}{s+t}|>$$

$$\leq K.\, S_\mu [|f(t)|].$$

This completes the proof.

Multiplication by an Exponential Function. Let μ be a real number such that $\mu \geq 0$. Now we prove that $\phi(t) \to e^{-\mu t}\phi(t)$ is a continuous and linear mapping from B_μ on to itself.

Proof. Let $\phi \in B_\mu$, We have

$$D^n[e^{-\mu t}\phi(t)] = \sum_{\upsilon=0}^{n} {}^n c_\upsilon D^{n-\upsilon} e^{-\mu t} D^\upsilon \phi(t)$$

$$= \sum_{\upsilon=0}^{n} {}^n c_\upsilon (-\mu)^{n-\upsilon} e^{-\mu t} D^\upsilon \phi(t).$$

Therefore, we get

$$|D^n[e^{-\mu t}\phi(t)]| \leq \sum_{\upsilon=0}^{n} K\,|D^\upsilon \phi(t)|$$

where $|\,{}^n c_\upsilon (-\mu)^{n-\upsilon} e^{-\mu t}| \leq K$ for $\mu \geq 0$ and $0 < t < \infty$.

Thus we get

(2.6) $\rho_n[e^{-\mu t}\phi(t)]\,K \leq \sum_{\upsilon=0}^{n} |\rho_\upsilon[\phi(t)]|$ $(n = 0,1,2,\ldots;\ \upsilon = 0,1,2,\ldots)$.

From (2.6), it follow that $\phi(t) \to e^{-\mu t}\phi(t)$ is a continuous and lineir mapping of B_μ on to itself, Therefore, from Theorem 1.10-1 due to Zemanian [2,p.29] the adjoint mapping $f \to e^{-\mu t} f$ is also a continuous and linear mapping of B'_μ on to itself and we get

(2.7) $< e^{-\mu t} f(t), \phi(t) > = < f(t), e^{-\mu t}\phi(t) >$.

An appeal to (2.7) and the generalised definition of S_μ-transform, we get

$$S_\mu[e^{-\mu t} f(t)] = < e^{-\mu t} f(t), \frac{e^{-\mu s/t}}{s+t} >$$

$$= < f(t), e^{-\mu t}\frac{e^{-\mu s/t}}{s+t} >.$$

Therefore,

$$|S_\mu[e^{-\mu t} f(t)]| \leq < |f(t)|, |e^{-\mu t}|.|\frac{e^{-\mu s/t}}{s+t}| >$$

by (2.5) if f is a regular generalised funciton

$$\leq M < |f(t)|, \frac{e^{-\mu s/t}}{s+t} >$$

$$\leq M S_\mu[\,|f(t)|\,]$$

where $|e^{-\mu t}| \leq M$

Thus we get an operation-transform formula

(2.8) $|S_\mu[e^{-\mu t} f(t)]| \leq M S_\mu[\,|f(t)|\,]$.

Multiplication by $(s+t)^{-\lambda}$ where $\lambda > 0$; $0 < t < \infty$ and $0 < s < \infty$.

We prove that $\phi(t) \to (s+t)^{-\lambda}\phi(t)$ is a continuous and linear mapping of B_μ on to itself, where $\lambda > 0;\ 0 < t < \infty$ and $0 < s < \infty$.

Proof. Let $\phi \in B_\mu$, we have

$$D_n[(s+t)^{-\lambda}\,\phi(t)] = \sum_{\upsilon=0}^{n} {}^nc_\upsilon\, D^{n-\upsilon}\,(s+t)^{-\lambda}\, D^\upsilon\,\phi(t)$$

$$= \sum_{\upsilon=0}^{n} {}^nc_\upsilon\, (-\lambda)(-\lambda-1)\ldots(-\lambda-(n-\upsilon-1)(s+t)^{\lambda-\upsilon+\upsilon}\, D^\upsilon\,\phi(t).$$

Therefore, we get

$$|D_n[(s+t)^{-\lambda}\,\phi(t)]| \le M \sum_{\upsilon=0}^{n} |D^\upsilon\,\phi(t)|$$

where $|{}^nc_\upsilon\,(-\lambda)(-\lambda-1)\ldots(-\lambda-(n-\upsilon-1)(s+t)^{\lambda-\upsilon+\upsilon}| \le M.$

$(n = 0,1,2,\ \ldots\ ;\ \upsilon = 0,1,2,\ldots)$

Thus, we get

(2.9) $\rho_n\,[(s+t)^{\lambda}\,\phi(t)] \le M \sum_{\upsilon=0}^{n} \rho_\upsilon\,[\phi(t)].$

From (2.9) it follows that $\phi\,(t) \to (s+t)^{-\lambda}\,\phi(t)$ is a continuous and linear mapping of B_μ on to itself. Therefore, from the theorem 1.10–1 due to Zemanian [2.p 29]. the adjoint mapping $f \to (s+t)^{-\lambda} f$ of $\phi \to (s+t)^{-\lambda}\,\phi$ is also a continuous and linear mapping of B'_μ on itself and we get

(2.10) $< (s+t)^{-\lambda}\, f(t).\ \phi(t),\quad f(t).(s+t)^{-\lambda}\,\phi(t)>.$

An appeal to (2.10) and the generalised definition of S_μ-tranform gives

$$S_\mu[(s+t)^{-\lambda}\, f(t)| = < (s+t)^{-\lambda}\, f(t), e^{-\mu s/t} >$$

$$= < f(t), (s+t)^{-\lambda}\, \frac{e^{-\mu s/t}}{s+t} > .$$

If be a regular generalised function then by using (2.5), *we get*

$$|S_\mu[(s+t)^{-\lambda}\, f(t)]| \le < |f(t)|, |(s+t)^{-\lambda}|\, |\frac{e^{-\mu s/t}}{s+t}| >$$

$$\le N < |f(t)|, \frac{e^{-\mu s/t}}{s+t} > \ \le N\, S_\mu\,[\,|f(t)|\,]$$

where $\quad 0 \le |(s+t)^{-\lambda}| \le N.$

Thus we get an operation-transform formula

(2.11) $|S_\mu[(s+t)^{-\lambda}\, f(t)]| \le N\, S_\mu\,[\,|f(t)|\,].$

Shifting. Let T be a fixed real number such that $0 < t + T < \infty$ and $0 < t < \infty$. Let $\phi\,(t) \in B_\mu$ Now we shall prove that $(t+T)$ is a continuous and linear mapping of B_μ on to itself.

Proof. Let us consider

$$D^n\,[\,\phi(t+T)] = (d/dt)^n\ |\phi\,(t+T)|$$

$$= [\frac{d}{d(1+T)}\ \frac{d(t+T)}{dt}]^n\ |\phi\,(t+T)|$$

$$= \left(\frac{d}{d(t+T)}\right)^n [\phi\ (t+T)]$$

$$= D^n_{t+T}\,\phi\ (t+T)$$

$$= D^n_{t_1}\,[\phi\ t_1],\ [t_1 = t+T]$$

where $0 < t + T < \infty$ and $0 < t < \infty$.

Therefore, we get

(2.12) $D^n_t\,[\,\phi(t+T)] = D^n_t\,[\,\phi(t)],\ [t_1 = t]$

i.e. $\rho_n\,[\,\phi(t+T)] = \rho_n\,[\,\phi(t)]$.

Thus from (2.12), it follows that $\phi\ (t) \to \phi(t+T)$ is a continuous and linear mapping of B_μ on to itself. Its inners sapping $\phi\ (t) \to \phi(t+T)$ is also a continuous and linear mapping of B_μ to on itself. Therefore $\phi\ (t) \to (t+T)$ is an isomorphiam of B_μ onto itself, the adjoint mapping of $\phi\ (t) \to \phi(t+T)$ is $f(t) \to f\ (t+T)$ which is also a continuous and linear mapping of B'_μ onto itself due to Theorem 1.10-1 of Zemanian [2,p. 29] and we get

(2.13) $< f\ (t+T),\ \phi\ (t) > = < f(t),\ \phi\ (t+T) >$

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FRACTIONAL INTEGRAL FORMULAE INVOLVING THE PRODUCT OF A GENERAL CLASS OF POLYNOMIALS AND *I*-FUNCTION OF TWO VARIABLES-II

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ABSTRACT

In the present paper we establish some fractional integral formulae involving the product of a general class of polynomials and the *I*-function of two variables. Our results are quite general in character and a number of known and new formulae can be obtained as their particular cases. Several such intersting special and confluent cases of our main results are mentioned breifly.

1. Introduction. The object of the present paper is to obtain a few fractional integral formulae involving the product of a general class of polynomials and the *I*-function of two variables.

The fractional integral operator investigated by Erdélyi [1] and Kober [4] is defined as

$$I_x^{\eta,\upsilon}\{f(x)\}=\frac{x^{-\eta-\upsilon+1}}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1}t^{\eta-1}f(t)dt,\ Re\,(\upsilon)>0,\ \ \eta>0\,. \qquad ...(1.1)$$

Since in the *I*-function of two variables we use the notation *I* therefore in the operator $I_x^{\eta,\upsilon}$, instead of *I* we shall use *U* and in place of η we shall use ς and therefore the operator $I_x^{(\eta,\upsilon)}$ will be represented by $U_x^{\varsigma,\upsilon}$.

The *I*-function of two variables defined by Goyal and Agrawal [2] in the following manner:

$$I^{m_1,n_1:\,m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)},p_i^{(2)},\,q_i^{(2)}:r}\left[\begin{matrix}Z_1\\Z_2\end{matrix}\left|\begin{matrix}[(e_p:E_p,E'_p)]:(a_j,\alpha_j)_{1,n_2}],(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}}];[(c_j,\gamma_j)_{1,n_3}],\\ [(f_q:F_q,F'_q)]:(b_j,\beta_j)_{1,m_2}],(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}];(d_j,\delta_j)_{i,m_3}],\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}]\\(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]\end{matrix}\right]\right.=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\,\Psi\,(\xi,\eta)Z_1^{\xi}\,Z_2^{\eta}\,d\xi d\eta \qquad ...(1.2)$$

where

$$\Phi_1(\xi)=\frac{\prod\limits_{j=1}^{m_2}\Gamma(b_j-\beta_j\xi)\prod\limits_{j=1}^{n_2}\Gamma(1-a_j+\alpha_j\xi)}{\sum\limits_{j=1}^{r}\left[\prod\limits_{j=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod\limits_{j=n_2+1}^{p_i^{(1)}}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}. \qquad ...\ (1.2.1)$$

$$\Phi_2(\eta) = \frac{\prod\limits_{j=1}^{m_3} \Gamma(d_j - \delta_j \eta) \prod\limits_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j \eta)}{\sum\limits_{i=1}^{r} \left[\prod\limits_{j=m_3+1}^{q_i^{(2)}} \Gamma(1 - d_{ji} + \delta_{ji} \eta) \prod\limits_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji} - \gamma_{ji} \eta) \right]}. \quad \ldots (1.2.2)$$

$$\Psi(\xi, \eta) = \frac{\prod\limits_{j=1}^{m_1} \Gamma(f_j - F_j \xi - F'_j \eta) \prod\limits_{j=1}^{n_1} \Gamma(1 - e_j + E_j \xi + E'_j \eta)}{\prod\limits_{j=m_1+1}^{q} \Gamma(1 - f_j + F_j \xi + F'_j \eta) \prod\limits_{j=n_1+1}^{p} \Gamma(e_j - E_j \xi - E'_j \eta)}. \quad \ldots (1.2.3)$$

Z_1, Z_2 are non zero complex variables; L_1, L_2 are two Mellin-Barne type contour integrals. Convergence conditions are

$$|arg\, z_1| < \frac{A_i \pi}{2}, \quad |arg\, z_2| < \frac{B_i \pi}{2}$$

$$A_i = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_2} \beta_j - \sum_{j=m_2+1}^{q_i^{(1)}} \beta_{ji} + \sum_{j=1}^{n_2} \alpha_j - \sum_{j=n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0$$

and

$$B_i = \sum_{j=1}^{n_1} E'_j - \sum_{j=n_1+1}^{p} E'_j + \sum_{j=1}^{m_1} F'_j - \sum_{j=m_1+1}^{q} F'_j + \sum_{j=1}^{m_3} \delta_j - \sum_{j=m_3+1}^{q_i^{(2)}} \delta_{ji} + \sum_{j=1}^{n_3} \gamma_j - \sum_{j=n_3+1}^{p_i^{(2)}} \gamma_{ji} > 0,$$

for $i = 1, \ldots, r$.

A general class of polynomials $S_n^m[x]$ defined by Srivastava [8] as

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k \qquad n = 0, 1, 2, \ldots \quad \ldots (1.3)$$

where m is an arbitrary positive integer and $A_{n,k}$ $(n, k \geq 0)$ are arbitrary constants, real or complex.

On specializing the coefficients of $A_{n,k}$; $S_n^m[x]$ yields a number of known polynomials as special cases.

Throughout the paper we shall use the following notations:

$P = m_2, n_2 ; m_3, n_3 \quad , \qquad Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r$

$V = (a_j, \alpha_j)_{1,n_2}], (a_{ji}, \alpha_{ji})_{n_2+1, p_i^{(1)}}]; (c_j, \gamma_j)_{i,n_3}], (c_{ji}, \gamma_{ji})_{n_3+1, p_i^{(2)}}]$

$V' = (b_j, \beta_j)_{1,m_2}], (b_{ji}, \beta_{ji})_{m_2+1, q_i^{(1)}}]; (d_j, \delta_j)_{1,m_3}], (d_{ji}, \delta_{ji})_{m_3+1, q_i^{(2)}}]$

and the result for binomial expansion

$$(x + a)^k = a^k \sum_{i=0}^{\infty} \binom{k}{i} (x/a)i \ , \quad |x/a| < 1 \quad \ldots (1.4)$$

The formula given by Ross [5] as:

$$U_x^{\zeta,\upsilon}\{x^{\lambda}\}=\frac{\Gamma(\lambda+\zeta)}{\Gamma(\lambda+\zeta+\upsilon)}\,x^{-1},\quad Re\,(\lambda)>-\zeta\,. \qquad \dots(1.5)$$

2. Main Results. In this section the following two fractional integral formulae for the *I*-function of two variables are established as :

(I)

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\,S_n^m[x^a\,(x+\alpha)^b(x+\beta)^c]\times I\left[\begin{matrix}x^{h_1}\,(x+\alpha)^{k_1}(x+\beta)^{I_1}\ z_1\\ x^{h_2}\,(x+\alpha)^{k_2}(x+\beta)^{I_2}\ z_2\end{matrix}\right]\right]$$

$$=\alpha^{\rho}\beta^{\mu}\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,\frac{\alpha^{bk-t_1}\,\beta^{ck-t_2}\,x^{\sigma+ak+t_1+t_2+ak}}{t_1!\,t_2!}\times$$

$$I_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{matrix}x^{h_1}\,\alpha^{k_1}\,\beta^{I_1}\ z_1\\ x^{h_2}\,\alpha^{k_2}\,\beta^{I_2}\ z_2\end{matrix}\,\middle|\,\begin{matrix}X:V\\ X':V'\end{matrix}\right] \qquad \dots(2.1)$$

where

$X=(1-\sigma-ak-t_1-t_2-\zeta:h_1,h_2),(-\rho-bk:k_1,k_2),(-\mu-ck:I_1,I_2),(e_p:E_p,E'_p)$

and

$X'=(f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon-t_1-t_2:h_1,h_2),(-\rho-bk+t_1:k_1,k_2),(-\mu-ck+t_2:I_1,I_2)$

Provided that

(i) $Re\,(\upsilon)>0$,

(ii) $Max\ \{\,|arg\,(x/\alpha)|;\ |arg\,(x/\beta)|\}<\pi,$

(iii) $Re\left[\sigma+ak+t_1+t_2+h_1\underset{1\le j\le m_2}{Min}\left(\frac{b_j}{\beta_j}\right)+h_2\underset{1\le j\le m_3}{Min}\left(\frac{d_j}{\delta_j}\right)+\zeta>0.\right]$

(II)

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\,S_n^m[x^a\,(x+\alpha)^b(x+\beta)^c]\times I\left[\begin{matrix}x^{-h_1}\,(x+\alpha)^{-k_1}(x+\beta)^{-I_1}\ z_1\\ x^{-h_2}\,(x+\alpha)^{-k_2}(x+\beta)^{-I_2}\ z_2\end{matrix}\right]\right]$$

$$=\alpha^{\rho}\beta^{\mu}\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,\frac{\alpha^{bk-t_1}\,\beta^{ck-t_2}\,x^{\sigma+ak+t_1+t_2}}{t_1!\,t_2!}\times$$

$$I_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{matrix}x^{-h_1}\,\alpha^{-k_1}\,\beta^{-I_1}\ z_1\\ x^{-h_2}\,\alpha^{-k_2}\,\beta^{-I_2}\ z_2\end{matrix}\,\middle|\,\begin{matrix}X:V\\ X':V'\end{matrix}\right] \qquad \dots(2.2)$$

where

$X=(e_p:E_p,E'_p),\ (\sigma+ak+t_1+t_2+\zeta+\upsilon:h_1,h_2),(1+\rho+bk-t_1:k_1,k_2),(1+\mu+ck-t_2:I_1,I_2),$

and

$X'=(\sigma+ak+t_1+t_2+\zeta:h_1,h_2),(1+\rho+bk_1:k_1,k_2),(1+\mu+ck:I_1,I_2),(f_q:F_q,F'_q),$

provided that

(i) $Re\,(\upsilon)>0,$

(ii) $Max\ \{\,|arg\,(x/\alpha)|,\ |arg\,(x/\beta)|\}<\pi,$

(iii) $$Re\left[\sigma+ak+t_1+t_2-h_1 \underset{1\le j\le m_2}{Min}\left(\frac{b_j}{\beta_j}\right)-h_2 \underset{1\le j\le m_3}{Min}\left(\frac{d_j}{\delta_j}\right)\right]+\zeta>0.$$

Proof. To establish the fractional integral formula (2.1), first we express the general class of polynomials in the series form given by (1.3) and *I*-function of two variables in terms of Mellin-Barne's type contour integrals on the left hand side of (2.1).

Let

$$A=U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,x^{ak}(x+\alpha)^{bk}(x+\beta)^{ck}\times\frac{1}{(2\pi\omega)^2}\right.$$
$$\left.\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\,\Psi(\xi,\eta)Z_1^{\xi}\,Z_2^{\eta}\,x^{h_1\xi+h_2\eta}(x+\alpha)^{k_1\xi+k_2\eta}(x+\beta)^{I_1\xi+I_2\eta}\,d\xi d\eta\right] \quad ..(2.3)$$

Interchanging the order of Barne's type contour integrals and the fractional integral involved in the expression (2.3). Collecting the powers of *x*, $(x+\alpha)$, $(x+\beta)$ in the expression thus obtained and expanding $(x+\alpha)^{\rho}$ and $(x+\beta)^{\mu}$ using the binomial expansion (1.4), changing the order of Barne's type contour integrals and series, which is permissible under conditions stated, we get

$$A=\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}\,\alpha^{\rho+bk+k_1\xi+k_2\eta}$$

$$\beta^{\mu+ck+I_1\xi+I_2\eta}\frac{\Gamma(\rho+bk+k_1\xi+k_2\eta+1)\,\Gamma(\mu+ck+I_1\xi+I_2\eta+1)}{t_1!\,t_2!\,(\rho+bk+k_1\xi+k_2\eta-t_1+1)\,\Gamma(\mu+ck+I_1\xi+I_2\eta-t_2+1)\,\alpha^{t_1}\alpha^{t_2}}\times$$

$$U_x^{\zeta,\upsilon}\,\{x^{\sigma+ck+t_1+t_2+h_1\xi+h_2\eta}\}\,d\xi d\eta \quad ...(2.4)$$

Now using the result (1.5), we arrive at the desired result (2.1). In the similar fashion we can easily establish the result (2.2).

3. Particular Cases

I. If we take $\alpha=\beta=1$ the result (2.1), we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+1)^{\rho+\mu}\,S_n^m[x^{a}(x+1)^{b+c}]\times I\left[\begin{matrix}x^{h_1}(x+1)^{k_1+I_1}\,z_1\\ x^{h_2}(x+1)^{k_2+I_2}\,z_2\end{matrix}\right]\right]$$

$$=\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{x^{\sigma+ak+t_1}}{t_1!}I_{p+2,q+2:Q}^{m_1,n_1+2:P}\left[\begin{matrix}(x)^{h_1}z_1\\(x)^{h_2}z_2\end{matrix}\,\middle|\,\begin{matrix}Y:V\\Y':V'\end{matrix}\right], \quad ...(3.1)$$

where

$Y=(1-\sigma-ak-\zeta:h_1,h_2),(-\rho-\mu-bk-ck:k_1+I_1,k_2+I_2),(e_p:E_p,E'_p),$

$Y'=(f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon-:h_1,h_2),(\rho-\mu-bk-ck+t_1:k_1+I_1,k_2+I_2),$

provided that

(i) $Re(\upsilon)>0$

(ii) $Max\ \{\ |arg\,(x/\alpha)|,\ |arg\,(x/\beta)|\} < \pi$

(iii) $Re\left[\sigma+ak+t_1+t_2+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right) + \zeta > 0\right]$

(II) In (2.1) if we put $\mu = c = I_1 = 1_2 = 0$, we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}\, S_n^m[x^a\,(x+\alpha)^b]\times I\begin{bmatrix} x^{h_1}(x+\alpha)^{k_1}\, z_1 \\ x^{h_2}(x+\alpha)^{k_2}\, z_2 \end{bmatrix}\right]$$

$$= \sum_{t_1=0}^{\infty}\sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, \frac{\alpha^{\rho+bk-t_1}\, x^{\sigma+ak+t_1}}{t_1!}\, I^{m_1,n_1+2:P}_{p+2,q+2:Q}\left[\begin{matrix} x^{h_1}\alpha^{k_1} z_1 \\ x^{h_2}\alpha^{k_2} z_2 \end{matrix}\middle|\begin{matrix} Y:V \\ Y':V' \end{matrix}\right] \qquad \text{... (3.2)}$$

where

$Y = (1-\sigma-ak-\zeta-t_1: h_1,h_2),(-\rho-bk : k_1,k_2),(e_p:E_p, E'_p)$,

$Y' = (f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon-t_1 : h_1,h_2),(-\rho-bk+t_1:k_1,k_2)$,

provided that

(i) $Re\,(\upsilon) > 0$,

(ii) $Re\left[\sigma+ak+t_1+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)\right] + \zeta > 0$

(iii) If we put $\rho = b = k_1 = k_2 = 0$ in (3.2), we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}\, S_n^m[x^a]\; I\begin{bmatrix} x^{h_1} & z_1 \\ x^{h_2} & z_2 \end{bmatrix}\right] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^{\sigma+ak}\, I^{m_1,n_1+1:P}_{p+1,q+1:Q}\left[\begin{matrix} x^{h_1} z_1 \\ x^{h_2} z_2 \end{matrix}\middle|\begin{matrix} Y:V \\ Y':V' \end{matrix}\right] \qquad \text{... (3.3)}$$

where

$Y = (1-\sigma-ak-\zeta : h_1,h_2)(e_p:E_p, E'_p)$

$Y' = (f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon : h_1,h_2)$

Provided that

(i) $Re\,(\upsilon) > 0$,

(ii) $\left[Re\ \ \sigma+ak+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right) + \zeta > 0\right]$

(iv) In (3.3), if we put $m_1 = n_1 = p = q = n_3 = p_i^{(2)} = 0$, $m_3 = q_i^{(2)} = 1$ and $z_2 \to 0$ the result converts in the form of Saxena's [6] *I*-function of one variable

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}\, S_n^m[x^a]\; I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}[x^{h_1},z_1]\right] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^{\sigma+ak}$$

$$I^{m_2,n_2+1}_{p_i^{(1)}+1,q_i^{(1)}+1:r}\left[x^{h_1} z_1\,\middle|\,\begin{matrix} (1-\sigma-ak-\zeta : h_1),V \\ V',\,(1-\sigma-ak-\zeta-\upsilon : h_1) \end{matrix}\right] \qquad \text{...(3.4)}$$

Provided that

(i) $Re\,(\upsilon) > 0,$

(ii) $Re\left[\sigma+ak+h_1 \underset{1\leq j\leq m_2}{Min}\left(\frac{b_j}{\beta_j}\right)\right]+\zeta > 0.$

(V) In (3.4) if we take $m=1$, $A_{n,k} = \binom{n+\gamma}{n} \frac{(\gamma+\delta+n+1)_k}{(\gamma+1)_k}$ then the general class of polynomials $S_n^m[x^a]$ reduces to Jacobi polynomials,i.e., $S_n^1[x^a] \rightarrow P_n^{(\gamma,\delta)}[1-2.x^a]$

$$U_x^{\zeta,\upsilon}\left[x^{\sigma} P_x^{(\gamma,\delta)}[1-2x^a]\, I_{p_i^{(1)},q_i^{(1)}:r}^{m_2,n_2}\,[x^{h_1},z_1]\right] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!}\binom{n+\gamma}{n}$$

$$\frac{(\gamma+\delta+n+1)_k}{(\gamma+1)_k} x^{\sigma+ak}\; I_{p_i^{(1)}+1,q_i^{(1)}+1:r}^{m_2,n_2+1}\left[x^{h_1} z_1 \,\middle|\, \begin{matrix}(1-\sigma-ak-\zeta:h_1),V\\ V',(1-\sigma-ak-\zeta-\upsilon:h_1)\end{matrix}\right] \tag{3.5}$$

Convergence conditions are same as in result (3.4).

In the similar manner we can establish the particular cases for the result (2.2).

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ON RELATIVELY SINGULAR MAPS

By

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ABSTRACT

In this paper we introduce the concept of relatively singular maps between locally compact spaces and obtain various results concerned to singular maps in this new setting.

1. Introduction. Throughout the paper a space will mean a locally compact Hausdorff space and a map will mean a continuous map.

The notion of the singular set of a mapping was defined and investigated by G.T. Whyburn [14] and G.L. Cain [2,3]. Later the idea became quite useful to obtain a compactification of a locally compact space X with a given compact space K as a remainder. The compactification obtained by this method is called a *singular compactification.* This laid a new step in the theory of compactifications, one of the most active areas of researches in general topology. A few celebrated names in this area are Banaschewski, Cain, Chandler, Conway, Faulkner, Magill Jr., Walker and Whyburn [2,3,4,5,6,7,8,12, 13,14].

A point y in Y is called a *singular point* of $f : X \to Y$ if the closure $cl\, f^{-1}(V)$ of $f^{-1}(V)$ is not compact for any neighbourhood V of y. The set of singular points of f is denoted by $S(f)$. Clearly it is a closed set of Y. If for a map $f:X\to Y$, $S(f) = Y$, f is called a *singular map.*

Let $f : X\to Y$ and $\alpha : Y\to Y$ be a self map. A point $y \varepsilon Y$ is called a *relatively singular point* of the map $f(w.r.t.\alpha)$ if the closure $cl\, (\alpha\, o\, f)^{-1}(U)$ of $(\alpha o f)^{-1}(U)$ is not compact for any neighbourhood U of y. If every point of Y is relatively singular, we call f to be *relatively singular map* ($w.r.t.\alpha$ $:Y\to Y$).

The singular map and relatively singular map may differ but when α is equal to identity map on Y, both concepts coincide. A singular map is always relatively singular but the converse is not true.

1.1 Example. The indentity map $I:R^2 \to R^2$ is not singular but I is relatively singular with respect to the first projection $\pi_1 : R^2 \to R$.

In section *2* we show that the product of a relatively singular map with any map is relatively singular. Also the induced map of a relatively singular map $f:X \to Y$ on respective cones remains relatively singular. Moreover an analogue of Pasting lemma is obtained for relatively singular maps.

A G-space is a Hausdorff space X on which a topological group G acts continuously. The orbit space of a G-space is denoted by X/G and the map which assigns to x in X its orbit is the orbit map on X. An equivariant map f from a G-space X to a G-space Y induces a continuous map f_G on orbit spaces which sends the orbit of x in X to the orbit of $f(x)$. Let H be a compact subgroup of G and let X be an H-space. Then for $h \varepsilon H$ and $(g,x) \varepsilon G \times X$, $h(g,x) = (gh^{-1}, h.x)$ defines an action of H on $G \times X$; the orbit space $G \times_H X$ of the H-space $G \times X$ is called the *twisted product* of G and X. For H-spaces X and Y, an equivariant map $f: X \to Y$ determines a continuous map $f_H: G \times_H X \to G \times_H Y$, which sends $[g,x]$ in $G \times_H X$ to $[g,f(x)]$ in $G \times_H X$ [1].

A map $f: X \to Y$ will be called *compact* if the inverse image of a compact set of Y is a compact set of X.

We consider G-spaces and equivariant maps in section 3 and obtain that an equivariant relatively singular map from a G-space X to a G-space Y induces a relatively singular map on orbit spaces.

2. Relatively Singular maps and Various Structures

In this section we obtain some results for relatively singular maps which are already established for singular maps [10].

2.1 Proposition. Let $f: X \to Y$, $h: X' \to Y'$ and $\alpha: Y \to Y$ be maps, If f is relatively singular (*w.r.t.* α), then $f \times h: X \times X' \to Y \times Y'$ defined by

$$(f \times h)(x, x') = \{f(x), h(x')\}$$

is relatively singular (*w.r.t.*$\alpha \times I$) where $I: Y' \to Y'$ is the idnetity map.

Proof. Suppose that $f \times h: X \times X' \to Y \times Y'$ is not relatively singular (*w.r.t.*$\alpha x I$). Then there exists a basic open set $U \times U'$ of $Y \times Y'$ such that $cl[(\alpha \times I) \circ (f \times h)]^{-1}(U \times U')$ is compact.

Since

$$cl[(\alpha \times I) \circ (f \times h)]^{-1}(U \times U')$$

$$= cl\,(\alpha \circ f)^{-1}(U) \times cl\,(I \times h)^{-1}(U')$$

$$\Rightarrow cl\,(\alpha \circ f)^{-1}(U) \text{ is compact.}$$

A contradiction to the hypothesis.

The following proposition gives an analogue of Pasting Lemma for relatively singular maps.

2.2 Proposition. Let X, Y be locally compact spaces and A,B be closed set of X such that $A \cup B = X$. Let $f{:}A \to Y$ and $g{:}B \to Y$ be two relatively singular maps ($w.r.t. \alpha{:}Y \to Y$) such that $f(x) = g(x)$ for $x \,\varepsilon\, A \cap B$. Then the map $h{:}X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \,\varepsilon\, A \\ g(x) & \text{if } x \,\varepsilon\, B \end{cases}$$

is relatively singular ($w.r.t. \alpha{:}Y \to Y$).

Proof. Let U be an open set of Y. Then $(\alpha o h)^{-1}(U) = (\alpha \circ f)^{-1}(U) \cup (\alpha o g)^{-1}(U) \Rightarrow cl\,(\alpha o h)^{-1}(U) = cl(\alpha \circ f)^{-1}(U) \cup cl\,(\alpha o g)^{-1}(U)$.

Since f and g are relatively singular maps, $cl\,(\alpha \circ f)^{-1}(U)$ and $cl\,(\alpha o g)^{-1}(U)$ both are noncompact.

$\Rightarrow (\alpha \circ h)^{-1}(U)$ is non-compact.

2.3 Definition. Let X be a topological space, the *Cone* $T_\times$ over X is the quotient space $(X \times I)/A$, where A is the subset of $X \times I$ given by $X \times \{1\}$.

2.4 Proposition. Let $f : X \to Y$ be a reltively singular map ($w.r.t. \alpha{:}Y \to Y$). If the quotient maps p, q are compact maps, then the induced map $Tf{:}\, Tx \to Ty$ is relatively singular ($w.r.t. T\alpha{:}Ty \to Ty$).

Proof Consider the following commutative diagram.

$$\begin{tikzcd}
X \times I \arrow[r, "(\alpha \times I)\, o\, (f \times I)"] \arrow[d, "p"'] & Y \times I \arrow[d, "q"] \\
Tx \arrow[r, "T\alpha\ o\ Tf"'] & Ty
\end{tikzcd}$$

If $f : X \to Y$ be relatively singular $w.r.t. \alpha{:}Y \to Y$, then $f \times I : X \times I\ Y \times I$ be relatively singular $w.r.t. \alpha \times I{:}\, Y \times I \to Y \times I$.

Suppose $Tf{:}\, Tx \to Ty$ is not relatively singular, then there is an open set U of Ty satisfying that $cl\,(T\alpha o Tf)^{-1}(U)$ is compact.

$\Rightarrow p^{-1}(cl(T\alpha o\ Tf)^{-1}(U))$ is compact. From continuity of q and commutativity of the above diagram, it follows that

$$\begin{aligned}
p^{-1}(cl(T\alpha o\ Tf)^{-1}(U)) &\supseteq cl\,(p^{-1}(T\alpha o\ Tf)^{-1}(U)) \\
&= cl\,(p^{-1}(Tf^{-1}(T\alpha^{-1}(U)))) \\
&= cl\,[(T\alpha \circ Tf) \circ (p)]^{-1}(U) \\
&= cl\,[q o \{(\alpha \times I) \circ (f \times I)\}]^{-1}(U) \\
&= cl\,[(\alpha \times I) \circ (f \times I)]^{-1}\, q^{-1}(U)
\end{aligned}$$

Thus $cl\,[(\alpha \times I) \circ (f \times I)]^{-1}\, q^{-1}(U)$ is compact, a contradiction to the hypothesis that $f \times I$ is relatively singular.

3. Relatively Singular Maps and G-spaces. In this section X and Y will denote G-spaces, where G is a compact topological group and the spaces X and Y are respectively locally compact and compact.

3.1 Proposition. Let $f{:}X\rightarrow Y$ be an equivariant and relatively singular map ($w.r.t.\alpha{:}Y\rightarrow Y$). Then the induced map $f_G{:}X/G\rightarrow Y/G$ is relatively singular ($w.r.t.\alpha_G{:}Y/G\rightarrow Y/G$).

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\alpha o f"] \arrow[d, "q_x"'] & Y \arrow[d, "q_y"] \\
X/G \arrow[r, "\alpha_G o f_G"'] & Y/G
\end{tikzcd}$$

where q_x and q_y are orbit maps.

Suppose that f_G is not relatively singular. Then there is a point $G(y)$ in Y and an open set U of Y/G containing $G(y)$ such that $cl(\alpha_G o f_G)^{-1}(U)$ is compact. Since q_x is a compact map, $q_x^{-1}(cl(\alpha_G o f_G)^{-1}(U))$ is compact. From continuity of q_x and commutativity of the above diagram, it follows that

$$\begin{aligned}
&q_x^{-1}(cl(\alpha_G o f_G)^{-1}(U)) \supseteq cl\,(q_x^{-1}\,(\alpha_G o f_G)^{-1}(U))\\
&= cl\,(q_x^{-1}(f_G^{-1}(\alpha_G^{-1}(U))))\\
&= cl\,((\alpha_G o f_G)\, o q_x)^{-1}(U)\\
&= cl\,(q_y\, o\, (\alpha o f))^{-1}(U)\\
&= cl\,(f^{-1}\,(\alpha^{-1}(q_y^{-1}\,(U)))).
\end{aligned}$$

Thus $cl(f^{-1}(\alpha^{-1}(q_y^{-1}(U))))$ is compact, a contradication to the hypothesis that f is relatively singular.

3.2. Proposition. Let $f{:}X\rightarrow Y$ be an equivariant and relatively singular map ($w.r.t.\alpha{:}Y\rightarrow Y$). Then the induced map $f_H{:}G\times_H X\rightarrow G\times_H Y$ is relatively singular $w.r.t.\alpha_H{:}G\times_H Y\rightarrow G\times_H Y$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
G\times X \arrow[r, "(I_G\times\alpha)\, o\, (I_G\times f)"] \arrow[d, "q_x"'] & G\times Y \arrow[d, "q_y"] \\
G\times_H X \arrow[r, "\alpha_H o f_H"'] & G\times_H Y
\end{tikzcd}$$

Suppose that f_H is not relatively singular. Then there is an open set U of $G\times_H Y$ such that $cl\ (\alpha_H o f_H)^{-1}(U))$ is compact, then $q_x^{-1}\ (cl\ (\alpha_H\ o\ f_H)^{-1}(U))$ is also compact. From continuity of q_x and commutativity of the above diagram it follows that

$$q_x^{-1}(cl(\alpha_H o f_H)^{-1}(U)) \supseteq cl\,(q_x^{-1}\,(\alpha_H o f_H)^{-1}(U))$$

$$= cl\,(q_x^{-1}(f_H{}^{-1}(\alpha_H{}^{-1}(U))))$$
$$= cl\,((\alpha_H o f_H)\, o q_x)^{-1}(U)$$
$$= cl\,(q_y\, o\, (((I_G \times \alpha)\, o\, (I_G \times f)))^{-1}(U)$$
$$= cl\,[((I_G \times \alpha)\, o\, (I_G \times f))^{-1}\, q_y{}^{-1}\,(U))]$$

is compact, a contradiction to the hypothesis that $I_G \times f$ is relatively singular.

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APPROXIMATING COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT

Let C be a nonempty subset closed convex subset of a real Banach space R and let U, T be nonexpansive mappings of C into itself. In this paper, we consider the following iteration procedure of Mann's type for approximating common fixed points of mappings U and T:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \frac{1}{n^2}\alpha_n \sum_{i,j=0}^{n-1} U^i T^j \text{ for every } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in [0,1]. Using some ideas in the nonlinear ergodic theory, we prove that the iterates converge weakly to a common fixed point of the nonexpansive mappings T and U in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable.

1. Introduction. Let C be a nonempty closed convex subset of a Banach space R. Then, a mapping $T : C \to C$ is called nonexpansive if

$$\| Tx - Ty \| \leq \|x - y\| \qquad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. Mann [5] introduced an iteration procedure for approximating fixed points of a mapping T in a Hilbert space as follows:

where $\{\alpha_n\}$ is a sequence in [0,1]. Later, Reich [7] discussed this iteration procedure in a uniformly convex Banach space whose norm is Frechet differentiable.

In this paper, we consider the following iteration procedure of Mann's type for approximating common fixed points of two nonexpansive mappings in a Banach space:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \frac{\alpha_n}{n^2} \sum_{i,j=0}^{n-1} U^i T^j x_n \text{ for every } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in [0,1] and U, T are nonexpansive mappings of C into itself. Using some ideas in the nonlinear ergodic theory and an inequality obtained by *Xu [9]*, we prove that the iterates converge weakly to a common fixed point of the two nonexpensive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable.

Through this paper, we assume that R is a real Banach space. We denote by R^* the dual space of R and also denote (y,x^*) the value of $x^* \in R^*$ at $y \in R$. We write $x_n \rightarrow x$ (or $\lim_{n\rightarrow\infty} x_n = x$) to indicate that the sequence $\{X_n\}$ of vectors converges weakly to x. Similarly $x_n \rightarrow x$ (or $\lim_{n\rightarrow\infty} x_n = x$) will symbolize strong convergence. We denote by N the set of all positive integers. For a subset A of R, coA and $\overline{coA}$ mean the convex hull of A and the closure of the convex hull of A, respectively. We say that R satisfies Opial's condition [6] if for any sequence $\{X_n\} \subset R$ with $x_n \rightarrow x \in R$, the inequality

$$\lim_{n\rightarrow\infty} inf\|x_n - x\| < \lim_{n\rightarrow\infty} inf \|x_n - y\|$$

holds for every $y \in R$ with $y \neq x$. It is known that all Hilbert spaces and l^p with $1 < p < \infty$ satisfy Opial's condition. It is also known that every separable Banach space can be equivalently renormed so that it satisfies Opial's condition [3]. We also know that if the Banach space R has a duality mappings which is weakly sequentially continuous at 0, then R satisfies Opial's condition [4]. However, the space L^p with $1 < p < \infty$ and $p \neq 2$ do not satisfy Opial's condition [6]. Let R be a Banach space. Then, the norm of R is said to be Gatcaux differentiable if

$$\lim_{t\rightarrow 0} (\| x + ty \| - \| x \|)/t$$

exists for each x and y in U_R, where $U_R = \{x \in R : \| x \| = 1\}$. It is said to be Frechet differentiable if for each x in U_R, this limit is attained uniformly for y in U_R. Let C be a closed convex subset of R and let T be a mapping of C into itself. Then, for each $\varepsilon > 0$, we define the set $F_\varepsilon(T)$ to be

$$F_\varepsilon(T) = \{x \in C : \| Tx - x \| \leq \varepsilon\}$$

The following lemmas were proved in [2].

Lemma1. Let R be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of R. Then, for each $\in > 0$, there exits $\delta > 0$ such that

$$co\, F_\delta(T) \subset F_\varepsilon(T)$$

for every nonexpansive mappings T of C into itself.

Lemma.2 Let R be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of R. Then,

$$\lim_{n\rightarrow\infty} sup_{\substack{x \in C \\ T \in N(C)}} \|1/(n+1) \sum_{i=0}^{n} T^i x - T(1/(n+1) \sum_{i=0}^{n} T^i x)\| = 0,$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

Lemma.3. Let C be a nonempty closed convex subset of a Banach space R. Let U and T be a nonexpansive mappings of C into itself such that $UT=TU$ and $F(S) \cap F(T) \neq \phi$. Now consider the following iteration scheme:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x_n \qquad \text{...(1)}$$

for every $n \in N$,

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Then, for any $n \in N$ putting

$$T_n\, x = (1-\alpha_n)\, x + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x \qquad \text{for every } n \in C,$$

the mapping T_n of C into itself is also nonexpansive. In fact, let $x,y \in C$. Then, we obtain

$$\begin{aligned} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j y \| &\leq 1/n^2 \sum_{i,j=0}^{n-1} \| U^i\, T^j x - U^i\, T^j y \| \\ &\leq 1/n^2 \sum_{i,j=0}^{n-1} \| x - y \| \\ &= \| x - y \|. \end{aligned}$$

$$\begin{aligned} \|T_n x - T_n y\| &= \|\{(1-\alpha_n)\, x + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x\} - \\ &\qquad \{(1-\alpha_n)\, y + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j y\}\| \\ &\leq (1-\alpha_n)\ \|x-y\| + \alpha_n\ \|1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x - \\ &\qquad 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j y \| \\ &\leq (1-\alpha_n)\, \| x-y \| + \alpha_n\, \| x-y \| \\ &= \| x-y \|. \end{aligned}$$

Further, we have $F(T) \cap F(U) \subset F(1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j) \subset F(T_n)$ for every $n \in N$ and hence $F(T) \cap F(U) \subset \bigcap_{n=1}^{\infty} F(T_n)$.

The iterates $\{x_n\}$ defined by (1) can be written as

$$x_{n+1} = T_n,\ T_{n-1} \ldots\ T_1\, x_1,. \qquad \text{... (2)}$$

Putting

$U_n = T_n\, T_{n-1} \ldots T_1\, x_{n+1}$, we get

$$x_{n+1} = U_n\, x_1 \qquad \text{... (3)}$$

Using Lemmas 1 and 2, we can prove the follwing lemma :

Lemma 3. Let C be a nonempty bounded closed convex Banach space R. Let U and T be nonexpansive mapping of C into itself with $UT = TU$. Then,

$$\lim_{n\to\infty} \sup_{x\in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y) \| = 0,$$

and

$$\lim_{n\to\infty} \sup_{x\in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y) \| = 0,$$

Proof. Let $\varepsilon > 0$. From Lemma 1, we know that there exits $\delta > 0$ such that

$$\overline{co}\, F_\delta(S) \subset F_\varepsilon(S) \qquad \dots (4)$$

for every nonexpansive mapping S of C into itself. From Lemma 2, we also have

$$\lim_{n\to\infty} \sup_{y\in C} \|1/n \sum_{i=0}^{n-1} U^i y - U(1/n \sum_{i=0}^{n-1} U^i y)\| = 0.$$

Then, there exits $n_1 \in N$ such that

$$\sup_{y\in C} \| 1/n \sum_{i=0}^{n-1} U^i y - U(1/n \sum_{i=0}^{n-1} U^i y) \| < \delta,$$

for every $n \geq n_1$. Then, we obtain that

$$1/n \sum_{i=0}^{n-1} U^i y \in F_\delta(U) \subset co\, F_\delta(U) \qquad \dots (5)$$

for every $y \in C$ and $n \geq n_1$. Let $l, p \in N$. Then, we have, for any $n \in N$ with $n > l, p$ and $x \in C$,

$$\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \|$$

$$\leq \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - (1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$+ \| 1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+p} x - U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$+ \| U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+l} x - U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+p} x) \|$$

$$\leq 2 \| 1/n^2 \sum_{i=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i=0}^{n-1} U^{i+l} T^{j+l} x) \|$$

$$+ \| 1/n^2 \sum_{i=0}^{n-1} U^{i+l} T^{j+p} x - U(1/n^2 \sum_{i=0}^{n-1} U^{i+p} T^{j+p} x) \|$$

and

$$I = \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$= \| 1/n^2 (\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} U^i T^j x + \sum_{i=0}^{l-1} \sum_{j=0}^{n-1} U^i T^j x - \sum_{i=n}^{n+l-1} \sum_{j=p}^{n-1} U^i T^j x - \sum_{i=1}^{n-1+l} \sum_{j=n}^{p+n-l} U^i T^j x) \|$$

$$\leq 1/n^2 (\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} \| U^i T^j x \| + \sum_{i=0}^{l-1} \sum_{j=0}^{n-1} \| U^i T^j x \| - \sum_{i=n}^{n+l-1} \sum_{j=p}^{n-1}$$

$$\| U^i T^j x \| - \sum_{i=n}^{n+l-1} \sum_{j=p}^{p+n-l} \| U^i T^j x \|)$$

$$\leq 1/n^2 \{ np + l(n-p) + l(n+p) + np \} M$$

$$\leq (2M(l+p)/n),$$

where $M = sup_{z \in C} \| z \|$. Then, there exits $n_0 \in N$ such that $n_0 > max \{n1, l, p\}$ and $(2M(l+p)/n) < \varepsilon$ for every $n \geq n_0$.

This is implies that

$$I = \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^{i+I} T^{j+p} x) \| < \in$$

for every $x \in C$. Next, we prove that

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \in \overline{co}\, F_\delta (U).$$

for every $x \in C$, $m, q \in N$ and $n \geq n_1$. If not, we have that for some $m, q \in N$, $n \geq n_1$ and $x \in C$

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \notin \overline{co}\, F_\delta (U).$$

From the separation theorem, there exits $y^*_1 \in R^*$ such that

$$(1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x y^*_1) < inf \{(z, y^*_1) : z \in \overline{co}\, F_\delta (U)\}. \qquad ...(6)$$

Then from (5) we obtain

$$inf \{(z, y^*_1) : z \in \overline{co}\, F_\delta (U)\} \leq inf \{(1/n \sum_{i=0}^{n-1} U^i x\, y^*_1) : x \in C\}$$

$$\leq (1/n \sum_{i=0}^{n-1} U^i y, y^*_1)$$

for all $y \in C$ and $n \geq n_1$. Then we have that for any $j \in \{0, 1, 2, 3, ..., n-1\}$

$$inf \{(z, y^*_1) : z \in \overline{co}\, F_\delta (U)\} \leq (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x), y^*_1)$$

and hence

$$inf \{(z, y^*_1) : z \in co\, F_\delta (U)\} \leq 1/n \sum_{i=0}^{n-1} (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x, y^*_1).$$

Therefore, From (6),

$$inf \{(z, y^*_1) : z \in \overline{co}\, F_\delta (U)\} \leq 1/n \sum_{j=0}^{n-1} (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x, y^*_1).$$

$$< inf \{(z, y^*_1) : z \in \overline{co}\, F_\delta (U)\}.$$

This is contraction. Hence, from (4), we have

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \in \overline{co}\, F_\delta (U) \subset F_\varepsilon (U) \qquad ... (7)$$

for every $m, q \in N$, $x \in C$ and $n \geq n_1$ then, it follows from (7) that

$$\sup_{\substack{x \in C \\ m,n \in N}} \| 1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x - U (1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x) \| < \varepsilon$$

for every $n \geq n_1$. Hence, we obtain

$$\| \, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U \left(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x\right) \| < 2\varepsilon + \varepsilon = 3\,\varepsilon$$

for every $x \in C$ and $n \geq n_0$. Therefore,

$$\lim_{n\to\infty} sup_{x\in C} \| \, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U \left(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x\right) \| = 0.$$

Similary, we have

$$\lim_{n\to\infty} sup_{x\in C} \| \, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - T \left(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x\right) \| = 0.$$

Lemma 4. Let R be a Banach space and let C be a nonempty closed convex subset of R. Let U and T be nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let we a common fixed point of T and U. Then, $lim \; \| \, x_n - \omega \, \|$ exists.

Proof. Let ω be a common fixed point of T and U. Then, we have

$$\begin{aligned} \| \, x_{n+1} - \omega \, \| &= \| \, (1-\alpha_n)x_n + \alpha_n \, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \, \| \\ &\leq (1-\alpha_n) \, \| \, x_n - \omega \, \| + \alpha_n \, \| \, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \, \| \\ &\leq (1-\alpha_n) \, \| \, x_n - \omega \, \| + \alpha_n \, \| \, x_n - \omega \, \| \\ &= \| \, x_n - \omega \, \| \end{aligned}$$

and hence $\lim_{n\to\infty} \| \, x_n - \omega \, \|$ exists.

Lemma 5. Let C be a nonempty closed convex subset of a uniformly convex Banach space R. Let U and T be nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $0 \leq \alpha_n \leq a$ for some a with $0 < a < 1$. Then,

$$\lim_{n\to\infty} \| \, Tx_n - x_n \, \| = \lim_{n\to\infty} (Ux_n - x_n \, \| = 0.$$

In particular, $x_n \to y_0$ implies $y_0 \in F(T) \cap F(U)$.

Proof. For $x \in C$ and $f \in F(U) \cap F(T)$, put $r = \| \, x - f \, \|$ and set $X = \{u \in R: \; \| \, u - f \, \| \leq r\} \cap C$. then, X is a nonempty bounded closed convex subset of C which is T, U-invariant and contains $x_1 = x$. So, without loss of generality, we may assume that C is bounded. Let w be a common fixed point of T and U. Then, there exits a continuous, strictly increasing and convex function $g : [0,\infty)\ [0,\infty)$ such that $g(0) = 0$ and

$$\| x_{n+1} - \omega \|^{2} = \| \ (1-\alpha_n)(x_n - \omega) + \alpha_n (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^{2}$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^{2} + \alpha_n \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^{2}$$

$$- \alpha_n (1-\alpha_n)\, g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|)$$

for all $n \in N$. Then, since $\alpha_n \leq a$, we have

$$\alpha_n (1-a)\, g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|$$

$$\leq \alpha_n (1-\alpha_n)\, g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|)$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^{2} + \alpha_n A \,|\, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^{2} - \| (x_{n+1} - \omega \|^{2}$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^{2} + \alpha_n \| x_n - \omega \|^{2} - \| (x_{n+1} - \omega \|^{2}$$

$$\leq | \, |(x_n - \omega) \|^{2} - \| (x_{n+1} - \omega \|^{2}.$$

So, from Lemma 4, we obtain

$$\lim_{n\to\infty} (1-\alpha_n)\, g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|) = 0.$$

Since g is continuous, strictly increasing and stisfies $g(0) = 0$, we have

$$\lim_{n\to\infty} (1-\alpha_n) \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \| = 0. \qquad \ldots (8)$$

It follows from the definition of $\{x_n\}$ that

$$x_{n+1} - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n = (1-\alpha_n)(x_n - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n).$$

Since

$$\begin{aligned} \| Tx_{n+1} - x_{n+1} \| \ &\leq \| Tx_{n+1} - T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n) \| \\ &+ \| T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \| \\ &+ \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_{n+1} \| \end{aligned}$$

$$\leq 2 \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_{n+1} \| + \| T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n)$$

$$- 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \|$$

from (8) and Lemma 3, we have

$$\lim_{n\to\infty} \| \; Tx_n - x_n \; \| \; = 0.$$

Similary, we have

$$\lim_{n\to\infty} \| \; Ux_n - x_n \; \| \; = 0.$$

Assume $x_n \to y_n$. Then, since I-T and I-U are semiclosed by [1], we obtain that y_0 is a common fixed point of T and U.

3. Main Result

Theorem 1. Let R be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable. Let C be a nonempty closed convex subset of R. Let U and T be a nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point z_0 of T and U.

Proof:. Let $x \in C$. We first assume that R satisfies Opial's condition. Let ω be a common fixed point of T and U. Then, from lemma 4, $\lim_{n\to\infty} \| \; x_n - \omega \; \|$ exits. As in the proof of Lemma 5, we may assume that C is bounded. Since R is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be a two subsequence of $\{x_n\}$ such that $x_{n_i} \to z_1$ and $x_{n_j} \to z_2$. Then, from Lemma 5, we have that z_1 and z_2 are common fixed points of T and U. Next, we show $z_1 = z_2$. If not, from Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \| \; x_n - z_1 \| &= \lim_{i\to\infty} \| \; x_{n_i} - z_1 \| \\ &< \lim_{i\to\infty} \| \; x_{n_j} - z_2 \; \| \; = \lim_{n\to\infty} \| \; x_n - z_2 \| \\ &\qquad\qquad = \lim_{j\to\infty} \| \; x_{n_j} - z_2 \| \\ &< \lim_{j\to\infty} \| \; x_{n_j} - z_1 \| - \lim_{n\to\infty} \| \; x_n - z_1 \| \end{aligned}$$

This is contradiction. Hence, we obtain $x_n \to y_0 \in F(T) \cap F(U)$.

Next, we assume that R has a Frechet differentiable norm. As in the proof of Lemma 5, we may assume that C is bounded. So, there exists a subsequence $\{x_n\}$ of $\{x_n\}$ such that $x_{n_i} \to y_0$. Then, from Lemma 5, we obtain $y_0 \in F(T) \cap F(U)$. Putting

$$T_n y = (1-\alpha_n)y + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j y$$

and

$U_n y = T_n \, T_{n-1} \, T_{n-2} \dots T_1 \, y$ for all $n \in N$ and $y \in$ C, form (3), we have

2. The Fractional Integral Operators. We recall here a few definitions and properties of operators used in solving the triple and quadruple integral equations. Lowndes [4] has defined the following operators:

$$I_{\eta,\alpha}(a, b;\sigma)\, f(x) = \frac{\sigma x^{-\sigma(x+\eta)}}{\Gamma\alpha} \int_a^b (x^\sigma - t^\sigma)^{\alpha-1}\, t^{\sigma(\eta+1)-1} f(t)\, dt,\ \alpha > 0, \tag{4}$$

$$= \frac{x^{1-\sigma(\alpha+\eta+1)}}{\Gamma(1+\alpha)}\, d/dx \int_a^b (x^\sigma - t^\sigma)^{\alpha}\, t^{\sigma(\eta+1)-1} f(t)\, dt,\ -1 < \alpha < 0, \tag{5}$$

$$K_{\eta,\alpha}(c, d;\sigma)\, f(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma\alpha} \int_c^d (t^\sigma - x^\sigma)^{\alpha-1}\, t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt,\ \alpha > 0, \tag{6}$$

$$= \frac{x^{\sigma(\eta-1)+1}}{\Gamma(1+\alpha)}\, d/dx \int_c^d (t^\sigma - x^\sigma)^{\alpha}\, t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt,\ -1 < \alpha < 0, \tag{7}$$

where $a < x < b,\quad \sigma > 0$.

From the theory of Abel integral equations it follows that the inverse operators are given by

$$I^{-1}_{\eta,\alpha}(a, b;\sigma)\, f(x) = I_{\eta+\alpha,-\alpha}(a, b;\sigma)\, f(x) \tag{8}$$

$$K^{-1}_{\eta,\alpha}(c, d;\sigma)\, f(x) = K^{-1}_{\eta+\alpha,-\alpha}(c, d;\sigma)\, f(x). \tag{9}$$

We require two lemmas given by Lowndes [4] which define the pairs of operators:

Lemma A. Let $I_{\eta,\alpha}(a, b;\sigma)$, $I^{-1}_{\eta,\alpha}(d, x;\sigma)$ be operators as defined earlier, then

$$I^{-1}_{\eta,\alpha}(d, x;\sigma)\, I_{\eta,\alpha}(a, b;\sigma)\, f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi}\ \frac{x^{-\sigma\eta}}{(x^\sigma - d^\sigma)} \int_a^b \frac{t^{\sigma(\eta+1)-1}(d^\sigma - t^\sigma)^\alpha}{(x^\sigma - t^\sigma)}\, f(t)\, dt, \text{ provided } x > d \geq b > a. \tag{10}$$

Lemma B. Let $K_{\eta,\alpha}(a, b;\sigma)$, $K^{-1}_{\eta,\alpha}(x,d;\sigma)$ be operators as defined earlier, then

$$K^{-1}_{\eta,\alpha}(c, d;\sigma)\, K_{\eta,\alpha}(a, b;\sigma)\, f(x) = \frac{\sigma \sin(\alpha\pi) . x^{\sigma(\alpha+\eta)}}{\pi\quad (d^\sigma - x^\sigma)}\, x \int_a^b \frac{t^{\sigma(1-\alpha-\eta)-1}(t^\sigma - d^\sigma)^\alpha}{(t^\sigma - x^\sigma)}\, f(t)\, dt, \text{ provided } x < d \leq a < b. \tag{11}$$

Two well-known results [4] which play an important part in our solution are :

$$M\,[I_{\eta,\alpha}(0, x;\sigma)\, f(x)\, ;\, s] = \frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\, M\,[f(x)\, ;\, s], \tag{12}$$

$$M\,[K_{\eta,\alpha}(x,\infty\, ;\, \sigma)\, f(x)\, ;\, s] = \frac{\Gamma(\eta+s/\sigma)}{\Gamma(\eta+\alpha+s/\sigma)}\, M\,[f(x)\, ;\, s]. \tag{13}$$

In what follows we are concerned with n-ranges of the variable x,

$y_0 \in \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\}$ Hence, we have

$$y_0 \in \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap F(T) \cap F(U) \subset \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

So from Lemma 4, we have

$$\{y_0\} = \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

Hence, we obtain $x_n \to y_0 \in F(T) \cap F(U)$.

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SOME *n*-TUPLE INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS

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ABSTRACT

The solution of n–tuple integral equations when n is even involving inverse Mellin transform has been obtained by reducing them to a system of Fredholm integral equations.

1. Introduction. In this paper we solve the n-tuple integral equations when n is even

$$(1) \quad M^{-1}\left[\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\,\phi(s)\,;\,x\right. = f_i(x),\ a_{i-1}<x<a_i,\ (i=1,3,\ldots,n-1) \text{ and } a_0=0,$$

$$(2) \quad M^{-1}\left[\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\,\phi(s)\,;\,x\right] = g_i(x),\ a_{i-1}<x<a_i,\ (i=2,4,\ldots,n) \text{ and } a_n=\infty,$$

where $\alpha, \beta, \xi, \eta, \delta, > 0$, $\sigma > 0$ are real parameters, g_i $(i = 2, 4 \ldots, n)$ are known functions, $\phi(s)$ is to be determined and

$$(3) \quad M[h(x)\,;\,s] = H(s),$$
$$M^{-1}[H(s)\,;\,x] = h(x),$$

denote the Mellin transform of $h(x)$ and its inversion formula respectively.

The above equations are an extension of the triple integral equations solved by Lowndes [4], quardruple integral equations solved by Dwivedi, Kushwaha and Trivedi [8], and six integral equations solved by Dwivedi, Shukla and Shukla [9] by means of a systematic applications of some slightly extended forms of Erdèlyi-Köber operators of fractional integration [2].

Using the properties of some slightly extended forms of the Erdèlyi-Köber operators given in [4]. We show in purely formal manner that the solution of the n-tuple integral equations can be expressed in terms of solution of Fredholm intergral equation of the second kind. The mothod of solution employed here will be seen to follow closely that used by Ahmad [7] to obtain the solution of som quadruple integral equation involving Bessel functions.

namely

(14) $I_i = \{x : a_{i-1} \leq x < a_i\}$, $(i = 1,2,3,..,n)$ and $(a_0 = 0,\ a_n \to \infty)$

and we shall write any function $f(x)$, $x \geq 0$, in the form

(15) $f(x) = \sum_{i=1}^{n} f_i(x)$,

where

(16) $f_i(x) = \begin{cases} f(x), & x \in I_i \\ 0, & \text{otherwise} \end{cases} \quad i = 1,2,3,\ldots, n.$

and similarly for g.

3. Solution of the Integral Equations. Using the notation of equations (15) and (16) we can write the n-tuple integral equations (when n is even) (1) and (2) as

(17) $M^{-1}\left[\dfrac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\ \phi(s)\ ;\ x\right] = f(x),$

(18) $M^{-1}\left[\dfrac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\ \phi(s)\ ;\ x\right] = g(x),$

where $f_1, f_3, \ldots, f_{n-1}$ and $g_2, g_4, \ldots, g_n$ are prescribed functions while f_2, $f_4, \ldots, f_n$ and $g_1, g_3, \ldots, g_{n-1}$ are unknown functions to be determined. If we write

(19) $\phi(s) = M[\phi(x); s]$,

and use the formulae (12) and (13) we find that equations (17) and (18) assume the operational form

(20) $I_{\eta,\alpha}(0, x; \sigma)\ \phi(x) = f(x)$

(21) $K_{\xi,\beta}(x,\infty\ ; \delta)\ \phi(x) = g(x).$

Using the formulae (8) and (9) and solving the above equations for $\phi(x)$ be obtain

(22) $\phi(x) = I_{\eta,\alpha}(0, x; \sigma)\ f(x)$

(23) $\phi(x) = K_{\xi+\beta,-\beta}(x,\infty\ ; \delta)\ g(x).$

We proceed to determine ϕ. The subscripts on all the operators I's will be supposed to have the subscripts $(\eta,\alpha;\sigma)$ understand and all K's to have subsript $(\xi, \beta; \delta)$. Evaluating (22) on $I_1, I_2, I_3, \ldots, I_{n-1}$ we get

(24) $\phi_1 = \binom{x}{a_0} I^{-1} f_1,$

(25) $\phi_2 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{x}{a_1} I^{-1} f_2,$

(26) $\phi_3 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} f_2 + \binom{x}{a_2} I^{-1} f_3,$

… … … …

…

(27) $\phi_{n-1} = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} f_2 + \dots + \binom{x}{a_{n-2}} I^{-1} f_{n-1},$

where $a_0 = 0$

Evaluating (23) on $I_2, I_3, \dots, I_n$ we get

(28) $\phi_2 = \binom{a_2}{x} K^{-1} g_2 + \binom{a_3}{a_2} K^{-1} g_3 + \dots + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

(29) $\phi_3 = \binom{a_3}{x} K^{-1} g_3 + \binom{a_4}{a_3} K^{-1} g_4 + \dots + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

... ...

... ...

(30) $\phi_{n-2} = \binom{a_{n-2}}{x} K^{-1} g_{n-2} + \binom{a_{n-1}}{a_{n-2}} K^{-1} g_{n-1} + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

(31) $\phi_{n-1} = \binom{a_{n-1}}{x} K^{-1} g_{n-1} + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

(32) $\phi_n = \binom{a_n}{x} K^{-1} g_n.$

where $a_n \to \infty$.

Since f_1 and g_n are known, ϕ_1 and ϕ_n can be determined by equations (24) and (32).

We now solve (25) for f_2 and substitute its value in (26) to get

(33) $\phi_3 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} \binom{x}{a_1} I \left[\phi_2 - \binom{a_1}{a_0} I^{-1} f_1 \right] + \binom{a^3}{a_2} I^{-1} f_3.$

Similarly

(34) $\phi_{n-1} = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} \binom{x}{a_1} I \left[\phi_2 - \binom{a_1}{a_0} I^{-1} f_1 \right] + \binom{a_3}{a_2} I^{-1} f_3 + \dots$

$+ \binom{a_{n-1}}{a_{n-2}} I^{-1} f_{n-1},$

We solve equation (31) for g_{n-1} and substitute its value in (30) to get

(35) $\phi_{n-2} = \binom{a_{n-2}}{x} K^{-1} g_{n-2} + \binom{a_{n-1}}{a_{n-2}} K^{-1} \binom{a_{n-1}}{x} K \left[\phi_{n-1} - \binom{a_n}{a_{n-1}} K^{-1} g_n \right]$

$+ \binom{a_n}{a_{n-1}} K^{-1} g_n.$

Similarly from equation (28) we find out the value of ϕ_2

(36) $\phi_2 = \binom{a_2}{x} K^{-1} g_2 + \binom{a_3}{a_2} K^{-1} \binom{a_3}{x} K \left[\phi_3 - \binom{a_4}{a_3} K^{-1} g_4 - \dots - \binom{a_n}{a_{n-1}} \right.$

$\left. K^{-1} g_n \right] + \dots + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

We have thus arrived at n–tuple simultaneous equations (24), (36), (33), ... , (35), (34) and (32) associated with n-tuple unknown functions $\phi_1, \phi_2, \phi_3, \dots, \phi_{n-2}, \phi_{n-1}$ and ϕ_n respectively.

From these equations we can obtain the values of these unknowns and hence the solution of the problem will be determined by equation (19).

4. An Application

We shall consider now the n-tuple integral equations, when n is even which are extension of triple integral equations solved by Lowndes [4] :

$$(37) \quad \int_0^\infty u^{-2n}\Psi(u)J_{2q}(ux)du = F_i(x),\ a_{i-1} < x < a_i,\ (i = 1,3,\dots n-1) \text{ and } a_0=0$$

$$(37) \quad \int_0^\infty \Psi(u)J_{2q}(ux)du = 0,\ a_{i-1} < x < a_i,\ (i = 2,4,\dots n) \text{ and } a_n \to \infty$$

where $J_{2p}(ux)$ is the Bessel function of the first kind of order $2p$, $F_1(x)$, $F_3(x),\dots, F_{n-1}(x)$ are prescribed functions and $\Psi(u)$ is to be determined. When $p = q$ and $a_{i>3,\dots n} \to \infty$ these are equations investigated by Ahmad [7]. We now show, in a fairly straight forward manner, that the above equations can be transformed into equations of the type (1) to (2) with $g_i = 0,\ (i = 2,4,\dots n,)$

Denoting the Mellin transforms of $\Psi(u)$ by

$$(39) \quad M[\Psi(u); s] = \Psi(s)$$

and using the result

$$(40) \quad M[\xi^{-2n} j_{2q}(\xi); s] = 2^{s-1-2n}\frac{\Gamma(q-n+s/2)}{\Gamma(1+n+q-s/2)},$$

we have, on applying the Faltung theorem for Mellin transforms that the integral equations (37) and (38) can be written in the form

$$(41) \quad M^{-1}\left[\frac{\Gamma(1+p-s/2)}{\Gamma(1+n+q-s/2)}\phi(s); x\right] = 2^{1+2n}x^{-2n}F_i(x),\ a_{i-1} < x < a_i, \\ (i = 1,3\dots n-1) \text{ and } a_0 = 0,$$

$$(41) \quad M^{-1}\left[\frac{\Gamma(p+s/2)}{\Gamma(q-n+s/2)}\phi(s); x\right] = 0,\ a_{i-1} < x < a_i,\ (i = 2,4,\dots,n) \,\&\, a_0 \to \infty$$

where

$$(43) \quad \phi(s) = 2^s\left[\frac{\Gamma(q-n+s/2)}{\Gamma(1+p-s/2)}\right]\Psi(1-s).$$

These are the same as equations (1) and (2) with

$\sigma = \delta = 2,\ \xi = \eta = p,\ \alpha = q-p+n,\ \beta = q-p-n,$

and

$$(44) \quad f_i(x) = 2^{1+2n}x^{-2n}F_i(x),\ i = 1,3,\dots, n-1 \\ g_i = 0,\ i = 2,4,\dots, n.$$

Using the results of the previous section we have therefore, that the solution of equations (41) and (42) can be found in terms of a function $\phi(x)$ by

$$(45) \quad \phi(s) = M[\phi(x); s],$$

where the functions $\phi_1, \phi_2, \phi_3, \dots, \phi_{n-2}, \phi_{n-1}$ and ϕ_n are obtained from equations (24), (36), (33), ..., (35), (34) and (32) with parameters ξ, η etc. given by equations (44).

Finally, in order to find the solution of the integral equations (37) and (38) in terms of $\phi(x)$, we proceed in the following way.

From equation (39) we have that the solution is

$$(46)\quad \psi(u) = M^{-1}\,[\psi(s); u],$$

$$= M^{-1}\left[2^{s-1}\,\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}\,M\,\{\,\phi(x);\ 1-s\};\ u\right],$$

on using equations (43) and (44). Inverting the order of integration in the last equation we get

$$(47)\quad \psi(u) = \int_0^\infty \phi(x)\,M^{-1}\left[2^{s-1}\,\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}\ ;\ ux\right] dx,$$

$$= \int_0^\infty \left(\frac{ux}{2}\right)^{1+n+p-q} \phi(x)\,J_{p+q-n}(ux)\,dx.$$

after applying the results (40). When $p = q$, this solution is exactly the same as that found by Ahmad [7].

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A UNIFIED PRESENTATION OF CERTAIN SUBCLASSES OF STARLIKE AND UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

A unified class $T(\alpha,\beta)$ of starlike and uniformly convex functions in the open disk has been introduced. Number of theorems involving for example, sharp distortion inequalities and modified Hamdard Product (or convolution) of functions belonging to the class $T(\alpha,\beta)$ have been obtained. It is also shown how these theorems would apply to yield various results given in the literature.

1. Introduction. Let T denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0 \tag{1}$$

Definition 1. The Class UCT. A function $f(z) \in T$ given by (1) is said to be uniformly convex, if it satisfies the inequality

$$1 + Re\,\{z f''(z)/f'(z)\} \geq |zf''(z)/f'(z)|,\ z \in U \tag{2}$$

We denote the class of such functions by UCT.

Definition 2. The Class $UCT(\alpha)$. A function $f(z)$ of the form (1) is said to be in $UCT(\alpha)$, $\alpha \geq 0$ if and only if

$$1 + Re\,\{z f''(z)/f'(z)\} \geq \alpha\,|z f''(z)/f'(z)|,\ z \in U \tag{3}$$

Remark:

$UCT(1) = UCT$

$UCT(0) = C$, the class of convex functions with negaive coefficients

Inclusion Results.

$UCT(\alpha)$	$\subseteq$	UCT	For $\alpha \geq 1$
UCT	$\subseteq$	$UCT(\alpha)$	For $0 < \alpha \leq 1$
$UCT(\alpha)$	$\subseteq$	$UCT(\gamma)$	For $\alpha \geq \gamma$

These results are straight forward from the definition.

Definition 3. The Class TS_p. A function $f(z)$ of the form (1) is said to be in TS_p if and only if

$$|zf'(z)/f(z) - 1| \leq Re\{zf'(z)/f(z)\} \; z \,\varepsilon\, U \tag{4}$$

Definition 4.The Class $TS_p(\alpha)$. A function $f(z)$ of the form (1) is said to be in $TS_p(\alpha)$ if and only if

$$\alpha\, |zf'(z)/f(z) - 1| \leq Re\{zf'(z)/f(z)\} \; z \,\varepsilon\, U \tag{5}$$

Remark.

$TS_p(1) = TSp$
$TS_p(0) = T$

Inclusion Results.

$TS_p(\alpha) \subseteq TS_p$ For $\alpha \geq 1$
$TS_p \subseteq TS_p(\alpha)$ For $0 \leq \alpha \leq 1$
$TS_p(\alpha) \subseteq TS_p(\gamma)$ For $\alpha \geq \gamma$

These results follow easily from definition.

The classes $UCT(\alpha)$ and $TS_p(\alpha)$ were introduced by Murugusundaramurthy [1].

Murugusundaramurthy [1] gave the following lemmas :

Lemma 1. A function $f(z)$ defined by (1) is in $UCT(\alpha)$, if and only if

$$\sum_{k=2}^{\infty} k\,[k(\alpha+1) - \alpha]\, a_k \leq 1 \tag{6}$$

The result (6) is *SHARP* for functions

$f_k(z) = z - z^k / k(\alpha+1) - \alpha$.

Lemma 2. A function $f(z)$ defined by (1) is in $TS_p(\alpha)$, if and only if,

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\, a_k \leq 1. \tag{7}$$

The result (7) is *SHARP* for functions

$f_k(z) = z - z^k / [k(\alpha+1) - \alpha]$.

In view of Lemma 1 and Lemma 2, it would seem to be natural to introduce and study an interesting unification of the class $UCT(\alpha)$ and $TS_p(\alpha)$.

Thus we say that a function $f(z)$ defined by (1) belongs to the class $T(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\,[1 - \beta + \beta k]\, a_k \leq 1 \tag{8}$$

for some $\alpha \geq 0$ and some β $(0 \leq \beta \leq 1)$.

Since,

$$1 - \beta + \beta k \geq 1 \qquad (0 \leq \beta \leq 1) \quad (k = 2, 3, \ldots)$$

Therefore,

$T(\alpha, 0) = TS_p(\alpha)$,

and $T(\alpha, 1) = UCT(\alpha)$.

The object of this paper is to present a unified study of the *UCT* (α) and $TS_p(\alpha)$ by proving various interesting properties and characteristics of the general class $T(\alpha, \beta)$.

2. Distortion Inequalities.

Theorem 1. If a function $f(z)$ defined by (1) is in the class $T(\alpha, \beta)$, then

$$r - \frac{r^2}{(\alpha+2)(1+\beta)} \leq |f(z)| \leq r + \frac{r^2}{(\alpha+2)(1+\beta)}. \tag{9}$$

and

$$1 - \frac{2r}{(\alpha+2)(1+\beta)} \leq |f(z)| \leq 1 + \frac{2r}{(\alpha+2)(1+\beta)}, \ |z| = r. \tag{10}$$

The estimates are sharp.

Proof. Since,

$$(a+2)(1+\beta) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} [k(\alpha+1) - \alpha][1-\beta+\beta k] a_k \leq 1,$$

we have,

$$\begin{aligned} |f(z)| &\leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \\ &\leq |z| + \frac{1}{(\alpha+2)(1+\beta)} |z|^2 \\ &\leq r + \frac{r^2}{(\alpha+2)(1+\beta)}. \end{aligned}$$

Likewise $|f(z)| \geq r - \dfrac{r^2}{(\alpha+2)(1+\beta)}$.

This proves the estimate (9) of the theorem 2.

We note that,

$$\begin{aligned} \frac{(\alpha+2)(1+\beta)}{2} \sum_{k=2}^{\infty} a_k \leq & \sum_{k=2}^{\infty} [1+\alpha/k][1-\beta+\beta k]k\, a_k \\ \leq & \sum_{k=2}^{\infty} \frac{[k(\alpha+1)-\alpha]}{k} [1-\beta+\beta k]k\, a_k \\ \leq & \ 1. \end{aligned}$$

Now we have

$$\begin{aligned} |f'(z)| &\leq 1 + |z| \sum_{k=2}^{\infty} k\, a_k \\ &\leq 1 + \frac{2}{(\alpha+2)(1+\beta)} |z| \\ &\leq 1 + \frac{2r}{(\alpha+2)(1+\beta)}. \end{aligned}$$

Likewise $|f'(z)| \geq 1 - \dfrac{2r}{(\alpha+2)(1+\beta)}$.

$$\Rightarrow \qquad \gamma\,(k+1) - k \;\le\; [\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]$$

$$\Rightarrow \qquad \gamma \;\le\; \frac{[\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k] - k}{\gamma+1}\,.$$

If we set

$$\phi\,(k) = \frac{[\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k] - k}{\gamma+1}\,.$$

We easily see that $\phi\,(k)$ is an increasing functionj of k.

on setting $k = 2$, we get

$$\gamma \;\le\; \phi\,(2) = (\alpha+2)^2\,(1-\beta) - 2.$$

This completes the proof.

Finally taking the function given by

$$f_j(z) = z - \frac{1}{(\alpha+2)\,(1+\beta)}\,z^2.$$

The result of the theorem 2 is *SHARP*.

Remark. Putting $\beta = 0$ and $\beta = 1$ in the theorem 2, we obtain the corresponding results for the classes $TS_p(\alpha)$ and $UCT\,(\alpha)$ given earlier by Murugusundaramurthy [1].

Theorem 3. Let the function $f_j(z)$ $(j = 1,2)$ be in the class $T(\alpha,\beta)$, then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} (a_{1,k}^2 + a_{2,k}^2)\,z^k$$

belongs to the class $T\,(\gamma,\beta)$, where

$$\gamma = \frac{(\alpha+2)^2\,(1+\beta) - 4}{2}\,.$$

Proof. Since,

$$\sum_{k=2}^{\infty} [\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]^2\,a_{j,k}^2$$

$$\le \Big[\sum_{k=2}^{\infty} [\,k\,(\alpha+1) - \alpha\,]\,[1-\beta+\beta k]\,a_{j,k}\Big]^2 \le 1 \;(j = 1,2)$$

we have,

$$\sum_{k=2}^{\infty} [\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]^2\,[a_{1,k}^2 + a_{2,k}^2\,] \le 2.$$

Thus, it is sufficient to find a largest γ such that

$$[k\,(\gamma+1) - \gamma]\,[1-\beta+\beta k] \le [k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]^2\,/2\,,$$

that is,

$$\gamma\,(k-1) - k \le [k\,(\alpha+1) - \alpha]^2\,[1-\beta+\beta k]\,/2,$$

$$\gamma\,(k-1) \;\le\; \frac{[\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k] - 2k}{2}$$

This proves the estimate (10) of the theorem 2.

The bounds are *SHARP* since the equalities are attained for the function

$$f(z) = z - \frac{1}{(\alpha+2)(1+\beta)} z^2 .$$

Remark. Putting $\beta = 0$ and $\beta = 1$ in theorem 2, we obtain the corresponding results for the classes $TS_p(\alpha)$ and $UCT(\alpha)$ given earlier by Murugusundaramurthy [1].

3. Modified Hadamard Product (Convolution Product)

Definition. Let the function $f_j(z)$ $(j = 1,2,)$ ε T is of the form

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{j,k} z^k, \ (a_{j,k} \geq 0, \ j = 1,2) \tag{11}$$

The modified Hadamard Product (or convolution) of $f_1(z)$ and $f_2(z)$ is denoted by $f_1 * f_2(z)$ and defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{1,k} \, a_{2,k} z^k \tag{12}$$

Theorem 2. Let the $f_j(z)$ $(j = 1,2)$ be in the class $T(\alpha,\beta)$. Then $f_1 * f_2(z)$ belongs to the class $T(\gamma,\beta)$, where $\gamma = (\alpha+2)^2 (1+\beta) - 2$.

Proof. Applying the technique used earlier by Schild and Silverman [2], we need to find the largest γ such that

$$\sum_{k=2}^{\infty} [\, k(\gamma+1) - \gamma \,] \, [1 - \beta + \beta k] \, a_{1,k} \, a_{2,k} \leq 1.$$

Since $f_j(z)$ $(j = 1,2)$ ε $T(\alpha,\beta)$, therefore, we have

$$\sum_{k=2}^{\infty} [\, k(\alpha+1) - \alpha \,] \, [1 - \beta + \beta k] \, a_{j,k} \leq 1 \ (j = 1,2).$$

Therefore, by Cauchy Schwarz inequality, we have

$$\sum_{k=2}^{\infty} [\, k(\alpha+1) - \alpha \,] \, [1 - \beta + \beta k] \sqrt{a_{1,k} \, a_{2,k}} \leq 1 \tag{13}$$

Thus, it is sufficient to show that

$$[\, k(\gamma+1) - \gamma \,] \, [1 - \beta + \beta k] \, a_{1,k} \, a_{2,k} \leq [\, k(\alpha+1) - \alpha] \, [1 - \beta + \beta k] \sqrt{(a_{1,k} \, a_{2,k})}$$

that is,

$$\sqrt{(a_{1,k} \, a_{2,k})} \leq \frac{k(\alpha+1) - \alpha}{k(\gamma+1) - \gamma}$$

that is, if

$$\frac{1}{[\, k(\alpha+1) - \alpha \,] \, [1 - \beta + \beta k]} \leq \frac{k(\alpha+1) - \alpha}{k(\gamma+1) - \gamma} \quad \text{(from (13))}$$

that is, if $\quad [\, k(\gamma+1) - \gamma \,] \ \leq \ [\, k(\alpha+1) - \alpha \,]^2 \, [1 - \beta + \beta k]$

that is,

$$\gamma \le \frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-2k}{2\,(k-1)}\ .$$

If we set,

$$\Psi\ (k) = \frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-2k}{2\,(k-1)}\ .$$

We easily see that $\Psi\ (k)$ is an increasing function of k. On setting $k = 2$, we get,

$$\gamma \le \Psi\ (z) = \frac{(\alpha+2)^2\,(1+\beta)-4}{2}\ .$$

This completes the proof.

Remark. Taking $\beta = 0$ and $\beta = 1$ we get the corresponding results for the classes $TS_p(\alpha)$ and $UCT\,(\alpha)$ given earlier by Murugusundaramurthy [1].

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SOME RESULTS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION, GENERALIZED PROLATE SPHEROIDAL WAVE FUNCTION, GENERALIZED POLYNOMIALS AND THE MULTI-VARIABLE *H*-FUNCTION

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ABSTRACT

In this paper an integral involving generalized prolate spheroidal wave function, generalized hypergeometric function, the generalized polynomials and the (Srivastava and Panda) *H*-function of several complex variables has been evaluated and an expansion formula for the product of the generalized hypergeometric function, generalized polynominals and the *H*-function of several complex variables has been established with the application of this integral. On account of the most general nature of the functions and the polynomials occuring in these results, our finding provide interesting unifications and extensions of a large number of (new an known) results. We record here only two special cases of our main integral, first involves the product of the generalized prolate spheroidal wave fuction, the generalized hypergeometric function, the generalized polynominals and the generalized Lauricella function of several complex variables ([13], p. 454), the other one involves the product of the generalized spheroidal wave function, the generalized hypergeometric function, the general class of polynomials and the multivariable *H*-function. With the application of the above two special cases of integrals we can also derive two expansion formulae for the product of the functions and polynomials stated in the integrals respectively. Out of several known results which follow as special cases of our main integral and expansion formula we refer here only to the results of Mishra [7], Gupta [6],

$$, \dots y_R (1-x)^{g_R}(1+x)^{w_R}]\ H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \dots z_r\,(1-x)^{h_r}(1+x)^{k_r}]$$

$$= \sum_{t,p,q=0}^{\infty} 2^{\rho+\sigma-\alpha-\beta+gq+wq}\, L\,(y_1,\dots,y_R)\, R^{\alpha,\beta}_{p,n}(s)\, \frac{(e_M)_q\, y^q}{q!\,(f_N)_q}$$

$$\sum_{m}^{t+p} \frac{(-t-p)_m(\alpha+\beta+t+n+p+1)_m}{m!\,(\alpha+1)_m}\, H^{0,\lambda+2\,:\,(u',v');\,\dots\,;\,(u^{(r)},v^{(r)})}_{A+2,C+1:\,(B',D');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-gq-g_1\alpha_1-\dots-g_R\alpha_R: h_1,\dots,h_r],\ [-\sigma-wq-w_1\alpha_1-\dots-w_R\alpha_R: k_1,\dots,k_r], \\ [(c):\psi',\dots,\psi^{(r)}]:[-1-m-\rho-\sigma-gq-wq-(g_1+w_1)\alpha_1-\dots-(g_R+w_R)\alpha_R: h_1+k_1,\dots,h_r+k_r]: \\ [(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1 2^{h_1+k_1},\dots,z_r 2^{h_r+k_r}\right)$$

$$\left\{\phi_t^{\alpha,\beta}(s,x)/(R^{\alpha,\beta}_{p,t}(s))^2\, \frac{\Gamma(t+\alpha+p+1)\Gamma(t+p+\beta+1)}{(2t+2p+\alpha+\beta+1)\Gamma(t+p+1)\Gamma\,(t+p+\alpha+\beta+1)}\right\} \qquad \dots (3.1)$$

where $L(y_1, \dots\, y_R)$ is same as given in (2.2), $Re\ \rho\left(+\sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$, $Re\left(\sigma + \sum_{i=1}^{r} k_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$, $h_i, k_i > 0\ \forall i \in \{1,\dots,r\}$; $g > 0$, $w = 0$, $g_{j'} > 0, w_{j'} > 0$, $\forall\, j' \in \{1,\dots,R\}$; $p_{j'}\ (j' = 1,\dots,R)$ is an arbitrary positive integer and the coefficients $A\,[q_1,\alpha_1; \dots ;q_R, \alpha_R]$ are arbitrary constants, real or complex, $T_i > 0$, $|arg\, z_i| < T_i\pi/2$, $i = 1,\dots,r$, $j = 1,\dots,u^{(i)}$, $\alpha > -1$, $\beta > -1$, $M \le N$ ($M = N+1$ and $|y| < 1$).

Proof. Let

$$f(x) = (1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\ {}_MF_N\left[\begin{matrix}e_M \\ f_N\end{matrix}; y(1-x)^g(1+x)^w\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}(1+x)^{w_1}$$

$$, \dots y_R (1-x)^{g_R}(1+x)^{w_R}]\ H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \dots z_r\,(1-x)^{h_r}(1+x)^{k_r}]$$

$$= \sum_{t=0}^{\infty} E_t\, \phi_t^{\alpha,\beta}(s,x), \qquad (-1 < x < 1) \qquad \dots (3.2)$$

Equation (3.2) is valid since $f(x)$ is continuous and of bounded variation in the open interval (–1,1).

Now mutliply both side of (3.2) by

$$(1-x)^\alpha (1+x)^\beta\, \phi_n^{\alpha,\beta}(s,x)$$

and integrate with respect to x from -1 to 1. Change the order of integration and summation (which is permissible) on the right, we obtain

$$\int_{-1}^{1} (1-x)^\rho(1+x)^\sigma\, \phi_n^{\alpha,\beta}(s,x)\ {}_MF_N\left[\begin{matrix}e_M \\ f_N\end{matrix}; y(1-x)^g(1+x)^w\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}$$

Srivastava and Singh [16].

1. Introduction. The solution of the following differential equation

$$(1-x^2)\frac{d^2y}{dx^2} + [(\beta-\alpha) - (\alpha+\beta+2)\,x]\,\frac{dy}{dx} + [\xi(s) - s^2x^2]\,y = 0 \qquad \ldots(1.1)$$

is the generalized prolate spheroidal wave function (Gupta [5]) which has been denoted as

$$\phi_n^{\alpha,\beta}(s,x) = \sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s)\, P_{p+n}^{(\alpha,\beta)}(x) \qquad \ldots(1.2)$$

where $s = 0$ and $\xi(0) = (n+p)\,(\alpha+\beta+n+p+1),\ p \geq 0.$

The generalized polynomials (multivariable) defined by srivastava [11] represented in the following manner :

$$S_{q_1,\ldots,q_R}^{p_1,\ldots,p_R}\begin{bmatrix} x_1 \\ \vdots \\ x_R \end{bmatrix} = S_{q_1,\ldots,q_R}^{p_1,\ldots,p_R}\,[x_1, \ldots ; x_R]$$

$$= \sum_{\alpha_1=0}^{[q_1/p_1]} \cdots \sum_{\alpha_R=0}^{[q_R/p_R]} \frac{(-q_1)p_1\alpha_1}{\alpha_1!} \cdots \frac{(-q_R)p_R\alpha_R}{\alpha_R!}.\, A\,[q_1,\alpha_1; \ldots ;q_R, \alpha_R]$$

$$x_1^{\alpha_1} \ldots x_R^{\alpha_R} \qquad \ldots(1.3)$$

where $q_j = 0, 1, 2, \ldots ; p_j\ (j = 1, \ldots, R)$ are non-zero arbitrary positive integer. The coefficients. $A\,[q_1,\alpha_1; \ldots ;q_R, \alpha_R]$ being arbitrary constants, real or complex. If we take $R = 1$ in the equation (3) and denote $A\,[q,\alpha]$ thus obtained by $A_{q,\alpha}$, we arrive at the well known general class of polynomials $S_q^p\,[x]$ intriduced by Srivastava [12].

The multivariable H-function of several complex variables occuring in the paper will be defined by Srivastava and Panda ([14], [15]) by means of the multiple Mellin-Barnes integral.

$$H\,[z_1, \ldots , z_r] =$$

$$= H_{A,\,C:\,(B',D');\ldots;(B^{(r)},D^{(r)})}^{0,\lambda\,:\,(u',\,v');\,\ldots\,;\,(u^{(r)},\,v^{(r)})}\left(\begin{matrix}[(a) : \theta',\ldots,\theta^{(r)}] :[(b'): \varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(c) : \psi',\ldots,\psi^{(r)}] :[(d'): \delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1, \ldots, z_r\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \phi_1(\xi_1) \ldots \phi_r(\xi_r)\,\psi(\xi_1, \ldots, \xi_r)\, z_1^{\xi_1} \ldots z_r^{\xi_r}\, d\xi_1 \ldots d\xi_r,\ \omega = \sqrt{-1} \qquad \ldots(1.4)$$

For the convergence of the integral given by (1.4) and other details of the multivariable H-function, we refer to the book by Srivastava et. al. ([10], p.251-3, eqn. (C-1)-(C-5)).

The orthogonality property of the generalized prolate spheroidal wave function (Gupta [5], p. 107, eqn. (3.1).

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, \phi_n^{\alpha,\beta}(s,x)\, \phi_t^{\alpha,\beta}(s,x)\, dx = N_{n,t}^{\alpha,\beta}\,\delta_{n,t} \qquad \ldots(1.5)$$

where

$$N_{n,t}^{\alpha,\beta} = \sum_{p=0}^{\infty} (R_{p,n}^{\alpha,\beta}(s))^2\, 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+p+1)\Gamma(n+p+\beta+1)}{(2n+2p+\alpha+\beta+1)\Gamma(n+p+1)\Gamma\,(n+p+\alpha+\beta+1)} \ldots(1.6)$$

and $\delta_{n,t}$ is the kronecker delta.

2. The main integral.

The following integral has been evaluated in this paper :

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, \phi_n^{\alpha,\beta}(s,x)\; {}_MF_N\left[\begin{matrix} e_M \\ f_N \end{matrix}\; ; y(1-x)^{g}(1+x)^{w}\right] S_{q_1,\dots,q_R}^{p_1,\dots,p_R}[y_1(1-x)^{g_1}$$

$$(1+x)^{w_1}, \dots y_R(1-x)^{g_R}(1+x)^{w_R}]\; H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \dots z_r(1-x)^{h_r}(1+x)^{k_r}]dx$$

$$= \sum_{p,q=0}^{\infty} 2^{\rho+\sigma+1+gq+wq}\, L(y_1,\dots,y_R)\, R_{p,n}^{\alpha,\beta}(s)\, \frac{(e_M)_q\, y^q}{q!\,(f_N)_q}$$

$$\sum_{m}^{n+p} \frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!\,(\alpha+1)_m}\, H_{A+2,C+1:\,(B',D');\dots;(B^{(r)},D^{(r)})}^{0,\lambda+2\,:\,(u',v');\,\dots\,;(u^{(r)},v^{(r)})}$$

$$\left(\begin{matrix}[-m-\rho\text{-}gq-g_1\alpha_1-\dots-g_R\alpha_R: h_1,\dots,h_r],\ [-\sigma-wq-w_1\alpha_1-\dots-w_R\alpha_R: k_1,\dots,k_r], \\ [(c):\psi',\dots,\psi^{(r)}]:[-1-m-\rho-\sigma-gq-wq-(g_1+w_1)\alpha_1-\dots-(g_R+w_R)\alpha_R: h_1+k_1,\dots,h_r+k_r]: \\ [(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}]; \end{matrix}\; z_1 2^{h_1+k_1},\dots,z_r 2^{h_r+k_r}\right) \quad \dots (2.1)$$

where

$$L(y_1,\dots,y_R) = \sum_{\alpha_1=0}^{[q_1/p_1]} \cdots \sum_{\alpha_R=0}^{[q_R/p_R]} \prod_{j'=1}^{R} \left[\frac{(-q_{j'})_{p_{j'}\alpha_{j'}}}{\alpha_{j'}!}\, y_j^{\alpha_j}\, 2^{(g_{j'}+w_{j'})\alpha_{j'}}\right.$$

$$A\,[q_1,\alpha_1;\dots;q_R,\alpha_R] \quad \dots (2.2)$$

provided that $h_i, k_i > 0\ \forall i \in \{1,\dots,r\}$; $g > 0$, $w = 0$, $g_{j'}>0$, $w_{j'}>0$, $\forall j' \in \{1,\dots,R\}$; $p_{i'}$, $(i' = 1,\dots,R)$ is an arbitrary positive integer and the coefficients $A\,[q_1,\alpha_1;\dots;q_R,\alpha_R]$ are arbitrary constant, real or complex.

$$Re\left(\rho + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ Re\left(\sigma + \sum_{i=1}^{r} k_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, T_i>0,\ |arg\, z_i| < 1/2$$

$T_i\pi$, $i = 1, \dots r$, $j= 1,\dots,u^{(i)}$, $\alpha > -1$, $\beta > -1$, $M=N$ ($M=N+1$ and $|y|<1$).

Proof. To establish (2.1), express the generalized prolate spheroidal wave function as given in (1.2), the generalized hypergeometric function as infinite series (Rainville [8], p.73. eqn. (2)), and the general polynomials with the help of equation (1.3), change the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and sums involved in the process) and then evaluate the inner integral by using a result of Gupta ([6], p.31, eqn. (2.2.1)), we arrive at the right hand side of (2.1).

3. Expansion Formula.

$$(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\; {}_MF_N\left[\begin{matrix} e_M \\ f_N \end{matrix}\; ; y(1-x)^{g}(1+x)^{w}\right] S_{q_1,\dots,q_R}^{p_1,\dots,p_R}[y_1(1-x)^{g_1}(1+x)^{w_1}$$

$$(1+x)^{w_1}, \ldots y_R (1-x)^{g_R}(1+x)^{w_R}]\; H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \ldots\, z_r (1-x)^{h_r} (1+x)^{k_r}]dx$$

$$= \sum_{l=0}^{\infty} E_l \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, \phi_n^{\alpha,\beta}(s,x), \phi_l^{\alpha,\beta}(s,x)\, dx \qquad \ldots (3.3)$$

Using the orthogonality property for the generalized prolate spheroidal wave functions (1.5) on the right-hand side and the result (2.1) on the left hand side of (3.3) we obtain

$$E_l = \sum_{p,q=0}^{\infty} 2^{\rho+\sigma+1+gq+wq} L(y_1,\ldots,y_R)\, R_{p,n}^{\alpha,\beta}(s)\, \frac{(e_M)_q\, y^q}{q!\,(f_N)_q}$$

$$\sum_{m}^{l+p} \frac{(-l-p)_m(\alpha+\beta+l+p+1)_m}{m!\,(\alpha+1)_m}\, H^{0,\lambda+2\,:\,(u',v');\,\ldots\,;\,(u^{(r)},v^{(r)})}_{A+2,C+1:\,(B',D');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-\rho-gq-g_1\alpha_1-\ldots-g_R\alpha_R: h_1,\ldots,h_r],\,[-\sigma-wq-w_1\alpha_1-\ldots-w_R\alpha_R: k_1,\ldots,k_r], \\ [(c):\psi',\ldots,\psi^{(r)}]:[-1-m-\rho-\sigma-gq-wq-(g_1+w_1)\alpha_1-\ldots-(g_R+w_R)\alpha_R: h_1+k_1,\ldots,h_r+k_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\quad z_1 2^{h_1+k_1},\ldots,z_r 2^{h_r+k_r}\right) \Big/ \{N_{n,l}^{\alpha,\beta}\delta_{n,l}\} \quad (3.4)$$

with the help of (3.3) and (3.4) in view of (1.6), the expansion formula (3.1) is obtained.

4. Special Cases.

(i) Letting $\lambda = A,\ u^{(i)}=1,\ v^{(i)}=B^{(i)},\ D^{(i)}=D^{(i)}+1\ \forall i=1,\ldots,r$ in (2.1), the multivariable *H*-function transforrms to the generalized Lauricella function of several complex variables ([13],p. 454).

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, \phi_n^{\alpha,\beta}(s,x)\; {}_M F_N\!\left[\begin{matrix}e_M \\ f_N\end{matrix};y(1-x)^{g}(1+x)^{w}\right] S_{q_1,\ldots,q_R}^{p_1,\ldots,p_R}[y_1(1-x)^{g_1}(1+x)^{w_1}$$

$$,\ldots y_R(1-x)^{g_R}(1+x)^{w_R}]\, F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\varphi'];\ldots;[1-(b^{(r)}):\varphi^{(r)}]; \\ [1-(c):\psi',\ldots,\psi^{(r)}]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix}\right.$$

$$\left. -z_1(1-x)^{h_1}(1+x)^{k_1}, \ldots\, z_r (1-x)^{h_r} (1+x)^{k_r}\right]dx$$

$$= \sum_{p,q=0}^{\infty} 2^{\rho+\sigma+1+gq+wq} L(y_1,\ldots,y_R)\, R_{p,n}^{\alpha,\beta}(s)\, \frac{(e_M)_q\, y^q}{q!\,(f_N)_q} \sum_{m=0}^{n+p}$$

$$\frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!\,(\alpha+1)_m}\; \frac{\Gamma(1+m+\rho+gq+g_1\alpha_1+..+g_R\alpha_R)\Gamma(1+\sigma+wq+w_1\alpha_1+\ldots+w_R\alpha_R)}{\Gamma(2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\ldots+(g_R+w_R)\alpha_R)}$$

$$.\,F^{A+2:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[1+m+\rho+gq+g_1\alpha_1+\ldots+g_R\alpha_R: h_1,\ldots,h_r],\,[1+\sigma+wq+w_1\alpha_1 \\ [1-(c):\psi',\ldots,\psi^{(r)}]:[2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\ldots+(g_R+w_R)\alpha_R\end{matrix}\right.$$

$$\left.\begin{matrix}+\ldots+w_R\alpha_R: k_1,\ldots,k_r],[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ :h_1+k_1,\ldots,h_r+k_r]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix}\quad -z_1 2^{h_1+k_1},\ldots,-z_r 2^{h_r+k_r}\right) \qquad \ldots (4.1)$$

valid under the same conditions as obtainable from (2.1)

(ii) If we take $R = 1$ in (2.1) we get an integral involving the product of general class of polynomials, the generalized prolate spheroidal wave function, generalized hypergeometric function and a multivariable H-funciton

(iii) Letting $\lambda = A,\ u^{(i)}=1,\ v^{(i)}=B^{(i)},\ D^{(i)}=D^{(i)}+1\ \forall i=1,\ldots,r$ in (3.1), we get an expansion formula for the product of hypergeometric function, generalized polynomials and the generalized Lauricella function of several complex variables ([13],p. 454)

$$(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\ {}_MF_N\left[\begin{matrix} e_M \\ f_N \end{matrix};y(1-x)^g(1+x)^w\right] S^{p_1,\ldots,p_R}_{q_1,\ldots,q_R}\left[y_1(1-x)^{g_1}(1+x)^{w_1},\ldots,\right.$$
$$\left. y_R(1-x)^{g_R}(1+x)^{w_R}\right] F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\varphi'];\ldots;[1-(b^{(r)}):\varphi^{(r)}]; \\ [1-(c):\psi',\ldots,\psi^{(r)}]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; \end{matrix}\right.$$
$$\left. -z_1(1-x)^{h_1}(1+x)^{k_1},\ldots,-z_r(1-x)^{h_r}(1+x)^{k_r}\right)$$

$$= \sum_{t,p,q=0}^{\infty} 2^{\rho+\sigma-\alpha-\beta+gq+wq} L(y_1,\ldots,y_R)\, R^{\alpha,\beta}_{p,t}(s)\frac{(e_M)_q\, y^q}{q!\,(f_N)_q}\sum_{m=0}^{n+p}$$

$$\frac{(-t-p)_m(\alpha+\beta+t+p+1)_m}{m!\,(\alpha+1)_m}\ \frac{\Gamma(1+m+\rho+gq+g_1\alpha_1+\ldots+g_R\alpha_R)\Gamma(1+\sigma+wq+w_1\alpha_1+\ldots+w_R\alpha_R)}{\Gamma(2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\ldots+(g_R+w_R)\alpha_R)}$$

$$F^{A+2:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[1+m+\rho+gq+g_1\alpha_1+\ldots+g_R\alpha_R: h_1,\ldots,h_r],[1+\sigma+wq+w_1\alpha_1 \\ [1-(c):\psi',\ldots,\psi^{(r)}]:[2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\ldots+(g_R+w_R)\alpha_R \end{matrix}\right.$$
$$\left.\begin{matrix}+\ldots+w_R\alpha_R: k_1,\ldots,k_r],[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ :h_1+k_1,\ldots,h_r+k_r]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; \end{matrix}\quad -z_1 2^{h_1+k_1},\ldots,-z_r 2^{h_r+k_r}\right)$$

$$\left\{\phi_t^{\alpha,\beta}(s,x)/(R^{\alpha,\beta}_{p,t}(s))^2\frac{\Gamma(t+\alpha+p+1)\Gamma(t+p+\beta+1)}{(2t+2p+\alpha+\beta+1)\Gamma(t+p+1)\Gamma(t+p+\alpha+\beta+1)}\right\} \quad \ldots (4.2)$$

valid under the same condition as obtainable from (3.1).

(iv) Taking $R = 1$ in (9), we get an expansion formula for the product of hypergeometric function, general class of polynomials and the multivariable H-function.

5. Known Results.

i. Letting $q_{j'}\ (j' = 1,\ldots,R) = 0$ in the equation (2.1) and (3.1), we get the results obtained by Gupta ([6], p.38, eqn. (2.3.2) and p. 44 eqn. (3.5.1)).

(ii) If we take $\alpha=\beta=n=0=s,\ w_1=0,\ g_1=1$ and $y=0$ in special case (ii) above, we get an integral similar to that obtained by Srivastava and Singh with , $\xi=-1$ and $\eta=1$ ([16],p.166, eqn. (2.2)).

(iii) Letting $y=0$ in known result (i) we get an integral obtained by Gupta ([6], p. 50, eqn. (2.6.1)).

(iv) Setting $s=0$, $x=1-2\eta$ and using *Saalschutz's* theorem (Slater [9], p. 49) in known results (iii) above we get a result given by Srivastava and Panda ([15], p. 131, eqn.(2.2)).

(v) Taking $r=2$, $y=0$ and $\theta'=\theta''=\psi'=\psi''=\varphi'=\varphi''=\delta'=\delta''=1$ in eqn. (2.1) we arrive at a known result obtained by Mishra ([7],p. 158, eqn. 5.6.2).

(vi) Setting $y=0$, $A=0=C=\lambda$, $\upsilon''=B''=d''=0, u''=D''=\delta''=1$, $z_2\to 0$ in (2.1) and applying a known transformation formula Chaurasia ([1], P. 18, eqn. (1.5.4)), we arrive at another result given by Mishra ([7],p.157, eqn. (5.6.1)).

(vii) Setting $r=2$, $y=0$ and $\theta'=\theta''=\psi'=\psi''=\varphi'=\varphi''=\delta'=\delta''=1$ in eqn. (3.1) and using a known formula (Rainville [8],p.24, eqn. (2)), we obtain a result recorded by Mishra ([7],p.166, eqn. (5.7.5)).

(viii) Setting $r=2$, $A=0=C=$, $\upsilon''=B''=d''=0$, $u''=D''=\delta''=1$, $z_2\to 0$ in eqn. (3.1) and using a transformation formula (Chaurasia [1],p,.18, eqn. (1.5.4)), we get an expansion formula involving Fox's H-function.

(ix) When $y=0$, the known results (viii) reduces to another known result due to Mishra ([7], p.165, eqn. (5.7.4)).

Several other interesting special cases of our results can be deduced.

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THE INTEGRATION OF CERTAIN PRODUCTS OF THE MULTIVARIABLE *H*-FUNCTION WITH GENERAL POLYNOMIALS AND EXTENDED JACOBI POLYNOMIALS

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ABSTRACT

In this paper the authors present four integral formulas for the *H*-function of several complex variables, which was introduced and studied by H.M. Srivastava and R. Panda ([16, 17]; see also [11]). Each of these integral formulas involves a product of the multivariable *H*-function, extended Jacobi polynomials and general polynomials with essentially arbitrary coefficients, which were considered else where by H.M. Srivastava [13]. By assigning suitable special values to these coefficients, the main results (contained in the theorem 1 and 2) can be reduced to integrals' involving the classical orthogonal polynomials including, for example, Hermite, Jacobi, [and, of course, Gegenbauer (or ultraspherical), Legendre, and Tchebycheff], and Laguerre polynomials, the Bessel polynomials considered by H.L. Krall and O. Frink [8], and such other classes of generalized hypergeometric polynomials as those studied earlier by F. Brafman [2] an by H.W. Gould and A.T. Hopper [7]. On the other hand, the multivariables *H*-function occuring in each of our main results can be reduced, under various special cases, to such simpler functions as the generalized Lauricella hypergeometric functions of several complex variables [due to H.M. Srivastava and M.C. Daoust (cf. [14] and [15])], which indeed include a great many of the useful functions (or the products of several such functions) of hypergeometric type (in one or more variables) as their particular cases (see, e.g., [1],[9]). We record here only two special cases of our main integrals, the first one involves the extended Jacobi

Polynomials, general polynomials and multivariable H-function, and the other one involves the extended Jacobi polynomials, general polynomials and Lauricella function of several complex variables. Out of several known results which follow as special cases of our integrals we refer here only to the results of Srivastava and Singh [18], Srivastava and Panda [17] and Sharma [10].

1 Introduction. The H-Function of several complex variables defined by Srivastava and Panda ([16], and [17]) by means of the multiple Mellin-Barnes type integral (see also [11], p. 251, eqn. (C–1) et. seq.)

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,\lambda\,:\,(u',\,\upsilon');\,\dots\,;\,(u^{(r)},\,\upsilon^{(r)})}_{A,\,C:\,(B',D');\dots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta];\dots;[d^{(r)}:\delta^{(r)}]; \end{matrix} z_1,\dots,z_r\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} U_1(\xi_1)\dots U_r(\xi_r)\,V(\xi_1,\dots,\xi_r)\,z_1^{\xi_1}\dots x_R^{\xi_r}\,d\xi_1\dots d\xi_r,\ \omega=\sqrt{-1} \qquad \dots(1.1)$$

where

$$U_i(\xi_i) = \frac{\prod\limits_{j=1}^{u^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\ \prod\limits_{j=1}^{\upsilon^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]}{\prod\limits_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_j)\ \prod\limits_{j=\upsilon^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}\xi_i)}, \qquad \forall i\in\{1,\dots,r\} \qquad \dots(1.2)$$

$$V(\xi_1,\dots,\xi_r) = \frac{\prod\limits_{j=0}^{\lambda}[1-a_j+\sum\limits_{i=1}^{r}\theta_j^{(i)}\xi_i]}{\prod\limits_{j=\lambda+1}^{A}[a_j-\sum\limits_{i=1}^{r}\theta_j^{(i)}\xi_i]\ \prod\limits_{j=1}^{C}\Gamma[1-c_j+\sum\limits_{i=1}^{r}\psi_j^{(i)}\xi_i]} \qquad \dots(1.3)$$

The multiple integral in (1.1) converges absolutely, if

$$T_i > 0 \text{ and } |arg\, z_i| < T_i\pi/2 \quad \forall i\,\{1,\dots r\} \qquad \dots(1.4)$$

where

$$T_i = \sum_{j=1}^{\lambda}\theta_j^{(i)} - \sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)} - \sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{u^{(i)}}\delta_j^{(i)}$$
$$-\sum_{j=u^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,\ \forall i\,\{1,\dots r\} \qquad \dots(1.5)$$

These conditions are assumed to be satisfied by the various H-function of several variables occuring in this paper.

The general polynomials (multivariable) defined by Srivastava [12] represented in the following manner :

$$S^{p_1,\,\dots\,,\,p_S}_{q_1,\,\dots\,,\,q_S}\begin{bmatrix} x_1 \\ \vdots \\ x_S \end{bmatrix} = S\,[x_1,\,\dots\,,\,x_S]$$

$$= \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_s=0}^{[q_s/p_s]} \frac{(-q_1)_{p_1k_1}}{k_1!} \cdots \frac{(-q_s)_{p_sk_s}}{k_s!} . A\,[q_1,k_1; \ldots ;q_s,k_s]$$

$$x_1^{k_1} \ldots x_s^{k_s} \qquad \ldots (1.6)$$

where $q_{i'} = 0,1,2,\ldots;\ p_{i'} \neq 0\ (i' = 1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A\,[q_1,k_1; \ldots ;q_s,k_s]$ are arbitrary constants real or complex.

If we take $s = 1$ in the equation (1.6) and denote $A\,[q,k]$ thus obtained by $A_{q,k}$, we arrive at the general class of polynomials $S_q^p\,(x)$ intorducted by Srivastava ([13], p.1, eqn. (1)).

Fujiwara [6] defined a generalized classical polynomials $R_n(x)$ on an interval (p,q) by means of the following Rodrigues' type formula:

$$R_n\,(x) = \frac{(-1)^n}{n!\,W(x)}\,\frac{d^n}{dx^n}\,[W(x)\{\upsilon(x)\}^n] \qquad \ldots(1.7)$$

where $\upsilon(x) = \mu\,(x-p)(q-x)$

$$W\,(x) = \frac{(x-p)^{\beta}(q-x)^{\alpha}}{(q-p)^{\alpha+\beta+1}\,B\,(\alpha+1,\,\beta+1)}\ \alpha > -1,\ \beta > -1 \qquad \ldots (1.8)$$

The class of polynomials $R_n(x)$ provides a unification of the classical orthogonal polynomials such as Jacobi, Laguerre and Hermite etc. The polynomials denoted by $F_n(\beta;\alpha;x)$ are called extended Jacobi Polynomials.

Thakare [20] studied the Fujiwara's polynomials extensively and obtained the following hypergeometric form of $F_n(\beta;\alpha;x)$.

$$F_n(\beta;\alpha;x) = \frac{(-\mu)^n\,(1+\beta)_n}{n\,!}\,(q-x)^n\ {}_2F_1\left(\begin{matrix}-n,-n-\alpha \\ 1+\beta\end{matrix};\frac{p-x}{q-x}\right),\ p < x < q \qquad \ldots(1.9)$$

When $p = -1,\ q = 1$ and $\mu = 1/2$ in (1.9), $F_n(\beta;\alpha;x)$ reduces to the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.

Lemma. If $Re\,(\rho) > -1,\ Re\,(\sigma) > -1$ and $p \neq q$, then

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma}\,F_n(\beta;\alpha;x)\,dx = \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!\,\Gamma\,(\rho+\sigma+n+2)}\,(q-p)^{\rho+\sigma+n+1}$$

$$\sum_{m=0}^{n} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(\rho+m+1)\,\Gamma\,(\sigma+n+1-m)}{m!\,(1+\beta+m)} \qquad \ldots (1.10)$$

Proof. The above lemma follows at once by applying the definition (1.9) in conjunction with the following form of the well known Eulerian integral for the Beta function

$$\int_p^q (x-p)^{\rho-1}(q-x)^{\sigma-1}\,dx = (q-p)^{\rho+\sigma+n-1}\,\frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma\,(\rho+\sigma)}, p \neq q \qquad \ldots (1.11)$$

where for convergence, $min\ \{Re\,(\rho),\ Re(\sigma)\} > 0$

2. The main integral formulas. Our main reults of the present paper are the integral formulas contained in the following theorems:

Theorem 1. With t_i defined by (1.5) let $|arg\, z_i| < T_i\pi/2\ \forall i\ \{1,\dots r\}$, where each of the equali Ties holds for suitably restricted values of the complex variables $z_1,\dots z_r$. Also let the polynomials $F_n(\beta;\alpha;x)$ be defined by (1.9) for every postive integer n.

Then

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\ S^{p_1,\dots,p_s}_{q_1,\dots,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ z_r(x-p)^{\rho_r}(q-x)^{\sigma_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\dots,y_s)$$

$$H^{0,\lambda+2:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+2,C+1:(B',D');\dots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-m-\rho-g_1k_1-\dots-g_sk_s:\rho_1,\dots,\rho_r], \\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\dots-(g_s+w_s)g_s:\rho_1+\sigma_1,\dots,\rho_r+\sigma_r]\end{matrix}\right.$$
$$[m-\sigma-n-w_1k_1-\dots-w_sk_s:\sigma_1,\dots,\sigma_r],[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];$$
$$[(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];[(d''):\delta''];\dots;[d^{(r)}:\delta^{(r)}];$$
$$\left. z_1(q-p)^{\sigma_1-\epsilon}, z_2(q-p)^{\sigma_2},\dots, z_r(q-p)^{\sigma_r}\right) \qquad \dots(2.1)$$

where

$$L(y_1,\dots,y_s) = \sum_{k_1=0}^{[q_1/p_1]} \dots \sum_{k_s=0}^{[q_s/p_s]} \prod_{j'=1}^{s}\left[\frac{(-q_{j'})p_{j'}k_{j'}}{k_{j'}!} y_{j'}^{k_{j'}}(q-p)^{(g_{j'}+w_{j'})k_{j'}}\right]$$
$$A[q_1,k_1;\dots;q_s,k_s] \qquad \dots(2.2)$$

provided that $p\neq q$, $\rho_i,\sigma_i > 0\ \forall i \in \{1,\dots,s\}$; $g_j, w_j > 0$, $\forall\, j \in \{1,\dots,s\}$; $p_{i'}$ $(i' = 1,\dots,s)$ is an arbitrary positive integer and the coefficients $A[q_1,k_1;\dots;q_s,k_s]$ are arbitrary constant, real or complex.

$$Re\left(\rho + \sum_{i=1}^{r}\rho_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ Re\left(\sigma + \sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, \qquad \dots(2.3)$$

and of course $T_i > 0$, $|arg\, z_i| < T_i\pi/2\ \ i = 1,\dots r$, $j = 1,\dots,u^{(r)}$.

Theorem 2. Under hypothesis preceding the assertion (2.1)of theorem 1,

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\ S^{p_1,\dots,p_s}_{q_1,\dots,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{\sigma_1} \\ z_2(q-x)^{\sigma_2} \\ \vdots \\ z_r(q-x)^{\sigma_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\dots,y_s)$$

$$H^{0,\lambda+1:(u',+1,v');(u'',v'');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:(B',D'+1);(B'',D'');\dots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-m-\sigma-w_1k_1-\dots-w_sk_s:\sigma_1,\dots,\sigma_r],\\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\dots-(g_s+w_s)\,k_s:\end{matrix}\right.$$

$$[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];[(b''):\varphi''];\dots;[b^{(r)}:\varphi^{(r)}];$$

$$\sigma_1-\epsilon,\sigma_2,\dots,\sigma_r],[(c):\psi',\dots,\psi^{(r)}]:[\rho+1+m+g_1k_1+\dots+g_sk_s:\epsilon],[(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];$$

$$\left. z_1(q-p)^{\sigma_1-\epsilon}, z_2(q-p)^{\sigma_2},\dots,z_r(q-p)^{\sigma_r}\right) \qquad \dots(2.4)$$

The Integral (2.4) holds provided that $p\neq q$, $\sigma_1 > \epsilon > 0$, $\sigma_j > 0\ j=2,\dots r$; $g_{j'},w_{j'} > 0\ \forall j' \in \{1,\dots,s\}$; $p_{j'}\ (i'=1,\dots,s)$ are arbitrary positive integers and the coefficients $A\,[q_1,k_1;\dots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho-\epsilon\frac{d_j'}{\delta_j'}\right) > -1,\ Re\left(\sigma+\sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1 \qquad \dots(2.5)$$

and of course $T_i > 0$, $|arg\ z_i| < T_i\pi/2$, $i=1,\dots r$, $j=1,\dots,u^{(r)}$.

$$\int_p^q (x-p)^\rho(q-x)^\sigma F_n(\beta;\alpha;x)\ S^{p_1,\dots,p_s}_{q_1,\dots,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i}\\ \vdots\\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{-\gamma}\\ z_2(x-p)^{\rho_2}\\ \vdots\\ z_r(x-p)^{\rho_r}\end{bmatrix}dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\dots,y^s)$$

$$H^{0,\lambda+2:(u',+1,v');(u'',v'');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:(B',D'+1);(B'',D'');\dots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-\rho-m-g_1k_1-\dots-g_sk_s:\rho_1,\dots,\rho_r],\\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\dots-(g_s+w_s)k_s:\end{matrix}\right.$$

$$[(a):\theta',\dots,\theta^{(r)}]:$$

$$\rho_1-\gamma,\rho_2,\dots,\rho_r],[(c):\psi_1,\dots,\psi^{(r)}]:[1+\sigma+n-m+w_1k_1+\dots+w_sk_s:\gamma];$$

$$\left.\begin{matrix}[(b'):\varphi'];[(b''):\varphi''];\dots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\delta'];[(d''):\delta''];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\quad z_1(q-p)^{\rho_1+\sigma_1},\dots,z_r(q-p)^{\rho_r+\sigma_r}\right) \qquad \dots(2.6)$$

The Integral (2.6) holds provided that $p\neq q$, $\rho_1 > \gamma > 0$, $\rho_j > 0\ j=2,\dots r$; $g_{j'},w_{j'} > 0\ \forall j' \in \{1,\dots,s\}$; $p_{j'}\ (j'=1,\dots,s)$ is an arbitrary positive integer and the coefficients $A\,[q_1,k_1;\dots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho + \sum_{i=1}^{r} \rho_i \frac{d_j'}{\delta_j'}\right) > -1,\ Re\left(\sigma - \gamma \frac{d'_j}{\delta'_j}\right) > -1 \qquad \text{...(2.7)}$$

and of course $T_i > 0$, $|arg\, z_i| < T_i \pi/2$, $i = 1, \ldots r$, $j = 1, \ldots, u^{(r)}$.

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s} \end{bmatrix} H \begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{-\gamma} \\ z_2 \\ \vdots \\ z_r . \end{bmatrix} dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\ldots,y_s)$$

$$H^{0,\lambda:\,(u'+2,v');\,(u'',v'');\ldots;\,(u^{(r)},v^{(r)})}_{A,C:\,(B'+1,D'+2);(B'',D'');\ldots;(B^{(r)},D^{(r)})} \Bigg(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi 1,\ldots,\psi^{(r)}]:[(d'):\delta']; \end{matrix}$$

$$[\rho+\sigma+n+2+(g_1+w_1)k_1+\ldots+(g_s+w_s)k_s:\gamma+\epsilon];\,[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];$$

$$[\rho+m+1+g_1k_1+\ldots g_sk_s:\epsilon],\,[\sigma+n-m+1+w_1k_1+\ldots+w_sk_s:\gamma];[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];$$

$$z_1(q-p)^{-\gamma-\epsilon}, z_2\ldots, z_r \Bigg) \qquad \text{...(2.8)}$$

The Integral (2.8) holds provided that $p \neq q$, $\gamma > 0$, $\epsilon > 0$, $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,\ldots,s\}$; $p_{j'}$ $(j' = 1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A[q_1,k_1;\ldots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho - \epsilon \frac{d_j'}{\delta_j'}\right) > -1,\ Re\left(\sigma - \gamma \frac{d'_j}{\delta'_j}\right) > -1 \qquad \text{...(2.9)}$$

and of course $T_i > 0$, $|arg\, z_i| < T_i \pi/2$, $i = 1, \ldots r$, $j = 1, \ldots, u^{(r)}$, where $L(y_1,\ldots,y_s)$ in (2.4), (2.6) and (2.8) is same as given in (2.2)

Proof. To establish (2.1) express the generalized polynomials with the help of equation (1.6) and the multivariable H-function in terms of Mellin-Barnes type Contour integral by virtue of (1.1), interchanging the order of intergration and summation (which is easily seen to be permissible under the conditions stated) and then evaluate the x-integral with the help of (1.10). On interpreting the result thus obtained by virtue of (1.1), we arrive at the right hand side of (2.1).

The remaining integrals (2.4), (2.6) and (2.8) can be evaluated in a similar manner.

3. Special Cases:

(i) Letting $s = 1$, in eqn. (2.1), (2.4), (2.6) and (2.8), we get interesting integrals involving extended Jacobi polynomials, general class of polynomials and multivariable H-function as follows :

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x) \; S_q^p\left[y\,(x-p)^{g}\,(q-x)^{w} \right] H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ z_r(x-p)^{\rho_r}(q-x)^{\sigma_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}\,.$$

$$A_{q,k}\left[y\,(q-p)^{g+w}\right]^k \; H^{0,\lambda+2\,:\,(u',\,v');\,\dots\,;\,(u^{(r)},\,v^{(r)})}_{A+2,C+1:\;(B',D');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix} [-m-\rho-gk:\rho_1,\,\dots\,,\rho_r],\;[m-\sigma-n-wk:\sigma_1,\dots,\sigma_r],\;[(a):\theta',\dots,\theta^{(r)}] \\ [1-n-\rho-\sigma-(g+w)k:\rho_1+\sigma_1,\dots,\rho_r+\sigma_r]\;[(c):\psi',\dots,\psi^{(r)}]: \end{matrix}\right.$$

$$\left.\begin{matrix} :[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}]; \end{matrix}\; z_1(q-p)^{\rho_1+\sigma_1},\dots,\,z_r(q-p)^{\rho_r+\sigma_r}\right) \qquad \dots(3.1)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x) \; S_q^p\left[y\,(x-p)^{g}\,(q-x)^{w} \right] H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{\sigma_1} \\ z_2\,(q-x)^{\sigma_2} \\ \vdots \\ z_r\,(q-x)^{\sigma_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\left[y\,(q-p)^{g+w}\right]^k \; H^{0,\lambda+1\,:\,(u',+1,v');\,(u'',\,v'');\dots\,;\,(u^{(r)},\,v^{(r)})}_{A+1,C+1:\;(B',D'+1);(B'',D'');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix} [m-\sigma-n-wk:\sigma_1,\,\dots\,,\sigma_r],\;[(a):\theta',\dots,\theta^{(r)}]: \\ [-1-n-\rho-\sigma-(g+w)k:\sigma_1-\epsilon,\,\sigma_2,\dots,\sigma_r],[(c):\psi',\dots,\psi^{(r)}]:\;[\rho+1+m+gk:\epsilon]; \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\,\dots;[d^{(r)}:\delta^{(r)}]; \end{matrix}\; z_1(q-p)^{\sigma_1-\epsilon}\, z_2(q-p)^{\sigma_2},\dots,\,z_r(q-p)^{\sigma_r}\right) \qquad \dots(3.2)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x) \; S_q^p\left[y\,(x-p)^{g}\,(q-x)^{w} \right] H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{-\gamma} \\ z_2\,(x-p)^{\rho_2} \\ \vdots \\ z_r\,(x-p)^{\rho_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}]^{k}\;H^{0,\lambda+1\,:\,(u',+1,v');\,(u'',\,v'');\dots;\,(u^{(r)},\,v^{(r)})}_{A+1,C+1:\;(B',D'+1);(B'',D'');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-\rho-m-gk:\,\rho_1,\,\dots\,,\rho_r],\;\;[(a):\theta',\dots,\theta^{(r)}]:\\ [-1-n-\rho-\sigma-(g+w)k:\rho_1-\gamma,\rho_2,\dots,\rho_r],[(c):\psi',\dots,\psi^{(r)}]:[1+\sigma+n-m+wk:\gamma],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\,\delta'\,];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\;\; z_1(q-p)^{\rho_1-\gamma},z_2(q-p)^{\rho_2},\dots,\,z_r(q-p)^{\rho_r}\right) \qquad \dots(3.3)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma}F_n(\beta;\alpha;x)\;\;S^{p}_{q}[\,y\,(x-p)^{g}\,(q-x)^{w}\,]\;H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{-\gamma}\\ z_2\\ \vdots\\ z_r\end{bmatrix}dx$$

$$=\frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1}\;\sum_{m=0}^{n}\sum_{k=0}^{[q/p]}\;\frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}]^{k}\;H^{0,\lambda\,:\,(u'+2,v');\,(u'',\,v'');\dots;\,(u^{(r)},\,v^{(r)})}_{A,C:\;(B'+1,D'+2);(B'',D'');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:\,[\rho+\sigma+n+2+\,(g+w)k:\,\gamma+\epsilon];\\ [(c):\,\psi',\dots,\psi^{(r)}]:\,[\rho+m+1+gk:\,\epsilon],\,[\sigma+n-m+1+wk:\gamma]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\,\delta'\,];\,\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\qquad z_1(q-p)^{-\gamma-\epsilon},\,z_2,\dots,\,z_r\right) \qquad \dots(3.4)$$

which are valid under essentially the same conditions as those of their parent formulas (2.1), (2.4), (2.6), and (2.8), respectively.

(ii) Giving suitable values to parameters and using a relation [17] in (2.1), (2.4), (2.6) and (2.8) we obtain the integrals involving generalized Lauricella function [15], but for the sake of brevity, they all are not presented here. (for example, putting $\lambda=A$, $u^{(i)}=1$, $v^{(i)}=B^{(i)}$, replecing $D^{(i)}$ by $D^{(i)}+1\ \forall i=1,\dots,r$ in (2.1), we have

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma}F_n(\beta;\alpha;x)\;\;S^{p_1,\dots,p_s}_{q_1,\dots,q_s}\begin{bmatrix} y_1(x-p)^{g_1}\,(q-x)^{w_1}\\ \vdots\\ y_s\,(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix}$$

$$F^{\,A:B',\dots,B^{(r)}}_{\,C:D',\dots,D^{(r)}}\begin{bmatrix} -z_1(x-p)^{\rho_1}(q-x)^{\sigma_1}\\ \vdots\\ -z_r(x-p)^{\rho_r}(q-x)^{\sigma_r}\end{bmatrix}dx\;=\;\frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1}$$

$$\sum_{m=0}^{n}\;\frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m}{m!\,\Gamma\,(1+\beta+m)}\,L\,(y_1,\dots,y_s)$$

$$\frac{\Gamma(1+\rho+m+g_1k_1+\ldots g_sk_s)\Gamma(1+\sigma+n-m+w_1k_1+\ldots+w_sk_s)}{\Gamma(2+n+\rho+\sigma+(g_1+w_1)k_1+\ldots+(g_s+w_s)k_s)} F^{A+2:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}$$

$$\left(\begin{array}{l} [1+m+\rho+g_1k_1+\ldots+g_sk_s:\rho_1,\ldots,\rho_r],[1-m+\sigma+n+w_1k_1+\ldots+w_sk_s:\sigma_1,\ldots,\sigma_r], \\ [2+n+\rho+\sigma+(g_1+w_1)k_1+\ldots+(g_s+w_s)k_s:\rho_1+\sigma_1,\ldots,\rho_s+\sigma_r]:[1-(c):\psi',\ldots,\psi^{(r)}]: \\ [1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\varphi'];\ldots;[1-b^{(r)}:\varphi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-d^{(r)}:\delta^{(r)}]; \end{array} -z_1(q-p)^{\rho_1+\sigma_1},\ldots,-z_r(q-p)^{\rho_s+\sigma_s}\right) \quad \ldots(3.5)$$

valid within the domain of convergence of the resulting series where n is non-negative integer, $Re\ (\rho) > -1$, $Re\ (\sigma) > -1, \rho_i, \sigma_i > 0$ $\forall i = \{1,..,r\}$, $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,\ldots,s\}$

(iii) Taking $g = 0$, $w=1$, $\alpha=\beta=n=0$ the results in eqn. (3.1), (3.2), (3.3) and (3.4) reduce to the known results given by Srivastava and Singh ([18],p. 166, eqn. (2.2), (2.4), (2.6) and (2.8).

(iv) Letting $p = -1$, $q = 1$, $\beta = \rho$, $\rho_1=\rho_2,\ldots,\rho_r=0$, $\mu=1$ the result (3.1) reduces to a result obtained by Srivastava and Panda ([17]).

(v) Letting $\rho = \beta$, q_j $(j = 1,\ldots,s)=0$ in (2.1) we obtained a result givin by Sharma ([10],p. 57, eqn. (2.2.3)).

(vi) Letting $\sigma_1=\sigma_2=\ldots=\sigma_r=0$ in case (v) above we obtained a result given by Sharma ([10], p. 56, eqn. (2.2.2)).

(vii) Letting $\rho_1=\rho_2=\ldots=\rho_r=0$ in case (v) above we obtained a result given by Sharma ([10], p. 55, eqn. (2.2.1)

Several other special case of the integral established here can be obtained.

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A REMARK ON SASAKIAN RECURRENT SPACES OF THE FIRST KIND

By

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1. Introduction. Let us consider an n-dimensional affinely connected sasakian space s_n with recurrent curvature R^i_{jkl} admitting an infinitesimal transformation :

$$\overline{x}^i = x^i + v^i(x)\,\delta t,$$

and we assume a basic condition $\underset{v}{\mathrm{L}}\lambda_m = 0$, where λ_m is a non-zero recurrence vector appeared in the equation

$$\nabla_m R^i_{jkl} = \lambda_m R^i_{jkl},$$

$\underset{v}{\mathrm{L}}$ denotes the Lie-derivative with respect to v^i and ∇ denotes the operator of covariant differentiation with respect to the Riemannian connection. In this case, the space has been called a Sasakian recurrent space of the first kind and is denoted in brief by an S^*_n – space. Under a condition $\alpha = \lambda_m v^m \neq$ constant, we have

$$Akl \overset{def}{=\!=} \nabla_l\lambda_k - \nabla_k\lambda_l \neq 0 .$$

The present space with $\lambda \neq$ constant is called a Sasakian recurrent spaces of the first kind, or in brief an $-A\,S^*_n$–space.

Takano (1966) has studied the existence of an affine motion of recurrent form and give a remark there and it is read as follows

In this space satisfying $\underset{v}{\mathrm{L}} R^i_{jkl} = 0$, under a docomposition of curvature tensor of the form :

$\alpha R^i_{jkl} = Akl\,\nabla_j v^i$ in order to have $\underset{v}{\mathrm{L}}\Gamma^i_{jk} = 0$, we can suppose formally a parallel property of contravariant vector $\beta v^i - \alpha\,\eta^i$, but this supposition yields a contradiction.

However, generally speaking, the present space itself does not admit intrinsically such a vector field. In this short report, we shall prove this fact.

2. Some Formulae. The space with recurrent curvature is characterized by a basic condition

$$\nabla_m R^i_{jkl} = \lambda_m R^i_{jkl}, \tag{2.1}$$

and the useful fundamental formulae are the first and second Bianchi's identities given by

(2.2) $R^i_{klj} + R^i_{ljk} + R^i_{jkl} = 0,$

(2.3) $\lambda_m R^i_{jkl} + \lambda_k R^i_{jlm} + \lambda_l R^i_{jmk} = 0.$

An $A\ S^*_n$–space has not only (2.1), (2.2) and (2.3) but also the following formulae :

(2.4) $A_{kl}\, v^l = -\alpha.\alpha_k$,

(2.5) $\alpha_i v^i = 0$, i.e. , $\underset{v}{\mathfrak{L}}\alpha = 0,$

and

(2.6) $A_{kl} = \lambda_k \alpha_l - \lambda_l\, \alpha_k.$

The well-known Ricci's identity in this space may be derived from (2.1) with ease and it is written as

(2.7) $A_{mn} R^i_{jkl} = R^\alpha_{jkl}\ R^i_{\alpha mn} - R^i_{\alpha kl}\, R^i_{jmn} - R^i_{j\alpha l}\, R^\alpha_{kmn} - R^i_{jk\alpha}\, R^i_{lmn}.$

3. Main Subject. Being $\alpha_k \neq 0$, we can introduce a vector η^k so as to satisfy

(3.1) $\alpha_k \eta^k = 1.$

Then, multiplying (2.7) by $\eta^m\ v^n$ side by side and summing over m and n, we get

(3.2) $\alpha R^i_{jkl} + R^\alpha_{jkl}\ A^i_\alpha\ - R^i_{\alpha kl}\, A^\alpha_j - R^i_{j\alpha l}\, A^\alpha_k - R^i_{jk\alpha}\, A^\alpha_l = 0$,

where we have used $A_{mn}\, \eta^m\ v^n\ = -\ \alpha$, derived from (2.4) and (3.1), or (2.5), (2.6) and (3.1) and put $A^i_j = R^i_{jkl}\, \eta^k v^l$.

Next, multiplying (2.3) by v^m and summing over m owing to $R^i_{jmk} = -\, R^i_{jkm}$, we find

(3.3) $\alpha R^i_{jkl} + \lambda_k\, R^i_{jlm}\, v^m - \lambda_l\, R^i_{jkm}\, v^m = 0,$

from which, putting $\beta = \lambda_m \eta^m$, we are able to have

(3.4) $\lambda_k A^i_j = (\ \beta v^l -\ \alpha\eta^l)\ R^i_{jkl}.$

Eliminating $\alpha\, R^i_{jkl}$ from (3.2) and (3.3), we have

$R^\alpha_{jkl} A^i_\alpha - R^i_{\alpha kl}\, A^\alpha_j = R^i_{j\alpha l}\, (A^\alpha_k - \lambda_k\ v^\alpha) - R^i_{j\alpha k}\, (A^\alpha_l - \lambda_l v^\alpha)$.

At this moment, if we multiply the last formula by $\beta v^l - \alpha\, \eta^l$ and sum over the index l, then by virtue of (3.4), it follows that

(3.5) $A^i_j\ \lambda_\alpha\ (A^\alpha_k - \lambda_k\ v^\alpha) = R^i_{j\alpha k}\, (A^\alpha_k - \lambda_l v^\alpha)\ (\beta v^l - \alpha\eta^l).$

Now, from $R^i_{jkl} = -R^i_{jlk}$ and (2.2), we see that $R^i_{jkl} - R_{kjl} = -R^i_{ljk}$, so from (3.4), we can derive the relation

$$\lambda_k A^i_j - \lambda_j A^i_k = -\ (\beta v^l - \alpha\eta^l)\ R^i_{ljk}.$$

Hereupon, let us assume the parallelism of $\beta v^i - \alpha\eta^i$, then we have immediately $(\beta v^l - \alpha\eta^l)\ R^i_{ljk} = 0$, so under the present assumption, it is concluded that

$\lambda_k A^i_j = \lambda_j A^i_k$.
Being $\lambda_k \neq 0$, from the last formula, we may put

(3.6) $A^i_k = \lambda_k A^i$,

where A^i indicates a suitable vector. Making use of (3.6) into the right hand side of (3.5) and taking care of $(\beta v^l - \alpha n^l)\,\lambda_l = 0$, we have $A^i_j(\mu_k - \alpha\lambda_k) = 0$, where we have defined μ_k by $A^i_k \lambda_i$,. It is easy to see that $A^i_j \neq 0$, and we have

(3.7) $\mu_k = \alpha\,\lambda_k$.

From (3.6), it follows that $\mu_k = \lambda_i A^i_k = \lambda_i A^i \lambda_k$ comparing the last result with (3.7), owing to, $\lambda_k \neq 0$ we get

(3.8) $\lambda_m A^m = \alpha$.

Substituting (3.6) into (3.2), we have

$$\alpha R^i_{jkl} - (\nabla_l \nabla_k - \nabla_k \nabla_l)\, A^i_j - A^\alpha\,(\lambda_k R^i_{j\alpha l} - \lambda_l R^i_{jk\alpha}) = 0$$

or by virtue of (2.3), we get

$$\alpha R^i_{jkl} + A^\alpha \lambda_\alpha R^i_{jlk} = (\nabla_l \nabla_k - \nabla_k \nabla_l)\, A^i_j.$$

Making use of $R^i_{jkl} = -R^i_{jlk}$ and (3.8), the above equation becomes

$$\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l\, A^i_j = 0.$$

In this case, for a suitable vector δ_k, we can regard A^i_j to be a recurrent tensor given by

(3.9) $\nabla_k A^i_j = \delta_k A^i_j$.

Differentiating (3.4) covariantly and making use of parallelism of $\beta v^i - \alpha\eta^i$, we find

$$A^i_j \nabla_l \lambda_k + \lambda_k \delta_l A^i_j = \lambda_l \lambda_k A^i_j,$$

where we have used (2.1) and (3.9).

Being $A^i_j \neq 0$, from the last formula, we get

(3.10) $\nabla_l \lambda_k = \lambda_k \lambda_l - \lambda_k \delta_l$.

Now, remembering (3.6), (3.9) may be rewritten as

$$\lambda_j \nabla_k A^i + A^i A^i{}_k \lambda_j = \delta_k \lambda_j A^i.$$

Consequently, substituting (3.10) into the last relation, we get

(3.11) $\nabla_k A^i = (2\delta_k - \lambda_k) A^i$,

where we have neglected non-vanishing λ_j. Multiplying (3.11) by λ_i and summing over i, according to

$$\begin{aligned} \lambda_i \nabla_k A^i &= \nabla_k(\lambda_i A^i) - A^i \nabla_k \lambda_i \\ &= \alpha\,\alpha_k - A^i(\lambda_i \lambda_k - \lambda_i \delta_k) \\ &= \alpha\,\alpha_k - \alpha\lambda_k + \alpha\delta_k, \end{aligned}$$

and making an appeal to (3.8) and (3.10), we have

$$\alpha\,\alpha_k - \alpha\lambda_k + \alpha\delta_k = \alpha\,(2\delta_k - \lambda_k),$$

or $\alpha(\delta_k - \lambda_k) = 0,$

i.e. $\delta_k = \lambda_k .$

In this way, (3.10) becomes

$$\nabla_l \lambda_k = \lambda_k \lambda_l - \lambda_k \alpha_l .$$

hence we can see that

$$Akl = \nabla_l \lambda_k - \nabla_k \lambda_l = - \lambda_k \alpha_l + \lambda_l \lambda_k .$$

Comparing the last result with (2.6), we have here $Akl = 0$ and so, we have

$$\alpha = c \text{ (constant)}.$$

This conclusing contradicts clearly with our preliminary assumption on $\alpha = \lambda_m \upsilon^m$, for the AS^*_n– space has been introduced orginally by basic condition $\underset{\upsilon}{\mathrm{L}} \lambda_m = 0$ and $\alpha \neq$ constant.

This completes the proof of the fact emphasized at the beginning of this report. Thus, we can state here a definite conclusion as follows :

The AS^*_n– space does not admit a parallel vector field defined by $\beta\upsilon^i - \alpha\eta^i$

4. Appendix. In the preceding section, under $\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l A^i_j = 0$ we have supposed formally a condition (3.9). We shall show here the possibility of (3.9).

If $\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l A^i_j = 0$ will be the case, the formula (3.2) is, simplified as

$$\alpha R^i_{jkl} = R^i_{j\alpha l} A^\alpha_k + R^i_{jk\alpha} A^\alpha_l .$$

Contracting $\eta^k \upsilon^l$ and using (3.6), we get

$$\alpha A^i_j = R^i_{j\alpha l} \upsilon^l (\eta^k \lambda_k) A^\alpha + R^i_{jk\alpha} \eta^k (\upsilon^l \lambda_l) A^\alpha,$$

say

$$\alpha A^i_j = R^i_{j\alpha l} (\beta \upsilon^l - \alpha \eta^l) A^\alpha .$$

Differentiating the last formula covariantly and making use of (2.1) and the parallelism of $\beta\upsilon^l - \alpha\eta^l$, we have

$$\alpha \alpha_m A^i_j + \alpha \nabla_m A^i_j = \alpha \lambda_m A^i_j + R^i_{j\alpha l} (\beta \upsilon^l - \alpha \eta^l) \nabla_m A^\alpha ,$$

i.e. $$\alpha \alpha_m A^i_j + \alpha \nabla_m A^i_j = \alpha \lambda_m A^i_j + A^i_j (\alpha \alpha_m - A^\alpha \nabla_m \lambda_\alpha) ,$$

That is to say, we have

$$\alpha \nabla_m A^i_j = (\alpha \lambda_m - A^\alpha \nabla_m \lambda_\alpha) A^i_j ,$$

or, being $a \neq 0$, we can see

$$\nabla_m A^i_j = \delta_m A^i_j , \qquad \delta_m \stackrel{def}{=\!=} \lambda_m - A^\alpha \nabla_m \lambda_\alpha / \alpha .$$

This completes the proof.

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ON SOME MULTIDIMENTIONAL INTEGRAL TRANSFORMS OF SRIVASTAVA AND PANDA'S *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

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ABSTRACT

In the present paper, we obtain multidimensional Laplace transforms and Whittaker transforms of Srivastava and Panda's *H*-function of several complex variables ([9], [10] ; also see [11], p. 251).

1. Introduction. Chandel [1] introduced multidimensional Laplacian operator to give integral representations of multiple hypergeometric functions of several variables $F^{(n)}_A$, $F^{(n)}_B$, $F^{(n)}_D$, of Lauricalla [7]. Chandel [2], further used this operator to give integral representations of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_D$, ${}^{(k)}_{(2)}E^{(n)}_D$. of Exton ([5],[6]). Also Chandel and Dwivedi ([3],[4]) introduced multidimensional Whittaker transforms of Lauricella multiple hypergeometric functions of sevreral variables [7], Exton ([5],[6]) and most generalized multiple hypergeometric function of Srivastava and Daoust [8] (also see Srivastava and Manocha [12, p. 64, [18], [19], [20]).

In the present paper, we further extend the above works to obtain the multidimensional Laplace transforms and Whittaker transforms of the most generalized Srivastava and Panda's *H*-function of several complex variables ([9], [10] ; also see [11, p.251]).

2. Laplacian Operator. Chandel [1] introduced multidimensional Laplacian operator

$$(2.1)\ L^{(\lambda,\mu)}_{\alpha_1,\dots,\alpha_n}\{\ \} = \frac{\Gamma(\alpha_1+\dots+\alpha_n)\,\lambda^{\alpha_1+\dots+\alpha_n+\mu}}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)\ \Gamma(\alpha_1+\dots+\alpha_n+\mu)}\int_0^\infty\!\!..\!\int_0^\infty e^{-\lambda(x_1+..+x_n)}(x_1+..+x_n)^{\mu}$$

$$x_1^{\alpha_1-1}\dots x_n^{\alpha_n-1}\{\ \}\,dx_1\dots dx_n\,,$$

where $Re\,(\alpha_j)>0,\ j=1,\dots,n,\ Re\,(\lambda)>0$ and $Re\,(\alpha_1+\dots+\alpha_n+\mu)>0$.
Here we give following additional applications of the above operator

$$(2.2)\ \ L^{(\lambda,\mu)}_{\alpha_1,\dots,\alpha_n}\Big\{H^{0,\gamma\,:\,(\mu',\upsilon')\,;\dots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,(B',D')\,;\dots;(B^{(n)},D^{(n)})}\Big(\begin{matrix}[(a):(\theta',\dots,\theta^{(n)})]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):(\psi',\dots,\psi^{(n)})]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}$$

$$u_1\,x_1^{\sigma_1}\,(x_1+\dots+x_n)^{\upsilon_1},\dots,u_n\,x_n^{\sigma_n}(x_1+\dots+x_n)^{\upsilon_n}\Big)\Big\}$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)\ \Gamma(\alpha_1+\ldots+\alpha_n+\mu)} H^{0,\gamma+1\,:\,(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+1,C+1\,:\,[B'+1,D'];\ldots;[B^{(n)}+1,D^{(n)}]} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}], \end{matrix} \right.$$

$$\begin{matrix} [1-(\alpha_1+\ldots+\alpha_n+\mu):\sigma_1+\upsilon_1,\ldots,\sigma_n+\upsilon_n] : [(b'):(\phi')],\ [1-\alpha_1:\sigma_1];\ldots;[(b^{(n)}):(\phi^{(n)})],\ [1-\alpha_n:\sigma_n]; \\ [1-(\alpha_1+\ldots+\alpha_n)\ :\sigma_1,\ldots,\sigma_n] : [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{matrix}$$

$$\left. u_1/\lambda^{\sigma_1+\upsilon_1},\ldots,u_n/\lambda^{\sigma_n+\upsilon_n} \right),$$

where $Re\ (\lambda) > 0$, $Re\ (\alpha_1+\ldots+\alpha_n+\mu) > 0$, $Re\ (\alpha_i) > 0$, σ_i, υ_i are positive numbers

$$\Delta_i = -\sum_{j=\gamma+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\upsilon^{(i)}} \phi_j^{(i)} - \sum_{j=\upsilon^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,$$

$$|arg\ u_i| < \pi\Delta_i/2,\ i=1,\ldots,n,$$

and H-is Srivastava and Panda's H-function of several complex variables ([9],[10] ; also see [11], p. 251).

3. Generalized Whittaker Tarnsforms. Chandel and Dwivedi ([3],[4]) introduced and stutied generalized Whittaker transform

$$(3.1)\quad \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\ldots,\alpha_n;\sigma}\{\}$$

$$= \frac{\lambda^{\sigma+\alpha_1+\ldots+\alpha_n}(\sigma+1-\mu+\alpha_1+\ldots+\alpha_n)\Gamma\left(\sum_{i=1}^{n}(\alpha_i)\right)}{\Gamma(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)\left(\prod_{i=1}^{n}\Gamma(\alpha_i)\right)} \int_0^\infty\ldots\int_0^\infty \eta_1^{\alpha_1-1}\ldots\eta_n^{\alpha_n-1}(\eta_1+\ldots+\eta_n)^\sigma$$

$$e^{-\lambda(\eta_1+\ldots+\eta_n)/2}\, W_{\mu,\upsilon}[\lambda(\eta_1+\ldots+\eta_n)]\{\}\, d\eta_1\ldots d\eta_n,$$

where $Re\ (\lambda) > 0$, $Re\ (\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon) > 0$ *and* $Re\ (\alpha_i) > 0$, $i=1,\ldots,n$. As an application of obove transform, we derive

$$(3.2)\quad \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\ldots,\alpha_n;\sigma}\left\{ H^{0,\gamma\,:\,(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\ldots;[B^{(n)},D^{(n)}]} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}] : [(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}] : [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \right.\right.$$

$$\left.\left. u_1\, x_1^{\sigma_1}(x_1+\ldots+x_n)^{\rho_1},\ldots,u_n\, x_n^{\sigma_n}(x_1+\ldots+x_n)^{\rho_n} \right)\right\}$$

$$= \frac{\Gamma(\sigma+1-\mu+\alpha_1+\ldots+\alpha_n)\,\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)\,\Gamma(\alpha_1+\ldots+\alpha_n)} H^{0,\gamma+2\,:\,(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+2,C+2\,:\,[B'+1,D'];\ldots;[B^{(n)}+1,D^{(n)}]}$$

$$\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)})],\ :[1/2-(\sigma\pm\upsilon):\sigma_1+\rho_1,\ldots,\sigma_n+\rho_n]; \\ [(c):\psi',\ldots,\psi^{(n)}],[\mu-(\sigma+\alpha_1+\ldots+\alpha_n):\sigma_1+\rho_1,\ldots,\sigma_n+\rho_n],[1-(\alpha_1+\ldots+\alpha_n):\sigma_1,\ldots,\sigma_n]; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):(\phi')],\ [1-\alpha_1:\sigma_1];\ldots;[(b^{(n)}):(\phi^{(n)})],\ [1-\alpha_n:\sigma_n]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \quad u_1/\lambda^{\sigma_1+\upsilon_1},\ldots,u_n/\lambda^{\sigma_n+\upsilon_n} \right)$$

provided $Re\ (\lambda) > 0$, $Re\ (\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon) > 0$, $Re\ (\alpha_i) > 0$, σ_i, ρ_i are positive numbers

$$\Delta_i = -\sum_{j=\gamma+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\upsilon^{(i)}} \phi_j^{(i)} - \sum_{j=\upsilon^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,$$

and $|arg\, u_i| < \pi \Delta_i/2\,,\; i = 1,\dots,n.$

Special cases of (3.2)

Case (a) For $\sigma_1 = \dots = \sigma_n = 0$, we derive from (3.2)

$$(3.3)\quad \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\dots,\alpha_n;\sigma}\Big\{H^{0,\gamma\,:\,(\mu',\upsilon');\dots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\dots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; u_1(x_1+\dots+x_n)^{\rho_1},\dots,u_n(x_1+\dots+x_n)^{\rho_n}\Big)\Big\}$$

$$= \frac{\Gamma(\sigma+1-\mu+\alpha_1+\dots+\alpha_n)\,\Gamma(\alpha_1+\dots+\alpha_n)}{\Gamma(\sigma+1/2+\alpha_1+\dots+\alpha_n\pm\upsilon)\,\Gamma(\alpha_1)\dots\Gamma(\alpha_n)}\; H^{0,\gamma+2\,:\,(\mu',\upsilon');\dots;(\mu^{(n)},\upsilon^{(n)})}_{A+2,C+1\,:\,[B',D'];\dots;[B^{(n)},D^{(n)}]}$$

$$\Big(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],[1/2-(\sigma\pm\upsilon):\rho_1,\dots,\rho_n]:[(b'):(\phi')];\dots;[(b^{(n)}):(\phi^{(n)})];\\ [(c):\psi',\dots,\psi^{(n)}],[\mu-(\sigma+\alpha_1+\dots+\alpha_n):\rho_1,\dots,\rho_n]:[(d'):(\delta')];\dots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}\; u_1/\lambda^{\rho_1},\dots,u_n/\lambda^{\rho_n}\Big).$$

where $Re\,(\lambda) > 0$, $Re\,(\sigma + 1/2 + \alpha_1+\dots+\alpha_n \pm \upsilon) > 0$, $Re\,(\alpha_i) > 0$, σ_i, ρ_i are positive numbers

$$\Delta_i = -\sum_{j=\gamma+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)} - \sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,$$

and $|arg\, u_i| < \pi\Delta_i/2\,;\; i = 1,\dots,n.$

Case (b) For $\rho_1 = \dots = \rho_n = 0$, (3.2) further gives

$$(3.4)\quad \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\dots,\alpha_n;\sigma}\Big\{H^{0,\gamma\,:\,(\mu',\upsilon');\dots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\dots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a):\theta';\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi';\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; u_1x_1^{\sigma_1},\dots,u_nx_n^{\sigma_n}\Big)\Big\}$$

$$= \frac{\Gamma(\sigma+1-\mu+\alpha_1+\dots+\alpha_n)\,\Gamma(\alpha_1+\dots+\alpha_n)}{\Gamma(\sigma+1/2+\alpha_1+\dots+\alpha_n\pm\upsilon)\,\Gamma(\alpha_1)\dots\Gamma(\alpha_n)}\; H^{0,\gamma+2\,:\,(\mu',\upsilon'+1);\dots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+2,C+2\,:\,[B'+1,D'];\dots;[B^{(n)}+1,\,D^{(n)}]}$$

$$\Big(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],[1/2-(\sigma\pm\upsilon):\sigma_1,\dots,\sigma_n]:\\ [(c):\psi',\dots,\psi^{(n)}],[\mu-(\sigma+\alpha_1+\dots+\alpha_n):\sigma_1,\dots,\sigma_n];[1-(\alpha_1+\dots+\alpha_n):\sigma_1,\dots,\sigma_n]:\end{matrix}$$

$$\begin{matrix}[(b'):\phi'],[1-\alpha_1:\sigma_1];\dots;[(b^{(n)}):\phi^{(n)}],[1-\alpha_n:\sigma_n];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; u_1/\lambda^{\sigma_1},\dots,u_n/\lambda^{\sigma_n}\Big).$$

4. Other multidimensional Whittaker transform $T^{\lambda,\mu,\upsilon}_{\beta_1,\dots,\beta_n;\sigma}$

Chandel and Dwivedi [4] introduced and studied the multidimensional Whittaker transform

$$T^{\lambda,\mu,\upsilon}_{\beta_1,\dots,\beta_n;\sigma}\{\;\},$$

$$= \frac{K\lambda^{\sigma+\beta_1+\dots+\beta_n}\Gamma(\sigma+1-\mu+\beta_1+\dots+\beta_n)\Gamma(\beta_1+\dots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\dots+\beta_n)\prod_{j=1}^{\upsilon}\Gamma\beta_i}$$

$$\int_0^\infty\dots\int_0^\infty e^{-\lambda/2\,[\sum_{j=1}^{n}(\alpha_1{}^j x_1+\dots+\alpha_n{}^j x_n)}\prod_{j=1}^{n}(\alpha_1{}^j x_1+\dots+\alpha^j{}_n x_n)^{\beta_j-1}[\sum_{j=1}^{n}(\alpha_1{}^j x_1+\dots+\alpha_n{}^j x_n)]^{\sigma}$$

$W_{\mu,\upsilon}\,[\lambda \sum_{j=1}^{n} (\alpha_1^{\,j}x_1+\ldots+\alpha_n^{\,j}x_n)]\ \{\}\ dx_1\ldots dx_n$.

where $Re\ (\lambda) > 0$, $Re\ (\sigma + 1/2 + \beta_1+\ldots+\beta_n \pm \upsilon) > 0$, $Re\ (\sigma + 1-\mu +\beta_1+\ldots+\beta_n) > 0$, $Re\ (\beta_j) > 0$ $j=1,\ldots,n$ and

$$K = \begin{vmatrix} \alpha_1^{\,1} & \alpha_2^{\,1} & \ldots & \alpha_n^{\,1} \\ \alpha_1^{\,2} & \alpha_2^{\,2} & \ldots & \alpha_n^{\,2} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{\,n} & \alpha_2^{\,n} & \ldots & \alpha_n^{\,n} \end{vmatrix} \neq 0.$$

As additional applications of above transform, we derive

$$(4.2)\quad T^{\lambda,\,\mu,\,\upsilon}_{\beta_1,\ldots,\beta_n\,;\,\sigma}\ \Big\{H^{0,\gamma\,:\,(\mu',\upsilon')\,;\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D']\,;\ldots;[B^{(n)},D^{(n)}]}\begin{pmatrix}[(a),:\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}:\phi^{(n)}];\\ [(c),:\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}:\delta^{(n)}];\end{pmatrix}$$

$$u_1\,(\alpha_1^{\,1}x_1+\ldots+\alpha_n^{\,1}x_n)^{\eta_1},\ldots,u_n(\alpha_1^{\,n}x_1+\ldots+\alpha_n^{\,n}x_n)^{\eta_n}\Big)\Big\}$$

$$= \frac{\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_n)\,\Gamma(\beta_1+\ldots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\ldots+\beta_n)\,\Gamma(\beta_1)\ldots\Gamma(\beta_n)}\ H^{0,\gamma+2\,:\,(\mu',\upsilon'+1)\,;\ldots;(\mu^{(n)},\upsilon^{(n)}+1);}_{A+2,C+2\,:\,[B'+1,D']\,;\ldots;[B^{(n)}+1,\,D^{(n)}];}$$

$$\Big(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)})],\ [1/2-(\sigma\pm\upsilon+\beta_1+\ldots+\beta_n)\ :\ \eta_1,\ldots,\eta_n\,]:\\ [(c):\psi',\ldots,\psi^{(n)}],\ [\mu-(\sigma+\beta_1+\ldots+\beta_n)\ :\ \eta_1,\ldots,\eta_n\,],\ [1-(\beta_1+\ldots+\beta_n):\eta_1,\ldots,\eta_n]:\end{matrix}$$

$$\begin{matrix}[(b'):\phi'],\ [1-\beta_1:\eta_1];\ldots;[b^{(n)}:\phi^{(n)}],[1-\beta_n:\eta_n\,];\\ [(d'):\delta'];\ldots;\ [(d^{(n)}):(\delta^{(n)})];\end{matrix}\qquad u_1/\lambda^{\eta_1},\ldots,u_n/\lambda^{\eta_n}\Big),$$

where $Re\ (\lambda) > 0$, $Re\ (\sigma + 1/2 + \beta_1+\ldots+\beta_n \pm \upsilon) > 0$, $Re\ (\sigma+1-\mu+\beta_1+\ldots+\beta_n) > 0$,

$$\kappa \qquad K = \begin{vmatrix} \alpha_1^{\,1} & \alpha_2^{\,1} & \ldots & \alpha_n^{\,1} \\ \alpha_1^{\,2} & \alpha_2^{\,2} & \ldots & \alpha_n^{\,2} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{\,n} & \alpha_2^{\,n} & \ldots & \alpha_n^{\,n} \end{vmatrix} \neq 0,$$

$Re\ (\beta_i) > 0$, η_i are positive numbers,

$$\Delta_i \;=\; -\sum_{j=\gamma+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)} - \sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,$$

and $\quad |arg\ u_i| < \pi\Delta_i/2$, $i=1,\ldots, n$.

$$(4.3)\ T^{\lambda,\,\mu,\,\upsilon}_{\beta_1,\ldots,\beta_n\,;\,\sigma}\ \Big\{\mathrm{H}^{0,\gamma\,:\,(\mu',\upsilon')\,;\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D']\,;\ldots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}:\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}:\delta^{(n)}];\end{matrix}$$

$$u_1[\sum_{j=1}^{n}(\alpha_1^{\,j}x_1+\ldots+\alpha_n^{\,j}x_n]^{\zeta_1},\ldots,u_n[\sum_{j=1}^{n}(\alpha_1^{\,j}x_1+\ldots+\alpha_n^{\,j}x_n]^{\zeta_n}\Big)\Big\}$$

$$= \frac{\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\ldots+\beta_n)}\ H^{0,\gamma+2\,:\,(\mu',\upsilon')\,;\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A+2,C+1\,:\,[B',D']\,;\ldots;[B^{(n)},\,D^{(n)}]}$$

$$\Big(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)})],\ [1/2-(\sigma\pm\upsilon+\beta_1+\ldots+\beta_n)\ :\ \zeta_1,\ldots,\zeta_n\,]:[(b'):(\phi')];\ldots;[b^{(n)}):(\phi^{(n)})];\\ [(c):\psi',\ldots,\psi^{(n)}],\ [\mu-(\sigma+\beta_1+\ldots+\beta_n)\ :\ \zeta_1,\ldots,\zeta_n\,]:[(d'):(\delta')];\ldots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}$$

$$u_1/\lambda^{\zeta_1},\dots,u_n/\lambda^{\zeta_n}\Big)\ ,$$

where all conditions of (4.2) are satisfied and ζ_i are positive numbers, $i=1,\dots,n$.

The results (4.2) and (4.3) can be further genealized as

$$(4.4)\ T^{\lambda,\,\mu,\,\upsilon}_{\beta_1,\dots,\beta_n\,;\,\sigma}\ \Big\{H^{0,\gamma\,:\,(\mu',\upsilon');\dots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\dots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}:\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}:\delta^{(n)}];\end{matrix}$$

$$u_1(\alpha_1^{\,1}x_1+\dots+\alpha_n^{\,1}x_n)^{\eta_1}[\textstyle\sum_{j=1}^{n}(\alpha_1^{\,j}x_1+\dots+\alpha_n^{\,j}x_n)]^{\zeta_1},\dots,\ u_n(\alpha_1^{\,n}x_1+\dots+\alpha_n^{\,n}x_n)^{\eta_n}[\textstyle\sum_{j=1}^{n}(\alpha_1^{\,j}x_1+\dots+\alpha_n^{\,j}x_n)]^{\zeta_n}\Big)\Big\}$$

$$=\frac{\Gamma(\sigma+1-\mu+\beta_1+\dots+\beta_n)\ \Gamma(\beta_1+\dots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\dots+\beta_n)\,\Gamma(\beta_1)\dots\Gamma(\beta_n)}\,H^{0,\gamma+2\,:\,(\mu',\upsilon'+1)\,;\dots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+2,C+2\,:\,[B'+1,D']\,;\dots;[B^{(n)}+1,\,D^{(n)}]}$$

$$\Big(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}),\ [1/2-(\sigma\pm\upsilon+\beta_1+\dots+\beta_n)\ :\eta_1+\zeta_1,\dots,\eta_n+\zeta_n\,]:[(b'):(\phi')],[1-\beta_1,\eta_1]\\ [(c):\psi',\dots,\psi^{(n)}],\ [\mu-(\beta_1+\dots+\beta_n+\sigma)\ :\eta_1+\zeta_1,\dots,\eta_n+\zeta_n\,],\ [1-(\beta_1+\dots+\beta_n):\eta_1,\dots,\eta_n]\end{matrix}$$

$$\begin{matrix};\dots;[b^{(n)}):(\phi^{(n)})],[1-\beta_n,\eta_n];\\ [(d'):(\delta')];\dots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}\qquad u_1/\lambda^{\eta_1+\zeta_1},\dots,u_n/\lambda^{\eta_n+\zeta_n}\Big),$$

where η_i and ζ_i , $i=1,\dots,n$ are all positive numbers, and all other conditions of (4.2) are satisfied.

5. Another Multidimensional Whittaker Transform :

Chandel and Dwivedi [4] introduced and studied another multidimensional Whittaker transform

$$(5.1)\ H^{\mu_1,\dots,\mu_n;\upsilon_1,\dots,\upsilon_n}_{\beta_1,\dots,\beta_n}\{\}=K\prod_{i=1}^{n}\frac{\Gamma(1+\beta_i-\mu_i)}{\Gamma(1/2+\beta_i\pm\upsilon_i)}\int_0^\infty\!\!\dots\int_0^\infty exp[-1/2\sum_{i=1}^{n}(\alpha_1^{\,i}x_1+\dots+\alpha_n^{\,i}x_n)]$$

$$\prod_{i=1}^{n}(\alpha_1^{\,i}x_1+\dots+\alpha_n^{\,i}x_n)^{\beta_i-1}\,W_{\mu_i,\upsilon_i}(\alpha_1^{\,i}x_1+\dots+\alpha_n^{\,i}x_n)\ \{\ \}\ dx_1\dots dx_n\,,$$

where $Re\ (1+\beta_i-\mu_i)>0$, $Re\ \ (1/2+\beta_i\pm\upsilon_i)>0,\ i=1,\dots,n$ and

$$K=\begin{vmatrix}\alpha_1^{\,1}\ \alpha_2^{\,1} & \dots & \alpha_n^{\,1}\\ \alpha_1^{\,2}\ \alpha_2^{\,2} & \dots & \alpha_n^{\,2}\\ \dots\ \ \dots & \dots & \dots\\ \alpha_1^{\,n}\ \alpha_2^{\,n} & \dots & \alpha_n^{\,n}\end{vmatrix}\neq 0,$$

As an its additional application, we derive

$$(5.2)\ H^{\mu_1,\dots,\mu_n;\upsilon_1,\dots,\upsilon_n}_{\beta_1,\dots,\beta_n}\ \Big\{H^{\ 0,\gamma\,:\,(\mu',\upsilon')\,;\dots;(\mu^{(n)},\upsilon^{(n)})}_{\ A,C\,:\,[B',D']\,;\dots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a),:\theta',\dots,\theta^{(n)}]:\\ [(c),:\psi',\dots,\psi^{(n)}]:\end{matrix}$$

$$\begin{matrix}[(b'):\phi'];\dots;[b^{(n)}):\phi^{(n)}]\ ;\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ u_1\,(\alpha^1{}_1x_1+\dots+\alpha^1{}_nx_n)^{\eta 1},\dots,u_n\,(\alpha_1^{\,n}x_1+\dots+\alpha_n^{\,n}x_n)^{\eta_n}\Big)\Big\}$$

$$=\prod_{i=1}^{n}\ \frac{\Gamma(1+\beta_i-\mu_i)}{\Gamma(1/2+\beta_i\pm\upsilon_i)}\ \Big\{\mathrm{H}^{0,\gamma\,:\,(\mu'\,:\,\upsilon'+2)\,;\dots;(\mu^{(n;},\upsilon^{(n)}+2)}_{A,C\,:\,(B'+2,D'+1)\,;\dots;(B^{(n)}+2,\,D^{(n)}+1)}\Big(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}\,]:\\ [(c):\psi',\dots,\psi^{(n)}\,]:\end{matrix}$$

$$\begin{matrix}[(b'):\phi'],\ [1/2-(\beta_1\pm\upsilon_1)\ :\eta_1];\dots;\ [b^{(n)}):\phi^{(n)}],[1/2-(\beta_n\pm\upsilon_n)\ :\eta_n];\\ [(d'):\delta'],\ [\mu_1-\beta_1:\eta_1];\dots;[(d^{(n)}):(\delta^{(n)})];[\ (\mu_n-\beta_n:\eta_n];\end{matrix}\qquad u_1,\dots,u_n\Big),$$

where $Re\ (1+\beta_i-\mu_i)>0$, $Re\ (1/2+\beta_i\pm\upsilon_i)>0$, $\eta_i>0$,

$$K=\begin{vmatrix}\alpha_1^{1}\,\alpha_2^{1} & \dots & \alpha_n^{1}\\ \alpha_1^{2}\,\alpha_2^{2} & \dots & \alpha_n^{2}\\ \dots\ \dots & \dots & \dots\\ \alpha_1^{n}\,\alpha_2^{n} & \dots & \alpha_n^{n}\end{vmatrix}\neq 0,$$

$$\Delta_i = -\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

and

$|arg\ u_i|<\pi\Delta_i/2\ ,i=1,..,n$

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ON DISTRIBUTIONAL STRUVE TRANSFORMATION

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ABSTRACT

The classical Struve transformation has been extended to a class of generalized functions. The generalized Struve transformation, so defined is seen to be a smooth function.

1. Introduction. The integral transfrom

$$(H_\upsilon f)(x) = \int_0^\infty \sqrt{(xt)}\; H_\upsilon(xt) f(t)\, dt \qquad \text{... (1.1)}$$

where $H_\upsilon(x)$ is the Struve function, has been studied briefly in [5, 8.4]. Recently, Love, [2], Rooney, [4] and Heywood and Rooney [1] have studied it in details.

The aim of the present work is to extend the Struve transform (1.1) to a class of generalized functions. It will be shown that the generalized Struve transform is a smooth function.

R.S. Pathak and J.N. Pandey [3] have extended the Hardy transformation given by

$$f(x) = \int_0^\infty F_\upsilon(tx)\, t dt\; C_\upsilon(ty) y\, f(y) dy, \qquad \text{... (1.2)}$$

where $C_\upsilon(z) = \cos p\pi\, J_\upsilon(z) + \sin p\pi\, Y_\upsilon(z)$

and $$F_\upsilon(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\upsilon+2p+2m}}{\Gamma(p+m+1)\,\Gamma(p+m+\upsilon+1)}.$$

It has been shown there that the Struve transform (1.1) is a special case of (1.2) when $p = 1/2$. The present paper deals the transform (1.1) Indepenently and differently.

2. Definition and properties of Struve's function. In this section, we quote some results on the Struve's function which we need hereafter of and on.

The Struve's function $H_\upsilon(z)$, of oreder υ. is defined by the equations

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[6, p. 328].

$$H_\upsilon(z) = \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1/2)\,\Gamma(1/2)} \int_0^1 (1-t^2)^{\upsilon-1/2} \sin zt.\, dt \qquad \text{... (2.1)}$$

$$= \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1/2)\,\Gamma(1/2)} \int_0^{\pi/2} \sin(z\cos\theta)\, \sin^{2\upsilon}\theta\, d\theta, \qquad \text{... (2.2)}$$

provided that $Re(\upsilon) > -1/2$, or equivalently

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\upsilon+2m+1}}{\Gamma(m+3/2)\,\Gamma(\upsilon+m+3/2)}. \qquad \text{... (2.3)}$$

Also, [6,p. 328]

$$\frac{d}{dz}\{z^\upsilon H_\upsilon(z)\} = z^\upsilon H_{\upsilon-1}(z), \qquad \text{... (2.4)}$$

$$\frac{d}{dz}\{z^{-\upsilon} H_\upsilon(z)\} = \frac{1}{2^\upsilon\,\Gamma(\upsilon+3/2)\,\Gamma(1/2)} - z^{-\upsilon} H_{\upsilon+1}(z). \qquad \text{... (2.5)}$$

From [6, p. 337], we have

$$H_\upsilon(z) = o(z^{-1/2}) \quad (\upsilon \le 1/2)$$

$$= o(z^{\upsilon-1}) \quad (\upsilon \ge 1/2) \text{ as } z \to \infty \qquad \text{.... (2.6)}$$

Also, $H_\upsilon(z) = o(z^{\upsilon+1})$ as $z \to 0$ .... (2.7)

3. The testing function space $H_a(I)$ and its dual $H'_a(I)$.

Let I be the interval $(0,\infty)$ and a, a fixed positive number. An infinitely differentiable function $\phi(x)$ defined over I is said to belong to $H_a(I)$ if

$$\gamma_k(\phi) = \sup_{0<t<\infty} |e^{-at} (tD)^k \phi(t)| < \infty \qquad \text{... (3.1)}$$

for $k = 0, 1, 2, \ldots$ and $D = d/dt$.

The topology on $H_a(I)$ is defined by means of the separating collection of semi-norms $\{\gamma_k\}_{k=0}^{\infty}$. A sequence $\{\phi_\upsilon\}_{\upsilon=1}^{\infty}$ is said to converge in $H(I)$ to the limit ϕ if $\gamma_k(\phi_\upsilon - \phi) \to 0$ as $\upsilon \to \infty$ for each $k = 0, 1, 2, \ldots$. A sequence $\{\phi_\upsilon\}_{\upsilon=1}^{\infty}$ is said to be a Cauchy sequence in $H(I)$ if $\gamma_k(\phi_\upsilon - \phi_\mu)$ goes to zero as υ, μ both go to infinity independently of each other. It can be readily seen that $H_a(I)$ is locally convex, sequentially complete, Hausdorff topological vector space. The dual of $H_a(I)$ will be represented by $H'_a(I)$.

Lemma 3.1 . For $a > 0$, $\upsilon > -1/2$ and $t, x > 0$, then for a fixed $x > 0$, $h(xt) = \sqrt{(xt)}\, H_\upsilon(xt) \in H_\upsilon(I)$, $H_\upsilon(z)$ being Struve fucntion as defined in Section 2.

Proof. From differential properties of Struve functions, we have

$$|e^{-at} (tD_t)^k h(xt)|$$

$$= |e^{-at} \sum_{j=0}^{k} a_j(\upsilon)\,(xt)^{(1/2)+j}\, H_{\upsilon-j}(xt)| < \infty,$$

where $a_j(\upsilon)$ is a polynomial in υ.

Lemma 3.2. For $a > 0$, $\upsilon = -1/2$ and $t, x > 0$,

$$\frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_\upsilon(xt)] \in H_a(I).$$

Proof. We have

$$|e^{-at}(tD_t)^k \frac{\partial^m}{\partial x^m}[\sqrt{}^n(xt)^k H_\upsilon(xt)]|$$

$$= |e^{-at}\Big(\sum_{j=0}\sum_{p=0} A_{j.p}(\upsilon).\, x^{-m}\,(xt)^{j+p+1/2}\, H_{\upsilon-j-p}(xt)\Big)| < \infty,$$

for a fixed $x > 0$, $\upsilon > -1/2$.

We now enlist some properties of the spaces $H_a(I)$ and its dual $H'_a(I)$.

Property 3.1. For $0 < a < b < 1/2$, $D(I) \subset H_b(I) \subset H_a(I) \subset E(I)$, all inclusions being continuous. Moreover as $D(I)$ is dense in $E(I)$, $H_a(I)$ is dense in $E(I)$.

Property 3.2. The dual $H'_a(I)$ equipped with the usual weak topology is Hausdorff locally convex, sequentially complete space of generalized functions.

For $f \in H'_a(I)$ there exists $C > 0$ and an integer r (C and r depending on f) such that

$$|<f, \phi>| \le C \max_{0 \le k \le r} \gamma_k(\phi).$$

Property 3.3. A locally integrable function f in I such that $e^{at}f(t)$ is absolutely integrable on I, gives rise to a regular generalized function of $H'_a(I)$ through

$$|<f, \phi>| = \int_0^\infty f(t)\,\phi(t)\,dt, \ \forall\, \phi \in H_a(I).$$

Property 3.4. The differential operator (tD_t) is a continuous linear mapping from $H_a(I)$ into itself. $(t\,D_t)'$, the adjoint of $(t\,D_t)$ maps continuously $H'_a(I)$ into itself.

4. The generalized transform. For $f \in H'_a(I)$, the generalized Struve transform is defined by

$$s[f] = F(x) = <f(t), \sqrt{(xt)}\, H_\upsilon(xt)> \qquad \text{... (4.1)}$$

where x is a non-zero real number and $t > 0$. From Lemma 3.1 we know that for fixed $x > 0$,

$$\sqrt{(xt)}\, H_\upsilon(xt) \in H_a(I),$$

where $\upsilon > -1/2$, $a > 0$. The relation (4.1) is meaningful.

Theorem 4.1. For real $x \ne 0$, let $F(x)$ be defined by (4.1). Then $F(x)$ is infinitely differentiable and that

$$F^m(x) = <f(t), \frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_a(xt)] \qquad \text{... (4.2)}$$

for all real $x \neq 0$ and $m = 1, 2, \ldots$.

Proof. By Lemma 3.2, it follows that $\frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_\alpha(xt)] \in H_\alpha(I)$. Hence, we need merely to prove (4.2), what we do through an inductive argument. We assume that (4.2) is true for m replaced by $(m-1)$. It is true by definition for $m = 0$. Keeping x fixed and $\partial x \neq 0$, consider

$$\frac{1}{\partial x}[D_x^{m-1} F(x+\partial x) - D_x^{m-1} F(x) - < f(t), D_x^m \sqrt{(xt)}\, H_\upsilon(xt) >$$

$$= \frac{1}{\partial x}[D_x^{m-1} < f(t), \sqrt{[(x+\partial x)t]}\, H_\upsilon(\overline{x+\partial x}\, t) > - D_x^{m-1} < f(t), \sqrt{(xt)}\, H_\upsilon(xt) >]$$

$$- < f(t), D_x^m [\sqrt{(xt)}\, H_\upsilon(xt)] >$$

$$= < f(t), \frac{D_x^{m-1} h(\overline{x+\partial x}\, t)\; D_x^{m-1} h(xt)}{\partial x} - D_x^m h(xt) >$$

(writing $h(xt)$ for $\sqrt{(xt)}\, H_\upsilon(xt)$.)

$$= < f(t), A_{\partial x}(t) > (say) \qquad \ldots (4.3)$$

where,

$$A_{\partial x}(t) = \frac{D_x^{m-1} h(\overline{x+\partial x}\, t)\; D_x^{m-1} h(xt)}{\partial x} - D_x^m h(xt)$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} + D_u^m h(ut)\, du - \frac{1}{\partial x}\int_x^{x+\partial x} D_x^m h(xt)\, du$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} [D_u^m h(ut) - D_x^m h(xt)]\, du$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1} h(\eta t)\, d\eta.$$

Now

$$|e^{-at}(tD_t)^k A_{\partial x}(t)|$$

$$= |e^{-at}\, \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1}\, \{(tD_t)^k h(\eta t)\}\, d\eta\, |$$

$$= |e^{-at}\, \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1}\, \{\Sigma\, a_j(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\, d\eta\, |$$

Let Δ denotes the interval $x - |\partial x| < n < x + |\partial x|$.

Then,

$$|e^{-at}(tD_t)^k A_{\partial x}(t)|$$

$$\leq \frac{|\partial x|}{2} e^{-at} \sup\, |D_\eta^{m+1} \sum_{j=0}^{k} a_j(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\, | \qquad \ldots (4.4)$$

Now, by Lemma 2.1,

$$e^{-al\, n\in\Delta}\ |D_\eta^{m+1} \sum_{j=0}^{k} a_j\,(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\ |$$

is bounded on $0 < t < \infty$, (taking $|\partial(x)| < 1$). Therefore, It follows from (4.4) that $A_{\partial x}(t)$ converges in $H(I)$ to zero as $\partial x \to 0$. Since $f \in H'(I)$, (4.3) converges to zero as $\partial x \to 0$. This completes our inductive proof of (4.2).

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* Born : September 7, 1923, Died : September 4, 2002

This Volume of

Jñānābha **ज्ञानाभ**

is

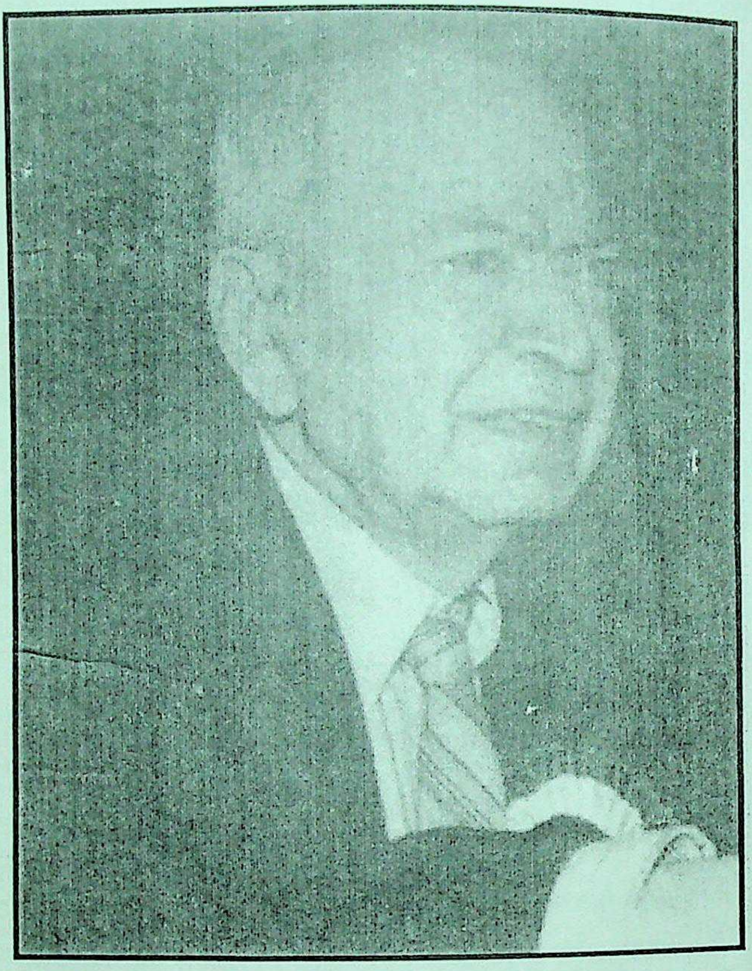
being dedicated to

the memory

of

PROFESSOR J.N. KAPUR

(September 7, 1923 - September 4, 2002)



Professor J. N. Kapur

(September 7, 1923 - September 4, 2002)

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(Dedicated to the memory of Professor J.N. Kapur)

PROFESSOR J.N. KAPUR

(1923-2002)

By

R.C. Singh Chandel
Secretary, Vijñāna Parishad of India
D.V. Postgraduate College, Orai -285 001, U.P.

After the Joint *9th* Annual Conference of **Vijñāna Parishad of India** and *5th* Annual Conference of **Indian Society of Information Theory and Applications** (Presided by Professor J.N. Kapur) held at Netaji Subhas Institute of Technology, Delhi on February 22-24,2002 and a meeting at his residence C-766 New Friends Colony, New Delhi on February 25,2002, I phoned Professor Kapur on September 20,2002 just after return of my U.S. visit of four months. Mrs. Kapur with full of sorrow replied that he had gone. In usual way , I asked "where?" She said "his own home where he had to go". I understood her feelings and shocked.

The world of Mathematics suffered an irreparable loss with the passing away of' Professor J.N. Kapur, *F.A.Sc., F.N.A.Sc., F.I.M.A. (U.K.), F.N.A., F.V.P.I,* who died in New Delhi on Wednesday, September 4,2002 at the age of 79 inspite of best possible treatment of continuously six months in New Delhi leaving behind a widow Mrs. Kapur and well established a son and two daughters with their families.

On behalf of Vijñāna Parishad of India, we pray to almighty for the peace of departed soul and God may grant fortitude to the Kapur family to bear irrreparable loss.

There are millions of students who appear in university examinations. Out of these, there are a few thousands who get first division in all examinations. Out of these, there are a few hundreds who get first positions in all their examinations. Out of these, there may be a few who beat earlier university records in both their first and second degree examinations. Out of these there was only one who beat these earlier records by wide margins of 4% and 15% marks. This man was Professor Kapur.

There are tens of thousands of persons who get *Ph.D.* degrees. A number of their theses are based on ten or more published papers. There was one thesis of more than 900 pages submitted for *Ph.D.* degree and this was based on 30 published papers. This was submitted by Professor Kapur.

There are tens of thousands of mathematicians who publish research papers. Out of these, a few hundred publish more than 100 research papers. Professor Kapur was one of them, having published more than 600 research papers.

There are research workers who work in one area in their life. Some work in two or three areas. Very few work in ten or more different areas. Professor Kapur was one of them, having published in 38 different areas.

There are many persons who write general scholarly articles on educational matters. There are a few who publish more than 500 such articles. Professor Kapur was one of them with about 1000 articles.

There are many authors who write dozens of non-fiction books. Professor Kapur was one of them, having written more than 160 books, more than 100 of which are innovative in nature and unique of their kind in the world.

There are a large number of authors whose books are published locally. There is a smaller number of authors whose books are published by all-India publishers. There are only a few whose books are published by international publishers. Professor Kapur was one of them. Four of his books have already been buplished in USA and Canada.

There are persons who have written books for primary level or secondary level or senior secondary level or undergraduate level or postgraduate level or research level or general books or books on education and ethical and moral values. Professor Kapur wrote books at all these levels

There are tens of thousands of scientists in the country. Out of these about a thousand are Fellows of one or the other of the three National Science Academies. There are about 300 who are Fellows of all the three Academies. Professor Kapur was one of them and was quite senior among them.

There are many mathematicians who have been president of indian

Mathematical Society or Calcutta Mathematical Society or Indian Society of Theoretical and Applied Mechanics or Indian Society of Agricultural Statistics, Mathematics Section of Indian Science Congress or Physical Sciences, Section of National Academy of Science or of Associations of Mathematics Teachers of India or of Vijñāna Parisad of India or of Mathematical Association of India or of Indian Society of Industrial and Applied Mathematics or of Indian Society of Information Theory and its Applications. Only Professor Kapur had been the President of all of them.

He had been Vice-President of Operation Research Society of India. Indian Society for History of Mathematics and Society of Scientific Values. This also reflects his wide interest in Pure and Applied Mathematics. Statistics, Theoretical and Applied Mechanics, Operations Research, Mathematics Education, History of Mathematics and Scientific Values.

There are many Indian Mathematicians who have been visiting professors abroad and out of them, some have been visiting professors in developed countries. He had been a visiting professor not only in departments of Pure and Applied Mathematics, but also in departments of Management Science Industrial Engineering, Schools of Business and System Engineering. Before his death he was honorary professor in a school of Computer and Systems Sciences at Jawaharlal Nehru University, Delhi and also in B.R. Ambedkar University, Agra and Netaji Subhas Institute of Technology, New Delhi. This also shows the wide range of his interests.

There are many persons who worked as editors of journals or on editorial boards of journals. Professor Kapur had worked as editor of four journals and had been on the editorial board of two dozen other journals.

Many pesons have directed summer schools organized by the UGC or NCERT. Some have directed a number of them. Professor Kapur had directed about 30 of them. What is more important is that he was one of those few perosns who has organized the summer schools on their own initiative.

There have been thousands of Vice-Chancellors of Indian Universities, but only a few who got this positions before they were even 47 years and fewer still who were requested to accept this position. In fact his Chancellor got leave of absence of him from IIT Kanpur for a second term without his even asking for it. He was a Vice-Chancellor who stood firmly for 100% fairness and impartiality in all admissions, appointments and examinations. It was he who conducted the CPMT examination inUP for four years without a single complaint. There were plenty of complaints before and after his time, but none during his time. In fact in an extraordinary resolution, the UP government congratulated him on his outstanding work which looked like a miracle at that time.

Again many eminent persons like their birthdays to be celebrated. However, when his students collected some funds for celebrating his 50th birthday, he prohibited them from doing so and these funds became the nucleus for a Mathematical Sciences Trust Society to work for the development of Mathematical Science in India.

When the UP Government wanted to give honorarium for the work for *CPMT*, he refused to accept it and at his instance the UP government donated the amount to *MSTS*. He had giving the royalty of all books to the Trust. The trust has published about 120 books so far and gives prizes worth about Rs. 100,000 per year to Mathematical Olympiad winners and to all those who stand first in mathematics in Board and University examinations all over the country.

Many persons can claim to have done some thing for mathematics education and research by using government money. He did not use government money. He believed in giving and not in taking. He believed that the Country will develop when the people depend on their own resources rather than on government resources for serving the country.

Many persons in India have personal libraries. Some mathematicians and Scientists in India have also their personal libraries, but few have a library of the size of Professor Kapur's Library which consists of more than 5000 books, 1000 journals and 300 *Ph.D.* Theses.

Professor Kapur won many awards including GP Chatterji Award of *ISCA*, Education Minister's Gold Medal of *N.A.Sc.*, Yogiji Maharaj Centenary Award and best paper awards in many international conferences. He won many distinguished service awards. He won honours (including Visiting Professorship, Fellowship of Academies, Presidentships of societies, Vice-chancellorship etc. but he never asked for them. He never hinted to anybody that he should be considered for any one of these.

He believed in joy of work. He enjoyed reading, writing lecturing, serving. He neither expected appreciation nor reward for his work. If there

Mathematical Modelling.

- Organizing Committee of First and Second International Congress of Teaching Mathematical Modelling.
- International Conference on Transportation (Chairman, Programme Committee)
- Central Board of Secondary Education (Member, Chairman, Mathmatics Committee)
- National Council of Teacher Education.
- University Grants Commission Mathematics Panel
- Innovations Committee of National Council of Educational Research and Training, India
- Review Committee of Regional Colleges of Education (Chairman)
- Council of Indian Society of History of Mathematics (Vice-President)
- Council of Operations Research Society of India (Vice-President).
- Science Education Forum of Indian Science Congress (Convener).
- Parmar Institute of Mathematical Sciences Shimla, India (Honorary Director).
- Indian Mathematical Society (Academic Secretary)
- National Committee on Mathematical Sciences *(DST)* (Chairman)
- National Committee on Mathematical Education and Research *(DST)*
- Governing bodies of Nine U.P. Engineering Colleges.
- Mathematics Section Committee and Council of *INSA*
- Indian National Commission on History of Science and Chairman of its Modern Period Committee.
- Chairman *INSA* Committee for Cooperation with International Commission on History and Philosophy and Science.
- Chairman *UGC* Committee for *USA* Programmes in Mathematics
- Member Selection Committee of Vice- Chancellorship of Agra, Meerut and Jiwaji Universities
- Member *UGC* Advisory Commitees for Mathematics Panjab, Roorkee and Jadavpur Universities.
- Chairman *CSIR* Post Doctoral Fellowship Committee.
- Chair Professor ACME
- Member *UGC* Committee of Three Vice-Chacellors for visiting 9 Universities of Germany to Discuss Collaboration between Indian and German Universities.

- Invited to Visit 11 Australian Universities at the invitation of Australian Vice-Chancellors Committee.
- Resource Person for two *UNESCO* conference in Bangladesh and Pakistan.
- Invited foreign lecturer at *NCTM* golden jubilee conference, Washington.

Publications:

Research Papers : 600 About 6000 Pages
General Articles : 1000 About 12,000 Pages
Books : 183 About 25,000 Pages

Administrative Experience

Vice-Chacellor, Meerut Universtiy, India (1971-74). Head of Postgraduate Mathematics Department (32 years), Acting Director, Indian Institute of Technology, Kanpur, on different occasions for approximately 250 days, Director of Thirty Summer Schools of Mathematics Organizer of a dozen national and International conferences.

Awards-

Gold Medal for best research National Academy	1980
National Lecturer, University Grants Commission	1982-83
G.P. Chatterjee Award, Indian Science Congress Association	1988
Distinguished Service Award, Mathematical Association of India	1983
Platinum Jubilee Lectureship, Indian Science Congress Association	1988
Best paper Award, Administrative Science Association, Canada	1986
Felicitation by Yogiji Maharaj Trust, Gandhigram	1992
Distinguished Service Award, Vijñāna Parishad of Inida	1996
Distinguished Service Award, Indian Society of Information Theory	1998
Distinguished Service Award, Mathematika Sciences Society	2000
Life Long Achievement Award, Indian Society of Industrial and Applied Mathematics	2000

Selected for Inclusion in more than a Dozen Biographies of American *Biographical Institute, Marquis* who is who, Distinguished Men of India and Many Others.

Hindu education trust award for being the best academician produced by Hindu College, Delhi in its first hundeed years. 2000

Distinguished Plaque of Industrial Engineering Department. *Manitoba* University, Canada for outstanding contributions to Industrial Engineering.

Universities Visited

USA (39), Canada (12), Italy (2),UK (12), West Germany (9), Australia (12), Netherlands (3), Singapore (1), Iran (2), Thailand (1), Bangladesh (1), Pakistan (3).

Mathematics Education Projects Visited

USA (14), UK (8), Netherlands (2), Italy (3), Australia (8).

Seminars/Lectures

Given in India (about 1000), Given in other countries (about 200).

Inaugural/Valedictory

At about 100 conferences.

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(Dedicated to the memory of Professor J.N. Kapur)

A COMMON FIXED POINT THEOREM INVOLVING BEST APPROXIMATION

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ABSTRACT

A fixed-point theorem due to Jungck is utilized to derive a common fixed-point theorem for six mappings in compact metric spaces, which is also used to prove another common fixed-point theorem involving best approximation. In process results due to Brosowski, Singh Hicks-Humphris and Sahab et al. are generalized and improved.

1. Introduction . A self-mapping T of a normed space X is said to be non-expansive (resp. I-non expansive) if $\|Tx-Ty\|\leq\|x-y\|$\{ resp.$\|Tx-Ty\| \leq \|Ix-Iy\|$\} for all x,y in X. If $\bar{x}$ is a point of X and C a subset of X, then set $B_c(\bar{x})$ of best C-approximant to $\bar{x}$ consist of the points y in C such that $\|y-\bar{x}\| = inf \| z - \bar{x} \| : z \in C \}$. A subset C of X is said to be starshaped (cf[2]) with respect to a point $q \in C$ if for all x in C and all $0 \leq \lambda \leq 1$,. $\lambda x + (1\text{-}\lambda)q$ is in C. Clearly, a convex set is starshaped with respect to each of its points but the converse is not always true.

Brosowski[1] proved that if T is non-expansive with $\bar{x} \in F(T)$, $T(C) \subset C$ and $B_c(\bar{x})$ is nonempty, compect convex, then T has a fixed point in $B_c(\bar{x})$. Subrahmanyam[13] substituted the nonempty requirement of $B_c(\bar{x})$ with the finite dimensionality of C (as a subspace of X) where as

Singh [10,11] noted that Brosowski's result remains true if $B_c(\bar{x})$ is only starshped, but soon noticed that non-expansive property of T on $B_c(\bar{x}) \cup \{\bar{x}\}$ is enough for his earlier result. In this continuation Hicks-Humphries[4] observed that Singh's first result remains true, if one replaces $T(C) \subset C$ by $T(\delta C) \subset C$, where δC denotes the boundary of C in X. Smoluk[12] substituted 'finite dimensionality of C' in Subrahmanyam's

result by 'linearity of T and compactness of $\overline{T(D)}$ for every bounded subset D of C which was later improved by Habiniak[3], by relaxing the linearity of T.

In what follows, $F(I, T)$ denotes the set common fixed points of I and T where as $F(A,B,S,T,I,J)$ denotes the set common fixed point of the mapping A,B,S,T,I, and J.

In an attempt to unify and generalize the results due to Hicks-Humphries[4] and Singh[188] Sahab et al [9] proved the following :

Theorem 1.1 : (Sahab et al.[9]) Let X be a normed space, I and T self-maps of X with $\bar{x} \in f(T,I)$, $C \subset X$ with $T(\delta C)$ and $q \in F(I)$. If $D = B_c(\bar{x})$ is compact and q-star shaped $I(D)$-D, I is continuous and linear on D, I, and T are commuting on D and T is I-non expansive on $D \cup (\bar{x})$, then I and T have a common fixed point in D.

We essentially require the following definitions :

Definition 1.1. (Jungck[6]) A pair of self-mappings $(B.I)$ of a normed *space* X is said to be compatible if

$$\lim_{n\to\infty} \|BIx_n - IBx_n\| = 0 ,$$

whenever $\{x_n\}$ is a sequence in X such that, $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ix_n = t \in X$

Definition 1.2. (Jungck and Rhoades[8]) A pair of self-mappings (B,I) on X is said to be coincidently commuting if (B,I) commute at the coincidence points of B and I.

In this note using a variant of a fixed point theorem due to Jungck[7], we first derive a common fixed point theorem in compact metric spaces involving six mappings, which is then used to prove yet another extension of Theorem1.1. In process relevent results due to Brosowski[1], Singh[10, 11], Hick-Humphries[4] and Sahab et al [9] are generalized and improved.

2. Main Results. Motivated from the observations explained in Jungck and Rhoades[8], one can state the following variant of a fixed point theorem, which is due to Jungck[7].

Theorem 2.1. (Jungck [7]) Let A, S, I and J be continuous self-mappings of a compact metric space (X,d) with $A(X) \subset J(X)$ and $S(X) \subset I(X)$. If either (A,I) are compatible and (S,J) coincidently commuting or (S,J) are compatible and (A,I) coincidently commuting and

$$d(Ax, Sy) < M(x, y) \text{ where}$$

$M(x, y) = max \{ d(Ix, Jy), d(Ix, Ax), d(Jy, Sy), 1/2 [d(Ix, Sy) + d(Jy, Ax)]\}$, for all $x, y \in X$ with $M(x, y) > 0$ then A,S,I and J have a unique common fixed point.

Proof. The proof is almost the same as that of Jungck's[7] theorem hence it is omitted

Remark 2.1. Theorem 2.1 was originally proved with compatibility of both the pairs in Jungck[7].

As an application of Theorem 2.1 we derive a common fixed point theorem for six mappings, as follows :

Theprem 2.2. Let A, B, S, T, I and J be self-mappings of a compact metric space (X, d) such that $A(X) \subset TJ(X)$, $S(X)$ with A, S, T, J and BI being continuous. If either (A, BI) are compatible and (S, TJ) coincidently commuting or (ST,J) are compatible and (A, BI) coincidently commuting and

$$d(Ax, Sy) < M(x, y),$$

where $M(x, y) = max \{ d(BIx, TJy,) d(BIx,Ax), d(TJy, Sy),$

$$\tfrac{1}{2}[d(BIx, Sy) + d(TJy, Ax)]\},$$

for all $x,y \in X$ with $M(x, y) > 0$, then A, S, BI and TJ have a unique common fixed point z in X. Moreover, if the pairs (B, I) (T, J) (A, B), (A, I), (S, T) and (S, J) commute at the fixed point z then z remains the unique common fixed point of A, B, S, T, I and J separately.

Proof. We begin by noting that the coutinuity of BI (resp. TJ) does not demand the continuity of B or I or both *(resp. T or J or both)*. But for maps A, S, BI and TJ all the conditions of Theorem 2.1 are satisfied ensuring the existence of unique common fixed point z of A, S, BI and TJ. Here it is worth noting that z is the common fixed point of borth pairs (A, BI) and (S, TJ) respectively.

Now it remains to show that z is also a common fixed point A, B, S, T, I and J. For this let z is the unique common fixed point of both the pairs (A, BI) and (S, TJ), then

$$Bz = B(BIz) = B(IBz) = BI(Bz), \qquad Bz = B(Az) = A(Bz),$$
$$Iz = I(BIz) = IB(Iz) = BI(Iz), \qquad Iz = I(Az) = A(Iz),$$

$$Tz = T(TJz) = T(JTz) = TJ(Tz), \qquad Tz = T(Sz) = S(Tz),$$
$$Jz = J(TJz) = JT(Jz) = TJ(Bz), \qquad Jz = J(Sz) = S(Jz),$$

which shows that Bz and Iz (*resp.* Tz and Jz) are other fixed points of the pair (A, BI) and (*resp.* S, TJ). Now in view of the uniqueness of common fixed point of the pairs (A, BI) and (S, TJ), we get

$$Z = Bz = Iz = Tz = Jz = BIz = TJz= =Az = Sz,$$

which shows that z also remains the common fixed point of A,B,S,T, I and J separately. This completes the proof.

Remark 2.2. By restricting $A,B,S,T,I,$ and J suitably and modifying the remaining hypothesis accordingly, one can derive a multitude of known and unknown fixed point theorems. So far we are not familiar of any fixed point theorem involving five or six mappings in compact metric spaces.

As an application of Theorem 2.2, we prove the following fixed point theorem (employing the notion of best approximation) which generalizes earlier results due to Brosowski[1], Hicks-Humphries[4], Singh[10], Sahab et al.[9] and others.

Theorem 2.3. Let A,B,S,T,I and J be self-mappings of a normed space X and C be a subset of X such that $A, S, : \delta C \to C$ with $\bar{x} \in F(A, B, S, T, I, J)$ if A, B, S, T, I and J satisfy the condition

$\|Ax - Sy\| < M(x, y)$, with A ans S bcing continuous where,

$$M(x, y) = \max \{\|BIx - TJy\|, \|BIx - Ax\|, \|TJy - Sy\|, \tfrac{1}{2}[\|BIx - Sy\| + \|TJy - Ax)\|]\},$$

for all $x, y \in D' = D \cup C \{\bar{x}\}$.

Futher suppose that the pairs (A, BI) and (S, TJ) are compatible with BI and TJ being linear and continuous on D. If D be a nonempty, compact and starshaped with respect to a point $q \in D$ and $BI(D) = D = TJ(D)$ then $\quad D \cap F(A, B, S, T, I, J) \neq \phi$,

provided the pairs (B, I), (T, J), (A, B), (S, TJ), (A, I) and (S, J) commute at the common fixed point of BI, TJ, A and S.

Proof : Let $y \in D$ then $BIy \in D$ as $BI(D) = D$ Also if $y \in \delta C$ then $Ay \in C$ as $A(\delta C) \subset C$

Using condition (2.5.1), we obtain

$$\|Ay - \bar{x}\| = \|Ay - S\bar{x}\| < M(y, \bar{x}),$$

giving thereby $Ay \in D$. Thus A is a self-mapping of D. Similarly S is also a self-mapping of D.

Let $\{t_n\}$ be a sequence of real nunbers such that $0 \leq t < 1$ and converging to 't'. We define sequences $\{A_n\}$ and of $\{S_n\}$ mapping by

$$A_n x = t_n\, Ax + (1 - t_n)\, q$$
$$0S_n x = t_n\, Sx + (1 - t_n)\, q$$

for all $x \in D$ and for each n.

Since D is starshaped with respect to q hence $\{A_n\}$ maps D into itself and so does $\{S_n\}$ also. Since I is linear, one can have

$$(A_n)\, Ix_n = t_n\, (AIx_n) + (1 - t_n)\, Iq,$$

and

$$I(A_n)\, x_n = t_n\, (IAx_n) + I\, (1 - t_n)\, q.$$

Since (A, BI) are compatible

$$0 \le \lim_{n\to\infty} \|(BI)\, A_n x_n - A_n (BI) x_n\|$$
$$\le \lim_{n\to\infty} \|(BI)\, Ax_n - A(BI)x_n\| + \lim_{n\to\infty} (1 - t_n)\, \|q - Aq\| = 0$$

whenever, $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} BIx_n = t \in D$ for all n

Hence (BI, A_n) on D. Similarly it can be shown that (TJ, S_n) are compatible on D.

Futher from

$$\|A_n x - S_n y\| = t_n \|Ax - Sy\| < t_n M(x, y) < M(x, y),$$

for all x,y$\in D$. Since I and J are continuous ans D is compact, therefore by Theorem 2.2.

$$F(A_n) \cap F(BI) \cap F(S_n) \cap F(TJ) = \{X_n\},$$

for each n Also since D is compact so $\{x_n\}$ has a convergent subseqence $\{x_{n_i}\}$ converging to z in D.

Now $x_{n_i} = A_{n_i} x_{n_i} = t_{n_i}\, Ax_{n_i} + (1 + t_{n_i})\, q$,

which on letting $n \to \infty$ reduces to $Az = z$ giving thereby $z \in D \cap F(A)$. Similarly, it can be shown that $z \in D \cap F(S)$. Since BI and Tj are continuous, we have

$$BIz = BI \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} BIx_{n_i} = \lim_{i\to\infty} x_{n_i} = z$$
$$TJz = TJ \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} TJ\, x_{n_i} = \lim_{i\to\infty} x_{n_i} = z$$

yielding thereby $BIz + TJz = Az = Sz = z$.

Let the pairs (A, BI) and (S, TJ) have different fixed point u and v respectively, then $\|u - v\| = \|Au - Sv\|$

$$< max\, \{\|BIu - TJv\|, \|Blu - Au\|\, TJv - Sv\|,$$
$$\tfrac{1}{2}\, [\|BIu - Sv\| + \|TJv - Au\|]\}$$

which is a contradiction, implying thereby $u = v$, thus both the pairs have same common unique fixed point $u = v = z$.

Now on the lines of the proof of Theorem 2.2, it can be easily shown that

z remains the unique common fixed point of A, B, S, T, I and J
Hence, we conclude that

$$D \cap F(A, B, S, T, I, J) \neq \phi .$$

This completes the proof.

Remarks 2.3

(i) Theorem 2.3 extends the result of Sahab et al[9] as we generalized contractions along with compatibility (cf [6]) instead of commutativity. Also Theorem 2.3 involves six mappings instead of two mappings. In Process related results due to Hicks-Humphries[4], Singh [10], Brosowski[1] and others are modified and improved either partially or completely.

(ii) If we use a fixed point theorem in complete spaces corresponding to Theorem 2.2 then the continuity requirement of any one of the maps A,S,BI or TJ can serve the purpose which is possible due to the fact that compact metric spaces are always complete. But due to a shorter proof we opt to use Theorem.

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A COMMON FIXED POINT THEOREM ON METRIC SPACE

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ABSTRACT

In this paper we improve the result of Jungck[8] obtained in metric space under the commutativity condition. A common fixed point theorem for a pair of coincidently self-mapping on metric space satisfying the contraction conditions.

1. Introduction. Let (X, d) be a metric space. The mapping $F : X \to X$ and $g : X \to X$ are said to be commutative if $(fg)(x) = (gf)(x)$, for all $x \in X$.

Dhage, B.C. (1999), [1] stated the following result :

TheoremA. Let f and g be two self-mappings of a metric space X stisfying:

$$d(fx, fy) \leq max\, \{d(gx, gy),\ d(gx, fx),\ d(gy, fy),$$
$$1/2[d(gx, fy) + d(gy, fx)]\}$$ for all $x, y \in X$ and for some $0 \leq \lambda < 1$.

Futher, suppose that

(i) $f(X) \subseteq g(X)$,

(ii) any one of $f(X)$ and $g(X)$ is complete and

(iii) f, g are coincidently commuting

Then f and g have a unique common fixed point z and the sequence $\{y_n\} \subset X$ is defined by $y_{n+1} = fx_n = gx_{n+1},\ n \geq 0$ ...(1)

for some sequence $\{x_n\} \subset X,\ x_0 = x \in X$ coverges to z. Further, if g is continous at z then f is continous at z.

Our related results are as follows

Theorem 1. Let f and g be two self-mappings of a metric space X satisfying :

$$d(fx, fy) < \lambda \ max \{d(gx, gy), \tfrac{1}{2}[d(gx, fx) + d(gx, fy)$$
$$\tfrac{1}{2}\{d(gy, fy) + d(gy, fx)]\} \qquad \ldots(*)$$

for all $x, y \in X$ and for some $0 \leq \lambda < 1$. Further, suppose that all the conditions of theorem A hold. Then f and g have a unique common fixed point z and futher if the sequence $\{y_n\} \subset X$ is defined by

$y_{n+1} = fx_n = gx_{n+1}, \ n \geq 0$

for some sequence $\{x_n\} \subset X, \ x_0 = x \in X$ converges to z mapping g is continous at that point z then f is also continuous at z.

Proof . *Let* $x_0 \in X$, be arbitrary element. We define sequnce $\{x_n\}$ such that $fx_n = gx_{n+1}$. Futher let sequence $\{y_n\}$ be such that

$y_{n+1} = f(x_n) = g(x_{n+1})$.

Now, the two cases arise.

Case 1. If $y_n = y_{n+1}$, for some $n \subset N$, then we have

$$z = y_n = gx_n = fx_n = gx_{n+1} = y_{n+1} \ . \qquad \ldots(2)$$

Taking $u = x_n$, *then* $fu = gu$.

Again, using (iii) we have $fg(u) = gf(u)$

or $\quad gg(u) = fg(u) = gf(u)$.

Now, $d(z, fz) = d(fu, fz)$

$\leq \lambda \ max. \{d(gu, gz), \tfrac{1}{2}[d(gu, fu) + d(gu, fz)], \tfrac{1}{2}[d(gz, fz) + d(gz, fu)]\}$

$= \lambda \ max. \{d(z, fz), \tfrac{1}{2}[d(fu, fu) + d(fu, fu)], \tfrac{1}{2}[d(gu, fu) + d(gu, gu)]\}$

$= \lambda \ max. \{d(z, fz), \tfrac{1}{2}[0 + 0], \tfrac{1}{2}[0 + 0]\}$

$= \lambda \ d(z, fz)$,

which is a contradicition. This shows that $fz = z$ and z is a fixed point of f. Again $z = fz = fgu = gfu = gz$.

So z is also fixed point of g

Case II . Let $y_n \neq y_{n+1}$, for all $n \in N$

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} \ .$$

As $z \in g(X)$, there exist a point $w \in X$ such that $z = gw$.

Using (*), we get

$d(fw, gw) \leq d(fw, fx_n) + d(fx_n, gw)$

$\leq \lambda \max \{d(gw, gx_n), \tfrac{1}{2}[d(gw, fw) + d(gw, fx_n)],$

$\tfrac{1}{2}[d(gx_n, fx_n) + d(gx_n, fw) + d(fx_n, gw)]\}$.

Taking $\lim\limits_{n\to\infty}$ we get

$d(fw, gw) \leq \lambda \max \{d(z, z), \tfrac{1}{2}[d(gw, fw) + d(gw, gw)],$

$\tfrac{1}{2}[d(z, z) + d(gw, fw)]\} + d(z, z)]\}$

$= \lambda \ . \ \max. \{0, \tfrac{1}{2}d(fw, gw), 0\} + 0$

$= \lambda \ . \ \tfrac{1}{2} d(fw, gw),$

which is a contradiction. This shows that $fw = gw$, Similarly if $f(X)$ is complete, then there is a point $z \in f(x)$ staisfying (1).

As $f(X) \subseteq g(X)$, there is a point $w \in X$ such that $z = gw$ and as shown above $fw = gw$. Also it is shown in the proof of case I that f and g have common fixed point z.

Again, for uniqueness,

let x be an other fixed point of f and g, then by condition (*), we have

$$D(x, z) = d\,(f(x), f(z))$$

$$\leq \lambda\, max.\ \{ d\,(gx, gz),\ ½[(d\,(gx, fx)],\ + d(gx, fz),\ ½[d\,(gz, fz) + d\,(gz, fx)]\}$$

$$= \lambda\, max.\ \{d\,(x, z),\ ½\{d\,(x,x) + d\,(x, z),\ ½[d\,(z, z) + d\,(z, x)]\}$$

$$= \lambda\, max.\ \{d\,(x, z),\ ½[0 + d\,(x, z)], ½[0 + d\,(z, x)]\}$$

$$= \lambda\, max.\ \{d\,(x, z),\ ½\,[d\,(x, z)],\ ½\,[d\,(z, x)]\}$$

$= \lambda\, d\,(x, z)$, which is a contradicition. This shows that $d\,(x, z)=0$ $\Leftrightarrow$ $x = z$. Thus z is a unique fixed point of f as well as of g.

Now, for the **proof of contiunity of *f***

we have, $f\,(X) \subseteq g\,(X)$ and g is continuous.

Also, $y_{n+1} = f\,z_{n+1} = gz_{n+1}$, $\quad n \geq 0$

$$\Rightarrow \quad \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_{n+1} = gz = z = fz$$

$$\Rightarrow \quad \lim_{n\to\infty} fz_n = fz$$

$\Rightarrow$ f is contiunous at z in X.

Theorem 2 . Let f and g be two self-mappings of a metric space X satisfying

$$d(f^p x, f^p y) \leq \lambda \text{ max. } \{d\,(g^q x, g^q y),\ ½\,[d\,(g^q x, f^p x) + d\,(g^q x, f^p y),$$
$$½[d\,(g^q y, f^p y) + d\,(g^q y, f^p x)]\} \qquad \ldots(**)$$

for all $x, y \in X$ and for some $0 \leq \lambda < 1$, where p and q are positive integers. Further, suppose that

(i) $f^p(X) \subseteq g^q(X)$,

(ii) any one of $f^p(X)$ and $g^q(X)$ is complete and

(iii) f, g are coincidently commuting.

Then f and g have a unique common fixed point z and the sequence $\{y_n\} \subset X$ is defined by

$$y_{n+1} = f^p x_n = g^q x_{n+1},\ \ n \geq 0 \qquad \ldots(1)$$

for some sequence $\{x_n\} \subset X$, $x_0 = x \in X$ coverges to z. Further, if g^q is continous at z, then f^p is also continous at z.

Proof . Same as of theorem.

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(Dedicated to the memory of Professor J.N. Kapur)

ANOTHER COMMON FIXED POINT THEOREM INVOLVING BEST APPROXIMATION

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Introduction. The concept of fuzzy sets was introduced initially by Zadeh [38] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications Especially, Deng[9], Erceg [11], kaleva and Seikkala [23], Kramosil and Michalek[24] have introduced the concept of fuzzy metric space in different ways. Grabiec[14], followed Kramosil and Michalek [24] and obtained the fuzzy version of Banach contraction principle Moreover, it appears that the study of Kramosil and Michalek[24] of fuzzy metric spaces paves the way for developing a smoothing machinery in the field of fixed point theorems, in particular for the study of contractive type maps. Fang[12] proved some fixed point theorems in fuzzy metric spaces, which improve generalize, unify and extend some main results of Banach[2], Edelstein[10], lstratescu[17], Sehgal and Bharucha-Reid [29]. Sessa[30] defined a generalization of commutativity, which is called weak commutativity. Further Jungck[21] introduced more generalized commutativity, so called compatibility. Mishra et. al.[26] obtained common fixed point theorems for compatible maps on fuzzy metric spaces. Recently Jungck et al.[22] introduced the concept of compatible mappings of type (α) metric spaces, which is equivalent to the concept of compatible mappings under some conditions and proved common fixed point theorems in metric spaces Cho[8] introduced the concept of compatible mappings of type (α) in fuzzy metric spaces.

Many authors have studied the fixed point theory in fuzzy metric spaces. The Most interesting references in this direction are Badard [1]. Chang[6], Fang[12], Grabiec[14], Hadyic[15] Jung, Cho, Chang and Kang[18], Jung, Cho and Kim[19], Mishra and Sharma [26] Sharma[32], [33], [34], [35], [36] and fuzzy mappings. Bose and Sahani[7], Butnariu[4], Chang[5], Chang, Cho, Lee, and Lee [7], Lee, Cho and Jung[25].

Recently, George and Veeramani[13] modified the concept of fuzzy metric space introduced by Kramosil and Michalek[24] and defined Hausdorff topology on the fuzzy metric spaces. They showed also that every metric induces a fuzzy metric.

In this paper, we prove common fixed point theorems for six mappings satisfying some conditions in fuzzy 2 and 3 metric spaces in the sense of Kramosil and Michalek[24], Our main theorem extend, generalize and fuzzify some known results in fuzzy metric spaces, probabilistic metric spaces and uniform spaces. Cho[8], Grabiec[14], Hadzyic[16]. Jungck[20], Jungck, Murthy and Cho[22], Mishra and Sharma[26], Rhoades[27] Sessa, Rhoades and Khan[31], Sharma[32, 33, 34, 35, 36], Singh and Pant[37]. We also give an example to illustrate our main theorem.

2. Preliminaries

Definition 2.1. A binary operation $* : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that

$a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$, whenever $a_1 \leq a_2, b_1 \leq b_2$, $c_1 \leq c_2$, for all a_1, a_2, b_1, b_2 and c_1, c_2, are in $[0,1]$.

Definition 2.2 . The 3-tuples $(X, M, *)$ is called a fuzzy 2-meric space if X is an arbitrary set, $*$ is continous t-norm and M is a Fuzzy set in$X^3 \times [0,\infty]$ satisfying the following conditions for all $x,y,z,u \in X$, and $t_1, t_2, t_3 > 0$:

(FM-1) $M(x, y, z, 0) = 0$,

(FM-2) $M(x, y, z, t)\ 1$, $t > 0$ and when at least two of the three points are equal

(FM-3) $M(x, y, z, t) = M(x, z, y, t) = M(y,z,x, t)$,
(Symmetry about three variables)

(FM-4) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$

(This corresponds to tetrahedron inequality in 2-metric space)

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.

(FM-5) $M(x, y, z, *) : [0,1] \rightarrow [0,1]$ is left continuous.

Definition 2.3. Let $(X, M,*)$ is a fuzzy 2-metric space.

(1) A sequence (x_n) in a fuzzy 2-metric space X is said to be convergent to a point $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, a, t) = 1 \text{ for all } a \in X \text{ and } t > 0.$$

(2) A sequence $\{X_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1 \text{ for all } a \in X \text{ and } t < 0,\ p>0.$$

(3) A fuzzy 2-metric space in which every Cauchy-sequence is convergent is said to be complete.

Remark 2.1. Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in fuzzy 2-metric space is uniquely determined. Let $(X, M, *)$ be a fuzzy 2-metric space with the following condition.

(FM-6) $\lim_{n\to\infty} M(x, y, a, t) = 1$, for all $x, y, a \in X$

Example 2.1. (George and Veeramani [13])

Let (X, d) be a 2-metric space, define $a * b = a\,b$ or ($a * b = \min\{a, b\}$) and for all x,

$$y \in X \text{ and } t > 0,\ M(x, y, a, t) = \frac{t}{t + d(x, y, a)} \quad ...(1.a)$$

Then $(X, M *)$ is a fuzzy metric space, we call this fuzzy metric M, induced by the metric d the standrad fuzzy metric. On the other hand, note that there exists no metric on X Satisfying (1. a).

Lemma 2.1. (Grabiec[14]). For all $x, y \in X$, $M(x, y, z,*)$ be non - decreasing.

Lemma 2.2. (Cho[8] and Mishra et al.[26]). Let $\{y_n\}$ be a sequence in a fuzzy 2-metric space $(X, M,*)$ with the condition (FM-6). If there exists a number $k \in (0, 1)$ such that

$$M(y_{n+2}, y_{n+1}, a, kt) \geq M(y_{n+1}, y_n, a, t) \quad ...(1.b)$$

for all $a \in X$ and $t > 0$ and $n = 1,2,\ldots$, then $\{y_n\}$ is a Cauchy sequence in X.

Proof. By simple induction with the condition (1. b), we have, for all $t > 0$ and $n=1,2,\ldots$

$$M(y_{n+1}, y_{n+2}, a, t) \geq M(y_1, y_2, a, t/k^n) \quad ...(1.c)$$

Thus by (I. c) and (FM-4), for any positive integer p and real number $t >$

0, we have

$$M(y_n, y_{n+p}, a, t) \geq M(y_n, y_{n+1}, a, t/p) * M(y_{n+1}, y_{n+2}, a, t/p) * \ldots$$

$$\ldots p\text{-}times \ldots * M(y_{n+p-1}, y_{n+p}, a, t/p)$$

$$\geq M(y_1, y_2, a, \frac{t}{pk^{n-1}}) * M(y_1, y_2, a, \frac{t}{pk^{n}}) * \ldots$$

$$\ldots p\text{-}times \ldots * M(y_1, y_2, a, \frac{t}{pk^{n+p-2}}).$$

Therefore, by (FM -6), we have

$$\lim_{n \to \infty} M(y_n, y_{n+p}, a, t,) \geq 1 * 1 * \ldots p\text{-}times \ldots * 1 \geq 1,$$

which implies that $\{y_n\}$ is a Cauchy sequence in X. This completes the proof.

3. Compatible Mappings of type (α). In this Section, we give the concept of compatible mappings of type (α) in fuzzy 2-metric spaces and some properties of these mappings for our main results.

Definition 3.1. (Mishra et al.[26]). Let A and B be mappings from a fuzzy 2-metric spacc $(X, M, *)$ into itself. The mappings A and B are said to be compatible if

$$\lim_{n \to \infty} M(ABx_n, BAx_n, a, t) = 1$$

for all $a \in X$ and $t > 0$, whereever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Ax_n = \lim Bx_n = z, \qquad \text{for some } z \in X$$

Definition 3.2. (Cho[8]) Let A and B be mappings from a fuzzy 2-metric space $(X, M, *)$ into itself. The mappings A and B are said to be compatible of type (α) if

$$\lim_{n \to \infty} M(ABx_n, BBx_n, a, t) = 1$$

$$\text{and} \quad \lim_{n \to \infty} M(BAx_n, AAx_n, a, t) = 1,$$

for all $a \in X$ and $t > 0$, whenever $\{x_n\}$ is a sequence in X such that,

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z, \quad \text{for some } z \in X.$$

Remark 3.1. In papers[21],[22], we can find the equivalent formulations of definitiois 3.1 and 3.2 and their examples in metric spaces. Such mappings are independent of each other and more general than commuting and weakly commuting mappings[20],[30].

Proposition 3.1. (Cho[8]). Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and A and B be continuous mappings from X into itself Then A and B are compatible if and only if they are compatible of type (α).

Proposition 3.2 (Cho[8]) Let $(X, M, *)$ be a fuzzy metric space with $t * t$

$\geq t$ for all $t \in [0,1]$ and A and B be mappings from X into itself. If A and B are compatible of type (α) and $Az = Bz$ for some $z \in X$, then

$$ABz = BBz = BAz = AAz.$$

Proposition 3.3. (Cho[8]) Let $(X,M,*)$ be a fuzzy metric space with $* \; t \geq t$ for all $t \in [0,1]$ and A and B be mappings from X into itself. If A and B are compatible of type (α) and $\{x_n\}$ is a sequence in X such that

(1) $\lim\limits_{n \to \infty} Ax_n = \lim Bx_n = z$ *for some* $z \in X$, *then*

(2) $\lim\limits_{n \to \infty} BAx_n = Az$ if A is continuous at z,

(3) $ABz = BAz$ and $Az = Bz$, if A and B are continuous at z.

Sharma [33] proved the following theorem for fuzzy metric space:

Theorem A. Let $(X, M,*)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and the condition (FM-6). Let A,B,S,T,P and Q be mappings from X into itself' such that

(i) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$

(ii) $AB = BA$, $ST = TS$, $PB = BP$, $QS = SQ$, $QT = TQ$,

(iii) A, B, S, and T are continuous,

(iv) The pairs (P, AB) and (Q, ST) are compatible of type (α),

(v) There exists a number $k \in (0, 1)$, such that

$$M(Px, Qy, kt) \geq M(ABx, Px, t) * M(Sty, Qy, t) * M(STy, Px, Bt) * M(ABx, Qy, (2-\beta)t) * M(ABx, STy, t).$$

for all $x, y, \in X$, $\beta \in (0,2)$ *and* $t > 0$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

In this paper we prove Theroem A for fuzzy 2-metric space. We prove the following

Theorem 4.1. Let $(X, M,*)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and the condition (FM-6). Let A,B,S,T,P and Q be mappings from X into itself' such that

(4.1) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$,

(4.2) $AB = BA$, $ST = TS$, $PB = BP$, $QS = SQ$, $QT = TQ$,

(4.3) A, B, S and T are continuous,

(4.4) The pairs (P, AB) and (Q, ST) are compatible of type (α),

(4.5) There exists a number $k \in (0, 1)$, such that

$$M(Px, Qy, a, kt) \geq M(ABx, Px, a, t) * M(Sty, Qy, a, t) \; M(STy, Px, a, t) * M(ABx, STy, a, t),$$

for all $x, y \in X$, $\beta \in (0,2)$ *and* $t > 0$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. By (4.1), since $P(X) \subset AB(X)$, for any $x_0 \in X$ there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$.

Since $Q(X) \subset ST(X)$, for this point $\{x_1\}$ we can choose a point $x_2 \in X$ such that $Qx_1 = STx_2$. Inductively, we can find a sequence $\{y_n\}$ in X as follows

$$y_{2n} = Px_{2n} = ABx_{2n+1}$$

and $\quad y_{n+1} = Qx_{2n+1} = STx_{2n+2} \qquad$ for all $n = 0, 1, 2, \ldots$

By (4.5), for all $t > 0$, we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, a, kt) = {} & M(Px_{2n+1}, Qx_{2n+2}, a, x\, t) \\ \geq {} & M(ABx_{2n+1}, Px_{2n+1}, a, t) * M(STx_{2n+2}, Qx_{2n+2}\, a, t) \\ & * M(STx_{2n+2}, Px_{2n+1}, a, t) * M(ABx_{2n+1}, STx_{2n+2}, a, t) \\ = {} & M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) \\ & * M(y_{2n+1}, y_{2n+1}, a, t) * M(y_{2n}, y_{2n+1}, a, t) \\ \geq {} & M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t). \qquad \ldots(1.d) \end{aligned}$$

Similarly we have also

$$M(y_{2n+2}, y_{2n+3}, a, kt) \geq (y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n+2}, y_{2n+3}, a, t) \qquad \ldots(1.e)$$

Thus from (1.d) and (1.e) it follows that

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t)$ *for* $n = 1, 2 \ldots$

and so for positive integers n, p

$M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t/k^p)$.

Thus, since $M(y_{n+1}, y_{n+2}, a, t/k^p) \to 1$ *as* $p \to \infty$, we have

$$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_{n+1}, y_{n+1}, a, t).$$

By Lemma 2.2, $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete $\{y_n\}$ converges to a point $z \in X$.

Since $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are sub-sequences of $\{y_n\}$, therefore they also converges to the point z that is as $n \to \infty$,

$$Px_{2n},\ Qx_{2n+1},\ STx_{2n+2} \text{ and } ABx_{2n+1} \to z.$$

Since A and B are continuous and the pair $\{P, AB\}$ is compatible mappings of type (α), by Proposition 3.3 (1), we have as $n \to \infty$

$$P(AB)x_{2n+1} \to ABz \text{ and } (AB)^2 x_{2n+1} \to ABz.$$

Similarly, Since S and T are continuous and pair $\{Q, ST\}$ is compatible mappings of type (α), by Proposition 3.3 - (1), We have as $n \to \infty$

$$Q(ST)x_{2n+2} \to STz \text{ and } (ST)^2 x_{2n+2}\ STz.$$

By putting $x = (AB)x_{2n+1}$ and $y = x_{2n+2}$ in (4.5), we have

$$\begin{aligned} M(P(AB)x_{2n+1}, Qx_{2n+2}, a, kt) \geq {} & M((AB)^2 x_{2n+1}, P(AB)x_{2n+1}, a, t) \\ & * M(STx_{2n+2}, Qx_{2n+2}, a, t) \end{aligned}$$

$$* M(STx_{2n+2}, P(AB)x_{2n+2}, a, t)$$
$$* M((AB)^2x_{2n+1}, STx_{2n+1}, a, t)$$

which implies that as $n \to \infty$

$$M(ABz, z, a, kt) \geq 1 * 1 * M(Z, ABz, a, t) * M(ABz, z, a, t)$$
$$\geq M(ABz, z, a, t)$$

Therefore, by Lemma 2.3, we have $ABz = z$

By Putting $x = Px_{2n+1}$ and $y = x_{2n+1}$ in (4.5), we have

$$M(Px_{2n}, Qx_{2n+1}, a, kt) \quad \geq \quad M((AB)^2(Px_{2n}), P(Px_{2n})\, a, t)$$
$$* M(STx_{2n+2}, Qx_{2n+1}, a, t)$$
$$* M(STx_{2n+1}, P(Px_{2n}), a, t)$$
$$* M((AB(Px_{2n})\, STx_{2n+1}, a, t).$$

Taking the limit $n \to \infty$,we have

$$M(Pz, z, a, kt) \geq \quad M(z, Pz, a, t) * M(z, z, a, t)$$
$$* M(z, Pz, a, t) * M(z, z, a, t)$$
$$\geq \quad M(Pz, z, a, t).$$

By Lemma 2.3, we have $Pz = z$. Therefore $ABz = z = Pz$.

Now we show that $Bz = z$.

By putting $x = Bz$ and $y = x_{2n+1}$ in (4.5), and using (4.2), we have

$$M(P(Bz), Qx_{2n+1}, a, kt) \geq M(AB(Bz), P(Bz), a, t)$$
$$* M(STx_{2n+1}, Qx_{2n+1}, a, t)$$
$$* M(STx_{2n+1}, P(Bz)\, a, t)$$
$$* M(AB(Bz), STx_{2n+1}, a, t)$$

which implies that, as $n \to \infty$

$$M(Bz, z, a, kt) \quad \geq \quad M(Bz, Bz, a, t) * M(z, z, a, t)$$
$$* M(z, Bz, a, t) * M(Bz, z, a, t)$$
$$= \quad 1 * 1 * M(z, Bz, a, t) * M(Bz, z, a, t)$$
$$* M(Bz, z, a, t).$$

Therefore, by Lemma 2.3, we have $Bz = z$.

Since $ABz = z$, therefore $Az = z$.

By putting $x = z$ and $y = STx_{2n+2}$ in (4.5), we have

$$M(Pz, Q(St)x_{2n+2}, a, kt) \geq M(ABz, Pz, a, t) * M((ST)^2x_{2n+2}, Q(ST)x_{2n+2}, a, t).$$
$$M((ST)^2x_{2n+2}, Pz, a, t) * M(ABz\, (ST)^2x_{2n+2}, a, t).$$

Taking the limit $n \to \infty$, we have

$$M(Z, STz, a, kt) \geq \quad M(z, z, a, t)M(STz, STz, a, t)$$
$$* M(STz, z, a, t) * M(z, STz, a, t)$$
$$= \quad 1 * 1 * M(STz, z, a, t) * M(z, STz, a, t)$$

$$\geq \quad M(z, STz, a, t).$$

By Lemma 2.3, this implies that $STz = z$.

Now by putting $x = z$ and $y = Qx_{2n+1}$ in (4.5) and using (4.2), we have

$$M(Pz, Q(Qx_{2n+1}), a, kt) \geq M(ABz, Pz, a, t) * M(ST(Qx_{2n+1}), Q(Qx_{2n+1}), a, t)$$
$$* M(ST(Qx_{2n+1}), Pz, a, t) * M(ABz, ST(Qx_{2n+1}), a, t).$$

Taking the limit $n \to \infty$,we have

$$\begin{aligned} M(z, Qz, a, kt) &\geq M(z, z, a, t) * M(z,z,a,t) * M(z,z,a,t) \\ &= 1 * M(z, Qz, a, t) * 1 * 1 \\ &\geq M(z, Qz, a, t). \end{aligned}$$

Therefore, by Lemma 2.3, this implies that $Qz = z$ and hence $STz = z\, Qz$.

Finally we show that $Tz = z$.

By putting x = z and $y = Tz$ in (4.5) and using (4.2), we have

$$\begin{aligned} M(Pz. Q(Tz), a, kt) \geq M(ABz, Pz, a, t) &* M(ST(Tz), Q(Tz), a, t) \\ &* M(ST(Tz), Pz, a,t) * M(ABz, ST(Tz), a, t) \\ &= M(z,z,a,t) * M(Tz, Tz, a, t) \\ &* M(Tz, z, a, t) * M(z, Tz, a, t). \end{aligned}$$

Therefore $M(z, Tz, a, kt) \geq$ Since $STz = z$, therefore $Sz = z$.

By combining the above results, we have

$Az = Bz = Sz = Tz = Pz = Qz = z$,

which shows that, z is a common fixed point of A, B, S, T, P and Q.

For uniqueness, let w $(w \neq z)$ be another common fixed of A,B,S,T,P and Q, then by (4.5), we have

$$\begin{aligned} M(Pz, Qw, a, kt) &\geq M(ABz, Pz, a, t) * M(STw, Qw, a, t) \\ &* M(STw, Pz, a,t) * M(ABz, STw, a, t) \\ &= M(z,z,a,t) * M(w, w, a, t) * M(w, z, a, t) * M(z, w, a, t) \\ &= 1 * 1 * M(w, z, a, t) * M(z, w, a, t). \end{aligned}$$

This yields

$$M(z, w, a, kt) \geq M(z, w, a, t).$$

Therefore by using Lemma 2.3, we have $z = w$. This completes the proof.

Example 4.1. Let the set $X = [0, 1]$ with the metric d defined by

$$d(x, y, a) = |x\text{-}y, a| \text{ and for each } t \in [0,1].$$

we define

$$M(x, y, a, t) = \frac{t}{t + |x\text{-}y, a|},$$

$$M(x, y, a, 0) = 0, \qquad \text{for all } a, x, y, \in X.$$

Clearly $(X, M, *)$ is a complete fuzzy metric space, where $*$ is defined by

$a * b = ab.$

Let A, B, S, T, P and Q be defined as

$Ax = x,\ Bx = x/2,\ Sx = x/5.$

$Tx = x/3,\ Px = x/6$ and $Qx = 0$, for all $x \in X$.

Then $P(X) = [0, 1/6] \subset [0, 1/2] = AB(X)$ and

$Q(X) = \{0\} \subset [0, 1/15] = ST(X)$.

If we have $x = 1/2,\ t = 1$, we see that condition (4.5) of Theorem 3.4.1 is satisfied. Clearly conditions (4.2) and (4.3) of Theorem 3.4.1 are also satisfied.

Further the pair $\{P, AB\}$ is compatible mappings of type (α),

If $\lim_{n\to\infty} x_n = 0$ where $\{x_n\}$ is a sequence in X, such that

$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = 0$, for some $0 \in X$.

Similarly, the pair $\{Q, ST\}$ is also compatible mappings of type (α).

Thus all the conditons of Theorem 4.1, are satisfied and 0 is the unique common fixed point of A, B, S, T, P and Q.

We give the following metric version of the condition (4.5).

For self mappings A, B, S, T, P, and Q of a metric space (X, d) there exists a number $k \in (0,1)$, such that.

$$d(Px, Qy, a) < k \max \{d(ABx, Px, a, d(STy, Qy, a)$$
$$\tfrac{1}{2}(d(ABx, Qy, a) + d(STy, PX, a)), d(ABx, STy, a)\} \quad ...(1.f)$$

for all $x, y, \in X$.

If we put $P = Q$ in (1 . f), we have

$$d(Px, Py, a) \le k \max \{d(ABx, Px, a), d(STy, Py, a)$$
$$\tfrac{1}{2}(d(ABx, Py, a) + d(STy, PX, a)), d(ABx, STy, a)\} \quad ...(1.g)$$

for all $x, y \in X$ and $k \in (0,1)$.

If we put $B{=}T{=}IX$ and $A = B = S = T = Ix$ in (1.g) respectivley, we have

$$d(Px, Py, a) \le k \max \{d(Ax, Px, a), d(Sy, Py, a)$$
$$\tfrac{1}{2}(d(Ax, Py, a) + d(Sy, Px, a)), d(Ax, Sy, a)\} \quad ...(1.h)$$

and
$$d(Px, Py, a) \le k \max \{d(x, Px, a), d(y, Py, a),$$
$$\tfrac{1}{2}(d(x, Py, a) + d(y, Px, a)), d(x, y,)\} \quad ...(1.i)$$

for all $x, y \in X$ where $k \in (0,1)$.

We can also find many fixed point theorems for three or four or five mappings on metric spaces satisfying the conditions (1.g),(1.h) and (1.i)

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(Dedicated to the memory of Professor J.N. Kapur)

$2\omega_p$ EMISSIONS BY NONLINEAR MIXING OF PLASMA AND ION-ACOUSTIC WAVES

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ABSTRACT

A large amplitude plasma wave $(\omega_p, \vec{k}_0)$ propagating through a plasma produces oscillatary electron velocity at the harmonics of ω_p. The electron velocity at second harmonic couples with the density oscillations of an ion acoustic wave $(\omega_s, \vec{k})$ to produce a nonlinear current at $(2\omega_p - \omega_s w \approx 2\omega_p)$, that produces $2\omega_p$, emissions. The process is sensitive to power of the plasma wave.

1. Introduction. Wave mixing and harmonic generation are the areas of intense research in plasma physics during past few decades [3, 7, 8, 11]. Wave mixing has applications in image processing, dispersion correction, tracking and pointing, optical computing and charged particle acceleration. Harmonic generation has applications in wavelength conversion, generation of coherent *XUV* and soft *X*-rays etc. Both of these processes also serves as diagnostics and much of the knowledge about plasma can be acquired [4]. These techniques are very attractive since they render possible the study of different processes also in cases when a local measurement in situ in the plasma is not possible for various reasons. Both of these processes involves excitation of natural plasma modes. Two such modes are plasma or Langmuir wave and ion-acoustic wave. Plasma wave is the high frequency mode obeying the dispersion relation $\omega^2 = \omega_p^2 + k^2 v_{th}^2$ and v_{th} are the plasma frequency and electron thermal velocity respectively. Ion acoustic wave is the low frequency mode with dispersion relation $\omega = kc_s$, c_s being the ion sound speed [2]. Second harmonic generation by an electromagnetic wave is often interpreted as a two-step process[1]. In the first step some electron density oscillations at a frequency equal to or very close to the fundamental frequency are produced. In the second step the incident fundamental is scattered from the electron density fluctuations or two electron plasma waves near fundamental frequency interact to produce second harmonic. Alternatively electron plasma waves can also be excited by parametric decay instability in which the incident electromagnetic wave decay into electron plasma wave and ion acoustic wave. The nonlinearity of ion acoustic wave may play an important role in the development of parametric instabilities leading to the excitation of ion acoustic waves. In particular, harmonic generation by a strong ion acoustic wave results in an increase of its effective dissipation and thus instability saturation. In various space and astrophysical plasmas, signatures if harmonic emissions have been reported. One of the possible scenario may be due to plasma wave coalescence [5,9]. In this paper we study generation of electromagnetic waves at $2\omega_p$ via mixing of plasma and ion acoustic waves. The electron velocity at $2\omega_p$, due to plasma wave couples with the density oscillations due to ion acoustics wave to produce nonlinear current at $2\omega_{p,}$ which results in $2\omega_{p,}$ emissions. In section we obtain electric field at $2\omega_{p,}$ and in section 3 we discuss our results.

2. Second Harmonic Field. Consider the propagation of a large amplitude Langmuir wave,

$$\phi_p = A_0 e^{-i(\omega_0 t - \vec{k}_0 \vec{r})} \tag{1}$$

and an ion-acoustic wave

$$\phi = A e^{-i(\omega t - \vec{k}.\vec{r})} \tag{2}$$

through a plasma of electron density n_0^0 and temperature Te. Here

$k_0 = \sqrt{\omega_0^2 - \omega_p^2} / \upsilon_{th}$, $k = \omega / c_s$, $\upsilon_{th} = k_B T_e / m_e$, $c_s = k_B T_e / m_i$, k_B is Boltzman constant,

$$\omega_p = \sqrt{4\pi n_0^0 - e^2 / m_e},$$

e and m_e being electronic charge and mass respectively and m_i is ion mass.

On solving equation of motion $m(d\vec{\upsilon}/dt) = -e\vec{E}$, electron velocity due to Langmuir wave, can be written as

$$\vec{\upsilon}_0 = \frac{e\vec{E}_0}{mi\omega_0} \quad , \vec{E}_0 = i\vec{k}_0 \phi_0 . \tag{3}$$

On integrating Eq. (3) we obtain,

$$\vec{r} = \vec{r}_0 - e\vec{E}_0 / m\omega_0^2 , \tag{4}$$

where r_0 is the electron's position at $t = 0$. Using (4) in (1) and applying the identity

$e^{if \sin\theta} = \sum_n J_n(f) e^{in\theta}$, we get the plasma wave potential as

$$\phi_0 = \sum_n A_0 J_n(\alpha) e^{-i[(n+1)\omega_0 t - (n+1)\vec{k_0}.\vec{r}]} \tag{5}$$

where $\alpha = k_0 v_{osc} / \omega_0$.

For $n=1$, we obtain the potential of the plasma wave at second harmonic as,

$$\phi_2 \approx A_0 J_1(\alpha) e^{-i(2\omega_0 t - \vec{k}_0.\vec{r})} \tag{6}$$

Electron velocity at second harmonic due to ϕ_{02} and self consistent potential $\phi_2 = \phi_2 e^{-i(2\omega_0 t - \vec{k}_0.\vec{r})}$ after solving the equation of motion is

$$\vec{v}_{02} = \frac{e\vec{k}_0}{m\omega_0} (\phi_{02} + \phi_2). \tag{7}$$

Using (7) in the equation of continuity $\partial n / \partial t + \nabla .(n\vec{v}) = 0$, we obtain the electron density oscillations at $(2\omega_0, 2\vec{k}_2)$ as

$$n_{02} = \frac{n_0^0 \, \vec{k} \, \vec{v}_{02}}{\omega_0} \tag{8}$$

Using (8) in the Poisson's equation $\nabla^2\phi = 4\pi ne$, we get

$$\varepsilon_2 \phi_2 = \chi_{e2} \phi_{02} , \tag{9}$$

where $\chi_{e2} = -\omega_p^2 / 4\omega_0^2$, $\varepsilon_2 = 1 + \chi_{e2}$.

Using (9) in (7) we get,

$$\vec{v}_{02} = \frac{e\vec{k}_0 \phi_{02}}{m\omega_0 \varepsilon_2} . \tag{10}$$

Electron density perturbation due to ion acoustic wave is given by

$$n = n_0^0 \; e\,\phi \,/ T_e . \tag{11}$$

From (10) and (11) we obtain the nonlinear current density at $2\omega_0 - \omega \approx 2\omega_p$ as

$$\vec{J}^{NL} = -\, n^* \, e\vec{v}_{02} . \tag{12}$$

Using (12) in the wave equation [10], we get

$$\nabla^2 \vec{E}_2 - \nabla (\nabla . \vec{E}_2) = \frac{4\pi i\omega_p}{c^2} n^* e\vec{v}_{02} - \frac{4\omega_p^2}{c^2} \vec{E}_2 - \frac{8\pi i\omega_p}{c^2} \; \vec{J}_2^L , \tag{13}$$

where, $\vec{J}_2^L$ is the linear component of current density. Taking $\nabla.$ of (13) we get

$$\nabla . \vec{E}_2 = \frac{(2\vec{k}_0 - \vec{k})}{\omega_p} n^* e\vec{v}_{02} , \tag{14}$$

Using (14) back in (13), we get

$$\nabla^2 \vec{E}_2 + \frac{3\omega_p^2}{c^2}\vec{E}_2 - \frac{4\pi i\omega_p}{c^2}\overset{*}{n}e\vec{v}_{02} - \frac{(2\vec{k}_0-\vec{k})}{\omega_p}n^* e\,\vec{k}_0\,.\vec{v}_{02}. \tag{15}$$

Considering the variation of $\vec{E} = \overrightarrow{A}_2\,(z)e^{-i[(2\omega_0-\omega)t-(2\vec{k}_0-\vec{k}).\vec{r}]}$ and applying the condition $(2\vec{k}_0-\vec{k})^2c^2+\omega_p^2=(2\omega_0-\omega)^2$ in (15), we obtain

$$c^2\frac{\partial A_2}{\partial z} = \frac{2\pi\omega_p n^* e v_{02}}{(2k_{0z}-k_z)} + \frac{n^* e\,\vec{k}_0\,.\vec{v}_{02}c^2}{2\omega_p}, \tag{16}$$

$$A_2 = J_1(\alpha)\,\frac{2\omega^3{}_p\,v_{02}k_z}{c^2k_z^2\,v^2{}_{th}}\,\phi^*. \tag{17}$$

In Fig.(1) we have plotted the efficiency of the power coupling with the parameter α.

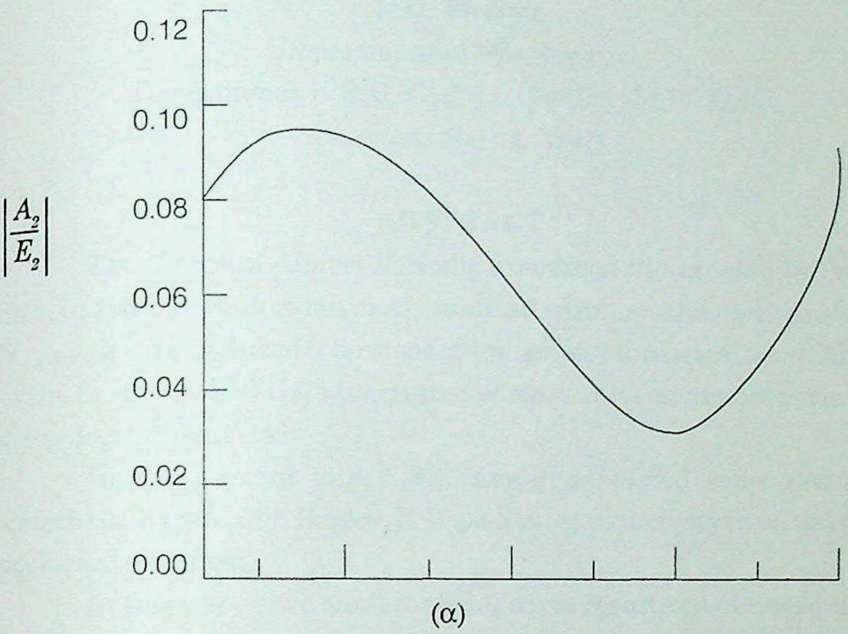
3. Results and discussions. The plasma wave on coupling with the ion acoustic wave can produce signatures of electromagnetic radiations at $2\omega_p$ frequencies. The efficiency of the process shows a peak around $\alpha = 2$ This may be one of the mechanisms observed in solar radio bursts. This mechanism could also be employed as a diagnostic tool in laboratory produced plasmas and also in the generation of high frequency waves. The scheme can be extended to study generation of higher harmonics also.

We can express Eq.(17) in terms of *I*-function of one variable[10], an extension of Mijer's *G*-function, Fox,s *H*-function of one variable.

$$A_2 = \frac{2\omega^3{}_p}{c^2k_z^2}\,\frac{v_{02}k_z}{v^2{}_{th}}\,\phi^* I^{1,0}{}_{0,2:1}\,\left[\frac{\alpha^2}{4}\middle|\left(\frac{1}{2},1\right),\left(-\frac{1}{2},1\right)\right]. \tag{18}$$

Figure Captions

Figure 1 : Effciency of the process variation with α.



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(Dedicated to the memory of Professor J.N. Kapur)

A NOTE ON ALMOST PERIODICITY

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ABSTRACT

The theory of Almost Periodic Functions was created by Bohr in 1924. In 1960 three Russian mathemaicians Antonovskii, Boltyanskii and Sarymsakov introduced a new space known as topological semifield and for the first time Iśeki[4] interpreted some results on almost periodicity on topological semifield.

In the present paper we have generalized some results of Gottschalk, Reddy, and Bagley, R.W on almost periodicity in context with topological semifield.

In this paper, we shall consider some results on almost periodic transformations in a metric space over a topological semi field [1].

Definition. If we consider a set F of uniformly continuous transformations defined on the real line, then a transformation f of F will be called almost periodic if to each positive number $\in$ there corresponds a positive integer L, so great that among each L successive positive integers there is an integer N satisfying

$$|f^N(x) - f(x)| < \in, \; -\infty < x < \infty.$$

Let X be a metric space over a topological semi field R. Let ρ be the metric defined on X. Let T be topological group and f be a mapping defined by

$$f : X \times T \to X \text{ such that } f(x,t) = xt \; (t \varepsilon T) \; for \; x \varepsilon X.$$

We write $f^t(x)$ in place of $f(x,t)$. We shall also suppose that the group $\{f^t : t \varepsilon T\}$ is a transformation group. We assume that T is generative and let S be a replete semi group [2] in T.

Definition. A subset B of T is said to be S syndetic [3] if there exists a compact subset K of T such that $S \subset BK$.

Definition. A transformation group $\{f^t : t \varepsilon T\}$ is said to be S-almost periodic if for each neighborhood *(nbd)* U of 0 in R and for each s in S, there exists a compact subset K of T such that either of the following conditions are satisfied :

i) $\rho\ (f^k(x), f^s(x))\ \varepsilon\ U$ for each $k \varepsilon K$, $x \varepsilon X$.

ii) $\rho\ (x, f^{ks}\ (x))\ \varepsilon\ U$ for each $k \varepsilon K$, $x \varepsilon X$.

Definition. The transformation group T is discretely almost periodic if and only if the transitive group $\{f^t : t \varepsilon T\}$ is totally bounded in its space index uniformity.

Theorem . Let X be a metric space over a topological semi field R. Then the following propositions are equivalent :

1) $\{f^t : t\varepsilon\ T\}$ is S-almost periodic,

2) $\{f^t : t\varepsilon\ T\}$ is discretely almost periodic,

3) The transformation group $\{f^s : s\varepsilon\ S\}$ is totally bounded by the natural topology of the metric space X over the topological semi field R.

Proof. (1)→ (2) : To prove $\{f^t : t\varepsilon\ T\}$ is discretely almost periodic, it is sufficient to show that $\{f^s : s\varepsilon\ S\}$ is totally bounded.

Let U be a *nbd of 0* in R. Take a nbd V of 0 in R such that $V+V \subset U$. Since $\{f^t : t\varepsilon\ T\}$ is S-almost periodic, there erists a compact subset K of T such that $\rho\ (f^k(x), f^s(s))\varepsilon\ U$ for some $k\varepsilon\ K$ and for each $s\varepsilon\ S\ (x\varepsilon X)$. Since the mapping $f : XxT \to X$ is uniformly continuous, therefore applying the theorem 5 of [4] for the compact set K, $\{f^k : k\ \varepsilon\ K\}$ is equi-continuous and is totally bounded also. Hence there exists a finite set P of K such that $\rho\ (f^p(x), f^k(x))\varepsilon V$ for some $p\varepsilon\ P$, $k\varepsilon\ K\ (x\varepsilon X)$. Thus for each $s\varepsilon\ S$, $k\varepsilon K$ and $p\varepsilon\ P$, we have

$$\rho\ (f^p(x), f^s(x)) << \rho\ (f^p(x), f^k(x)) + \rho\ (f^k(x), f^s(x))$$

$$\varepsilon\ \ V + V \subset U.$$

which shows that $\{f^t : t\varepsilon\ T\}$ is discretely S-almost periodic.

(2) → (3) : It is obvious by the definition of discretely S-almost periodic function.

(3)→ (4) : For a given *nbd* U of 0 in R, we choose a *nbd* V of 0 in R such *that* $V+V+V \subset U$.

Since $\{f^s : s\varepsilon\ S\}$ is totally bounded, there exists a finite subset P of T such that

$\rho\ (f^p(x), f^s(x))\varepsilon V$ for $s\varepsilon\ S$ and for some $p\varepsilon\ P\ (x\varepsilon X)$.

We select a nbd W of 0 in R such that $\rho\ (x,y)\ \varepsilon\ W$ implies

$\rho\ (f^p(x), f^s(y))\varepsilon V$ for $x, y\varepsilon\ X$.

We shall show that $\rho(x,y)\varepsilon W$ implies

$\rho(f^{s}(x), f^{s}(y))\varepsilon U$.

Let $\rho(x,y)\varepsilon W$ and let $s\varepsilon S$. We select $p\varepsilon P$ such that

$\rho(f^{p}(x), f^{s}(x))\varepsilon V$.

Thus for $\rho(x,y)\ \varepsilon\ W$ and for some $s\varepsilon S$, we have

$$\rho(f^{s}(x), f^{s}(y)) << \rho(f^{s}(x), f^{p}(x)) + \rho(f^{p}(x),f^{p}(y)) + \rho(f^{p}(y), f^{s}(y))$$
$$\varepsilon V+V+V \subset U,$$

which shows that $\{f^{s}: s\varepsilon S\}$ is uniformly equicontinuous.

Theorem [5]. Let X be a metric space over a topological semi field R. If $\{f^{t}: t\varepsilon T\}$ be an abelian transformation group defined on X and is almost periodic, then $\{f^{t}: t\varepsilon T\}$ is equicontinuous.

Remark: $\{f^{t}: t\varepsilon T\}$ is said to be equi continuous at $x\varepsilon X$, if for ever *nbd* U of 0 in R, there is a *nbd* V of 0 in R such that $\rho(x,y)\varepsilon V$ implies

$\rho(f^{t}(x), f^{t}(y))\ \varepsilon\ U$ for all $t\ \varepsilon\ T$.

If $\{f^{t}: t\varepsilon T\}$ is equi continuous at each point of X, then

$\{f^{t}: t\varepsilon T\}$ is said to be equi continuous on X.

Proof of Theorem. Let U be a *nbd of* 0 in R. Select a *nbd* V of 0 in R such that $V+V+V\subset U$.

since $\{f^{t}: t\varepsilon T\}$ is almost periodic, there exists a compact subset K of T and a set A of T such that $T = AK$ and

$\rho(x, f^{a}(x))\varepsilon$ V for each aεA and $x\varepsilon X$.

Since K is compact therefore $\{f^{k}: k\varepsilon K\}$ is equi continuous, thus for all $x,y\varepsilon X$ and for each *nbd* V of 0 in R there exists *nbd* W of 0 in R such that

$\rho(x,y)\ \varepsilon\ W$ implies

$\rho(f^{k}(x),f^{k}(y))\ \varepsilon\ V(k\ \varepsilon\ K)$.

Now for $t\varepsilon T$ there are elements $a\varepsilon A$ and $k\varepsilon K$ such that $t = ak$. Hence for $\rho(x,y)\varepsilon W$, we have

$$= \rho(f^{t}(x),f^{t}(y) = \rho(f^{ak}(x),f^{ak}(y)$$
$$\rho(f^{ka}(x),f^{ka}(y))$$
$$<< \rho(f^{ka}(x),f^{k}(x) + \rho(f^{k}(x),f^{k}(y) + \rho(f^{k}(y),f^{ka}(y))$$
$$\varepsilon V+V+V\subset U.$$

Hence the proof is completed.

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(Dedicated to the memory of Professor J.N. Kapur)

ON A CLASS OF ENTIRE DOUBIE DIRICHLET SEQUENCES

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ABSTRACT

In this paper, we have considered a class U of Entire Double Dirichlet Sequences. With suitably Gelfand binary compositions, norm and involution, U has been provided with a Gelfand algebraic structure which is not a division algebra. Self adjoint and unitary elements in U. have been characterised. *Sym (U)*, the symmetric set of U, has been shown to be a commutative B^* –subalgebra of U. *Sym (U)* has also been provided with a Banach lattice structure and has been shown to be an Archimedean function ring.

1. Introduction. Let U be a class of complex double sequences, defined as

$$(1.1)\qquad U = \{ f : f = \{a_{mn}\} \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} |a_{mn}| < \infty \}$$

where $\{\lambda_m\}$ and $\{\mu_n\}$ are strictly increasing and unbounded sequences, satisfying

$$(1.2)\qquad \lim_{m+n \to \infty} \frac{\log (m+n)}{\lambda_m + \mu_n} = D < \infty.$$

Note that every element of U satisfies

$$(1.3)\qquad \lim_{m+n \to \infty} \frac{\log |a_{mn}|}{\lambda_m + \mu_n} = -\infty.$$

Hence every element of U is the sequence of coefficients of an entire function represented by Double Dirichlet series, i.e.

$$f(s, w) = \Sigma a_{mn} e^{s\lambda_m + w\mu_n}.$$

For this reason, we call any double sequence satisfying (1.3) as an Entire Double Dirichiet Sequence.

Srivastava ([4], [5]) studied a class of entire Dirichlet series defined as

$$S = \{f : f(s) = \Sigma\, a_n e^{s\lambda_n},\ a_n = o\,(e^{-n\lambda_n})\}\,.$$

Daoud [2] generalised his class to the case of several complex variables and studied a class X of integral functions represented by double Dirichlet series defined as

$$X = \{f(s_1, s_2) = \Sigma\, a_{mn} e^{s_1\lambda_m + s_2\mu_n},\ \Sigma\,|\,a_{mn}|\, e^{m\lambda_m + n\mu_n} < \infty\,\}.$$

We have considered in [6] the class X of entire double Dirichlet sequences corresponding to the class X of Daoud. In this paper, we consider a superset of X which we denote as U and define as (1.1).

In section 2, we have provided a B^*– algebraic structure to U. Self adjoint and unitary elements in U are characterised. Symmetric set of U has been given an algebraic structure. In section 3, lattice structure has been developed on $Sym(U)$. $Sym(U)$ has been shown to be a Banach lattice as well as an Archimedean 1– ring.

2. Algbraic Structure For U. For definitions of this section we refer to [3]. We define binary compositions on U as follows:

(i) $f + g = \{c_{mn}\}$ *where* $c_{mn} = a_{mn} + b_{mn}$, œ $m,n \geq 1$ . .

(ii) $\alpha.f = \{d_{mn}\}$ *where* $d_{mn} = \alpha.a_{mn}$, $\alpha \in C$; *and*

(iii) $f.\,g = \{e^{m\lambda_m + n\mu_n}\, a_{mn}.b_{mn}\}$.

With these binary compositions, U becomes an algebra with identity element u defined as follows:

(2.1) $u = \{e_{mn}\},\ e_{mn} = e^{-m\lambda_m - n\mu_n}$ œ $m,n \geq 1$.

We define norm on U as follows:

$$\|f\| = \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} |\,a_{mn}\,|,\ \forall\, f = \{a_{mn}\} \in U.$$

Obviously U becomes a normed linear space.

Also, since

$$\begin{aligned} \|f.g\| &= \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} |e^{m\lambda_m + n\mu_n}\, a_{mn}.b_{mn}| \\ &\leq \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} |a_{mn}|\, \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} |b_{mn}| \\ &= \|f\|.\|g\|, \end{aligned}$$

hence U becomes a normed algebra.

In fact, we have

Theorem 1. U is a Banach Algebra.

Proof. We only have to show that U is complete.

Let $\{f_p\}$ be a Cauchy sequence in U,

$f_p\,;\,\{\,a^{(p)}_{mn}.$ œ $p \geq 1$

Given $\in > 0 \; \exists$ some $p_0 > 1$ such that

(2.2) $\quad \| f_p . g_p \| < \in, \forall \; p, q \geq p_0$

$$\Rightarrow \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \,|\, a^{(p)}_{mn} - a^{(q)}_{mn} | < \in \forall p, q \geq p_0 .$$

This implies that $\{a^{(p)}_{mn}\}$ forms a Cauchy sequence in C for every m, n and hence, owing to the completeness of C, converges to a complex number, say a_{mn}.

In (2.2), letting $q \to \infty$, we have

(2.3) $\quad \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \,|\, a^{(p)}_{mn} - a_{mn} | < \in \forall \; p \geq p_0$

$$\Rightarrow \; \| f_p - f \| \to 0 \; as \; p \to \infty$$

$$\Rightarrow \; f_p \to f \; as \; p \to \infty .$$

Further, owing to (2.3) and (1.1), f belongs to U, since

$$\sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} |a_{mn}| \leq \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \,|\, a^{(p)}_{mn} - a_{mn} | + \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \,|\, a^{(p)}_{mn} | .$$

Hence the theorem.

Next we define an involution * on U, as follows :

$$*(f) = f^* = (\bar{a}_{mn}) \; \forall \; f = \{a_{mn}\} \in U,$$

where $\bar{a}_{mn}$ is the complex conjugate of a_{mn}.

Note that

(i) $\quad (f^*)^* = \overline{\{\bar{a}_{mn}\}}, = \{\bar{\bar{a}}_{mn}\} = \{a_{mn}\} = f$

$\Rightarrow (f^*)^* = f.$

(ii) $\quad (f+g)^* = \{\overline{a_{mn} + b_{mn}}\}, = \{\bar{a}_{mn}\} = \{\bar{b}_{mn}\} = f^* + g^*$

$\Rightarrow (f+g)^* = f^* + g^*.$

(iii) $\quad (\alpha . f)^* = \{\overline{\alpha . a_{mn}}\} = \{\bar{\alpha} . \bar{a}_{mn}\} = \bar{\alpha} . f^*$

$\Rightarrow (\alpha . f)^* = \bar{\alpha} . f^*$

(iv) $\quad (f.g)^* = \{\overline{e^{m\lambda_m + n\mu_n} . a_{mn} . b_{mn}}\}, = \{\overline{e^{m\lambda_m + n\mu_n}} . \bar{a}_{mn} . \bar{b}_{mn}\}$

$$= \{e^{m\lambda_m + n\mu_n} . \bar{a}_{mn} . \bar{b}_{mn}\} = g^* . f^*$$

$\Rightarrow (f.g)^* = f^* . g^*.$

Further

$$\| f.f^* \| = \{ \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} | e^{m\lambda_m + n\mu_n} . a_{mn} . \bar{a}_{mn} |$$

$$= \{ \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} | a_{mn} | \}^2 = \| f \|^2$$

$$\Rightarrow \; \| f.f^* \| = \| f \|^2 .$$

Thus we get,

Theorem 2. U is a commutative B^* – algebra with indentity.

Corollary1. *U* is a Gelfand Algebra.

Proof. Note that, for the identity element *u* of *U* defined by (2. 1), we have

$$\|u\| = \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} \left| e^{-m\lambda_m - n\mu_n} \right| = 1.$$

Hence the result

Corollary 2. *U* is not a division algebra.

Proof. Consider, for instance, $f = \{a_{mn}\}$ where

$$a_{mn} = \left\{ \frac{e^{-m\lambda_m - n\mu_n}}{m+n} \right\} .$$

Obviously, *f* belongs to *U*. Let, if possible, there exists $g = \{b_{mn}\}$ such that $f.g = u$

$$\Rightarrow \quad e^{m\lambda_m + n\mu_n} \frac{e^{-(m\lambda_m + n\mu_n)}}{m+n} \; b_{mn} = e^{-(m\lambda_m + n\mu_n)}, \text{œ } m,n$$

$$\Rightarrow \; b_{mn} = (m+n)\, e^{-(m\lambda_m + n\mu_n)} .$$

Hence

$$\|g\| = \sup_{m+n \geq 2} \; e^{m\lambda_m + n\mu_n} \left| b_{mn} \right|$$

$$= \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} (m+n) e^{-(m\lambda_m + n\mu_n)}$$

$$= \sup_{m+n \geq 2} (m+n).$$

This implies that *g* does not belong to *U*. Hence f does not possess inverse in *U*. Hence the result.

Theorem 3. Let $f = \{a_{mn}\} \in U$. Then *f* is self adjoint if and only if

$e^{m\lambda_m + n\mu_n} . a_{mn}$ is real for each m,n.

Also f is unitary if any only if

$\left| e^{m\lambda_m + n\mu_n} . \, a_{mn} \right| = 1$, for each m,n.

Proof. Recall that in a normed * algebra, an element *f* is said to be self adjoint if $f = f^*$ in *U*, we have

$$f^* = f \quad \Leftrightarrow \bar{a}_{mn} = a_{mn}$$

$\Leftrightarrow a_{mn}$ is real œ m,n

$\Leftrightarrow e^{m\lambda_m + n\mu_n} \; a_{mn}$ is real œ m,n.

Again, recall that an element *f* of a normed * algebra with identity is called unitary if $f.f^* = e$, the identity element.

In *U*, we have

$$f.\,f^* = u \iff e^{m\lambda_m + n\mu_n}\, a_{mn} \,.\, a_{mn} = e^{-(m\lambda_m + n\mu_n)}$$
$$\iff \left| e^{m\lambda_m + n\mu_n}\, a_{mn} \right|^2 = 1,$$
$$\iff \left| e^{m\lambda_m + n\mu_n}\, a_{mn} \right| = 1.$$

Hence the theorem.

Recall that the set of all self adjoint elements of a normed* algebra X is called the symmetric set of X and is denoted as *Sym (X)*.

The above characterisation of self adjoint elements of U suggests an alternative expression for *Sym (U)* as follows:

Sym (U) = $\{f = \{a_{mn}\},\ f \in U,\ e^{m\lambda_m + n\mu_n}\, a_{mn}$ is real œ $m,n \geq 1\}$.

Sym (U) is obviously a subclass of U. Equipped with same binary compositions as in U and same norm as on U, we get

Theorem 4 . *Sym (U)* is a commutative B^* sub algebra of U with identity.

Proof. We only prove the completeness of *Sym(U)*.

Let $\{f_p\}$ be a Cauchy sequence in *Sym (U)*, $f_p = \{a^{(p)}_{mn}\},\ p \geq 1$.

Given $\in > 0$, $\exists$ some $p_0 \geq 1$ such that

$$\|\mathrm{f}_\mathrm{p} - \mathrm{f}_\mathrm{q}\| < \in, \text{ œ } \ p,q \geq p_0,$$

i.e.

$$(2.4) \qquad \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \left| a^{(p)}_{mn} - a^{(q)}_{mn} \right| < \in \forall \quad p,q \geq p_0.$$

This implies that $\{a^{(p)}_{mn}\}$ is a real Cauchy sequence. Owing to the completeness of R, therefore, it converges to a real number, say a_{mn}.

In equation (2.4), as $q \to \infty$, we get

$$(2.5) \quad \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \left| a^{(p)}_{mn} - a_{mn} \right| < \in \ as\ p \geq p_0$$
$$\Rightarrow \sup_{m+n \geq 2} = e^{m\lambda_m + n\mu_n} \left| a^{(p)}_{mn} - a_{mn} \right| \to 0 \in \ as\ p \to \infty$$
$$\Rightarrow \|f_p - f\| \to 0 \ as\ p \to \infty \text{ where } f = \{a_{mn}\}$$
$$\Rightarrow f_p \to f \ as\ p \to \infty,$$

The limit f belongs to *Sym(U)*, owing to (2.5) and (1.1), since

$$\sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} \left| a_{mn} \right| \leq \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} \left| a^{(p)}_{mn} - a_{mn} \right| + \sup_{m+n \geq 2} e^{m\lambda_m + n\mu_n} \left| a^{(p)}_{mn} \right|.$$

Hence the theorem.

3. Lattice Structure For *Sym (U)*.

For the definitions of this section, we refer to Birkhoff [1]. In this section, we provide a lattice structure to *Sym (U)* by defining a partial ordering on it as follows:

If $f = \{a_{mn}\}$ and $g = \{b_{mn}\}$ are arbitrary elements of *Sym (U)*, we say that

$f \le g$, *if* $e^{m\lambda_m + n\mu_n} a_{mn} \le e^{m\lambda_m + n\mu_n} b_{mn}$ œ m, n.

Clearly the relation "$\le$" on *Sym (U)* satisfies the following properties :

(i) Reflexivity: $f \le f$ œ $f \in Sym(U)$,

(ii) Anti Symmetry : $f \le g$ and $g \le f \Rightarrow f = g$,

(iii)Transitivity: $f \le g$ and $g \le h \Rightarrow f \le h$.

Thus the relation "$\le$" defines a partial ordering on *Sym (U)* and *{Sym (U), ≤}* becomes a partially ordered set or a poset. In fact, we have

Theorem 5. *{Sym (U), ≤}* is a Banach lattice.

Proof. Let $f = \{a_{mn}\}$ and $g = \{b_{mn}\}$ be two arbitrary elements of *Sym(U)*.

Define $f \wedge g = \{c_{mn}\}$and $f \vee g = \{b_{mn}\}$ where,œ $m,n \ge 1$,

$$e^{m\lambda_m + n\mu_n} c_{mn} = \min \{e^{m\lambda_m + n\mu_n} a_{mn},\ e^{m\lambda_m + n\mu_n} b_{mn}\}$$

and $$e^{m\lambda_m + n\mu_n} d_{mn} = \max \{e^{m\lambda_m + n\mu_n} a_{mn},\ e^{m\lambda_m + n\mu_n} b_{mn}\}.$$

It can be verified that $f \wedge g$ and $f \vee g$ belong to *Sym (U)* and are the greatest lower bound *(g.l.b.)* and least upper bound *(l.u.b.)* respectively of *f* and *g* in *Sym (U)*.

With these definitions of *l.u.b.* and *g.l.b.*, *{Sym (U),≤}* becomes a lattice.

Define $|f| = f \vee (-f)$ where $(-f) = \{-a_{mn}\}$. Then

$$|f| \le |g| \Rightarrow f \vee (-f) \le g \vee (-g)$$

$$\Rightarrow \max \{e^{m\lambda_m + n\mu_n} a_{mn},\ -e^{m\lambda_m + n\mu_n} a_{mn}\} \le \max \{e^{m\lambda_m + n\mu_n} b_{mn}, -e^{m\lambda_m + n\mu_n} b_{mn}\}$$

$$\Rightarrow e^{m\lambda_m + n\mu_n} |a_{mn}| \le e^{m\lambda_m + n\mu_n} |b_{mn}|$$

$$\Rightarrow \sup_{m+n \ge 2} e^{m\lambda_m + n\mu_n} |a_{mn}| \le \sup_{m+n \ge 2} \le e^{m\lambda_m + n\mu_n} |b_{mn}|$$

$$\Rightarrow \|f\| \le \|g\| \ \text{œ}\ f, g \in Sym\ (U).$$

So that *{Sym (U), ≤}* becomes a Banach lattice.

Theorem 6. The lattice *{Sym (U), ≤}* is modular.

Proof. Recall that a lattice is called modular if it satisfies the modular identity, *viz.*

$$f \le h \Rightarrow f \vee (g \wedge h) = (f \vee g) \wedge h \ \text{œ}\ g.$$

Let us assume $f \le h$, where $f = \{a_{mn}\}$ and $h = \{c_{mn}\}$, so that

$$e^{m\lambda_m + n\mu_n} a_{mn} \le e^{m\lambda_m + n\mu_n} c_{mn} \ \text{œ}\ m, n \ge 1,$$

(3.1) $$e^{m\lambda_m + n\mu_n} c_{mn} = \max \{e^{m\lambda_m + n\mu_n} a_{mn},\ e^{m\lambda_m + n\mu_n} c_{mn}\} \ \text{œ}\ m, n \ge 1.$$

Now for every $g \in Sym\ (U)$, with $g\ \{b_{mn}\}$, we have

$f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h)$ (by distribution)

$$= min \left\{ \begin{matrix} max\ \{e^{m\lambda_m + n\mu_n} a_{mn},\ e^{m\lambda_m + n\mu_n} b_{mn}\}, & \\ & max\ \{e^{m\lambda_m + n\mu_n} a_{mn},\ e^{m\lambda_m + n\mu_n} c_{mn}\} \end{matrix} \right\}$$

$$= min\ \{max\ \{e^{m\lambda_m + n\mu_n} a_{mn},\ e^{m\lambda_m + n\mu_n} b_{mn}\},\ e^{m\lambda_m + n\mu_n} c_{mn}\} \text{ due to (3.1)}$$

$$= (f \vee g) \wedge h.$$

This shows that $\{Sym\ (U), \leq\}$ is modular.

Theorem 7. $\{Sym\ (U), \leq\}$ is an Archimedean function ring.

Proof. We first prove two lemmas.

Lemma 1. The lattice $\{Sym\ (U), \leq\}$ is an Archimedean 1-group.

Proof. Since $\{Sym(U), \leq\}$ is a commutative B^*- algebra with identity. and $\{Sym\ (U), \leq\}$ is a partially ordered set and lattice, therefore to show that $Sym\ (U)$ is an l-group, it only remains to show that every group translation in $Sym\ (U)$ is isotone.

i.e. $f \rightarrow f_1 + f + f_2$ is isotone $\text{œ}\ f_1, f_2 \in Sym(U)$

Or, in other words, we have to show that, $\text{œ}\ f_1, f_2 \in Sym(U)$

$f \leq g \Rightarrow f_1 + f + f_2 \leq f \rightarrow f_1 + g + f_2.$

Let us suppose that $f \leq g$ and let $f_1 = \{a^{(1)}_{mn}\}$, $f_2 = \{\ a^{(2)}_{mn})$ be arbitrary elements of $Sym\ (U)$. Then

$$f \leq g \Rightarrow e^{m\lambda_m + n\mu_n} a_{mn} \leq e^{m\lambda_m + n\mu_n} b_{mn}\ \text{œ}\ m,n$$

$$\Rightarrow e^{m\lambda_m + n\mu_n} (a^{(1)}_{mn}) + e^{m\lambda_m + n\mu_n} (a_{mn}) + e^{m\lambda_m + n\mu_n} (a^{(2)}_{mn}) \leq$$

$$e^{m\lambda_m + n\mu_n} (a^{(1)}_{mn}) + e^{m\lambda_m + n\mu_n} (b_{mn}) + e^{m\lambda_m + n\mu_n} (a^{(2)}_{mn})$$

$$\Rightarrow f_1 + f + f_2 \leq f_1 + g + f_2.$$

Therefore the mapping $f \rightarrow f_1 + f + f_2$ is isotone.

Thus $\{Sym\ (U), \leq\}$ is an l-group.

It remains to show that $\{Sym\ (U), \leq\}$ is Archimedean, i.e.

$$f << g \Rightarrow f \equiv 0.$$

Let $f << g$, i.e., f is incomparably smaller than g. This implies that for every positive k

$$k.\ f \leq g \Rightarrow k\ e^{m\lambda_m + n\mu_n} a_{mn} \leq e^{m\lambda_m + n\mu_n} b_{mn}\ \text{œ}\ m,n$$

$$\Rightarrow k\ a_{mn} \leq b_{mn}\ \text{œ}\ m,n$$

$$\Rightarrow a_{mn} = 0\ \text{œ}\ m,n\ (\therefore\ a_{mn} \in R\ \&\ R \text{ is Archimedean})$$

$$\Rightarrow f \equiv 0.$$

Thus $\{Sym\ (U) \leq\}$ is Archimedean.

Lemma2. The lattice $\{Sym\ (U), \leq\}$ is an Archimedean l-ring.

Proof. $\{Sym\ (U), \leq\}$ is already shown to be a lattice as well as a ring. Further note that

(a) $f \le g \Rightarrow e^{m\lambda_m + n\mu_n}\, a_{mn} \le e^{m\lambda_m + n\mu_n}\, b_{mn} \text{ œ } m,n \ge 1$

$\Rightarrow e^{m\lambda_m + n\mu_n}\,(c_{mn} + a_{mn}) \le e^{m\lambda_m + n\mu_n}(c_{mn} + b_{mn})$

$\Rightarrow h + f \le h + g \text{ œ } h = \{c_{mn}\} \text{ in } sym\ (U)$

and

(b) $0 \le f,\ 0 \le g \Rightarrow \le e^{m\lambda_m + n\mu_n}\, a_{mn},\ 0 \le e^{m\lambda_m + n\mu_n}\, b_{mn} \text{ œ } m,n \ge 1$

$\Rightarrow 0 \le e^{m\lambda_m + n\mu_n}\, a_{mn}.b_{mn}) \text{ œ } m,n \ge 1$

$\Rightarrow 0 \le f.g.$

This pioves that $\{Sym\ U, \le\}$ is an Archimedean l-ring.

Proof of Main Theorem. Suppose that $f \wedge g = 0$ and h is any element of $Sym\ (U)$ such that $h \ge 0$. Then

$$h \ge 0 \Rightarrow e^{m\lambda_m + n\mu_n}\, c_{mn} \ge 0 \text{ œ } m,n \ge 1$$

and

$$(f \wedge g) = 0 \Rightarrow min\ \{e^{m\lambda_m + n\mu_n}\, a_{mn}, e^{m\lambda_m + n\mu_n}\, b_{mn}\} = 0 \text{ œ } m,n \ge 1.$$

Together, we have

$$= min\ \left\{ \begin{array}{l} e^{m\lambda_m + n\mu_n}\,(e^{m\lambda_m + n\mu_n}\, c_{mn}.a_{mn}), \\ \qquad\qquad e^{m\lambda_m + n\mu_n}\,(e^{m\lambda_m + n\mu_n}\, c_{mn}.a_{mn}) \end{array} \right\} = 0,\ m,\ n \ge 1.$$

Hence the theorem.

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(Dedicated to the memory of Professor J.N. Kapur)

Bd-EXPANDABILITY PROPERTY AND BOUNDEDLY PARACOMPACT SPACES

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ABSTRACT

The notions of *Bd*-expandability and bounded *Bd*-expandability properties are introduced and studied. It is proved that *Bd*-expandability property is a topological property other than finite dimensionality coupled with Hausdorffness which strengthens paracompactness to bounded paracompactness. Not only this, *Bd*-expandability property strengthens each of the following properties to bounded paracompactness : θ-refinability, metacompactness, bounded metacompactness, subparacompactness. screenability and Lindelöness.

Introduction. In 1970 Fletcher, McCoy and Slover [3] introduced the notion of boundedly paracompact spaces. A topological space X is said to be boundedly paracompact if every open cover of X has a bounded locally finite open refinement. The class of boundedly paracompact spaces contains the class of compact spaces and is contained in the class of paracompact spaces. But a paracompact space need not be boundedly paracompact and a boundedly paracompact space need not be compact. However, Fletcher, McCoy and Slover [3] have proved that finite dimensionality together with Hausdorffness strengthens paracompactness to bounded paracompactness. Also, Fletcher. McCoy and Slover [4] have proved that bounded countable paracompactness together with Hausdorffness strengthens hereditary paracompactness to bounded paracompactness. But it is not known, whether any other topological property strengthens paracompactness to bounded paracompactness. Therefore, the following question arises:

Is there any topological property other than finite dimensionality coupled with Hausdorffness which strengthens paracompactness to bounded paracompactness?

All the known expandability properties being weaker than paracompactness cannot strengthen paracompactness to bounded paracompactness. In the present paper. we introduce the concept of *Bd*-expandability as a stronger form of expandability and answer the above question in the affirmative. The whole paper is divided into two sections. In section 1 the concept of *Bd*-expandability is introduced, discussed and the above question is answered in the affirmative by proving that every *Bd*-expandable paracompact space is boundedly paracompact. The notion of bounded *Bd*-expandability is also introduced by generalizing the notion of *Bd*-expandability. Various inclusion relations are obtained and several counter examples are given which demonstrate the invalidity of a number of plausible conjectures concerning these properties. In section 2, open problems are given.

1. *Bd*-Expandability Property

1.1 Definitions. A topological space X is said to be *Bd*-expandable (respectively, boundedly *Bd*-expandable) if for each locally finite (respectively, bounded locally finite) collection $\{F_\alpha : \alpha \varepsilon A\}$ of subsets of X, there exists a bounded locally finite collection $\{G_\alpha : \alpha \varepsilon A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \varepsilon A$.

1 .2 Theorem. A topological space X is Bd-expandable (respectively, boundedly *Bd*-expandable) iff for each locally finite (respectively, bounded locally finite) collection $\{F_\alpha : \alpha \varepsilon A\}$ of closed subsets of X, there exists a bounded locally finite collection $\{G_\alpha : \alpha \varepsilon A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \varepsilon A$.

Proof . Obvious.

1.3 Theorem. A topological space X is boundedly *Bd*-expandable iff for each discrete collection $\{F_\alpha : \alpha\varepsilon A\}$ of subsets of X, there exists a bounded locally finite collection $\{G_\alpha : \alpha\varepsilon A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each of $\alpha\varepsilon A$.

Proof. The proof follows from a simple modification of the proof of the Theorem 2.7 in [10].

1.4 *Lemma.* Let $\{F_\alpha : \alpha\varepsilon A\}$ be a bounded locally finite collection of subsets of a topological space X. Let $\Gamma = \{\gamma \subset A : \gamma \text{ is finite}\}$. Then $\{\cap_{\alpha\varepsilon\gamma} F_\alpha : \gamma\varepsilon\Gamma\}$ is bounded locally finite.

Proof. Obvious.

Using Lemma 1.4 above we give a characterization of *Bd*-expandable spaces as follows :

1.5 Theorem. A topological space X is *Bd*-expandable iff for each locally finite collection F of (closed) subsets of X, there exists a bounded locally finite open cover U of X and a positive integer n such that each member of U intersects atmost n members of F.

Proof. Let X be a *Bd*-expandable space. Let $\boldsymbol{F} = \{F_\alpha : \alpha\varepsilon A\}$ be a locally finite collection of (closed) subsets of X. Then $Cl\, F = \{Cl F_\alpha : \alpha\varepsilon A\}$ is locally finite. Therefore, there exists a bounded locally finite collection $\{G_\alpha : \alpha\varepsilon A\}$ of open subsets of X such that $Cl\, F_\alpha \subset G_\alpha$ for each $\alpha\varepsilon A$. Then there exists a positive integer n such that each point $x\varepsilon X$ belongs to at most n members of $\{G_\alpha : \alpha\varepsilon A\}$.

Let. $\Gamma = \{\gamma \subset A : |\gamma| \leq n\}$. For each $\gamma\in\Gamma$,define $V_\gamma = \cap_{\alpha\varepsilon\gamma} G_\alpha - \cap_{\alpha\varepsilon\gamma} Cl F_\alpha$. It can be proved easily that $\boldsymbol{V} = \{V_\gamma : \gamma\varepsilon\Gamma\}$ is an open cover of X such that each member of $\boldsymbol{V}$ intersects atmost n members of $\boldsymbol{F}$. Now by Lemma 1.4 above and the fact that $V_\gamma \subset \cap_{\alpha\varepsilon\gamma} G_\alpha$, for each $\gamma\varepsilon\Gamma$. it follows that V is bounded locally finite.

Conversely suppose that $\boldsymbol{F} = \{F_\alpha : \alpha\varepsilon A\}$ is a locally finite collection of (closed) subsets of X. There exists then a bounded locally finite open cover $U = \{U_\beta : \beta\varepsilon B\}$ of X and a positive integer n such that each member of U intersects atmost n members of $\boldsymbol{F}$. For each $\alpha\varepsilon A$, define $B_\alpha = \{\beta\varepsilon B : F_\alpha \cap U_\beta \neq \phi\}$ and $G_\alpha = \cup_{\beta\varepsilon B\alpha} U_\beta$. Then clearly $\boldsymbol{G} = \{G_\alpha : \alpha\varepsilon A\}$ is a collection of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha\varepsilon A$. Since $\boldsymbol{U}$ is bounded locally finite, therefore, there exists a positive integer m such that each point $x\in X$ has a neighbourhood which intersects atmost m members of U. Let $m.n=k$. It can be shown that $\boldsymbol{G}$ is bounded locally finite of order k.

1.6 Theorem. A topological space X is boundedly *Bd*-expandable if and only if for each discrete collection $\boldsymbol{F}$ of (closed) subsets of X, there exists a bounded locally finite open cover U of X such that each member of U intersects atmost one member of F.

Proof. The proof follows from the similar arguments used in the proof of theorem 1.5 above.

1.7 Theorem. Every countably compact space is Bd-expandable.

Proof. Obvious in view of Theorem 2.7 in [5]

The converse of Theorem 1.7 above is not known.

1.8 Theorem. (i) Every Bd-expandable space is boundedly Bd-expandable.

(ii) Every Bd-expandable space is expandable.

(iii) Every boundedly Bd-expandable space is boundedly expandable.

(iv) Every collectionwise normal space is boundedly Bd-expandable.

Proof. Obvious.

The converse of Theorem 1.8(i) is not true. Following is an example of a boundedly Bd-expandable space which is not Bd-expandable:

1.9 Example. A boundedly Bd-expandable space which is not Bd-expandable. The topological space X defined in [8, page 179-180] being collectionwise normal is boundedly Bd-expandable but it is not Bd-expandable because it is not countably paracompact.

The converse of Theorem 1.8 (ii) is not true. Following is an example of an expandable space which is not Bd-expandable:

1.10 Example . An expandable space which is not Bd -expandable. Hilbert space being metrizable is expandable but it is not Bd-expandable.

The converse of Theorem 1.8 (iii) is not known.

The converse of Theorem 1.8 (iv) is not true. Following is an example of a boundedly Bd-expandable space which is not collectionwise normal:

1.11 Example. A boundedly Bd-exapnadable space which is not collectionwise normal :

Let (R,τ) be the real line with the usual topology. We define a new topology by defining $\tau^* = \{X \subset R : X = \phi$ or $R\text{-}X$ is compact in $(R,\tau)\}$. Surely τ^* is a topology on R. Then (R, τ^*) being compact is boundedly Bd-expandable but it is not normal and hence not collectionwise normal.

1.12 Note. The example 1.10 also shows that a paracompat space need not be Bd-expandable. In fact, the notion of Bd-expandability is independent of that of paracompactness. Following is an example of Bd-expandable space which is not paracompact.

1.13 Example. A normal Hausdorff Bd-expandable space which is not paracompact. Let Ω be the first uncountable ordinal. Then $X=[0, \Omega[$ with the usual order topology is a normal Hausdorff Bd-expandable space which is not paracompact.

1.14 Theorem. Every *Bd*-expandable space is boundedly countably paracompact.

Proof. Let X be a *Bd*-expandable space. Let $R = \{R_i : i = 1,2,\ldots\}$ be a countable open cover of X. Put $Si = U^{i}_{p=1} R_p$ for each i. Let $A_1 = S_1$ and $A_i = S_i - S_{i-1}$ for i=2,3,.... Surely $A = \{A_i : i=1,2,\ldots\}$ is a locally finite refinement of R such that $A_i \subset R_i$ for each *i*. Since X is *Bd*-expandable, therefore, there exists a bounded locally finite collection $\{G_i : i=1,2,\ldots\}$ of open subsets of X such that $A_i \subset G_i$ for each *i*. Let $Ui=R_i \cap G_i$ for each i. Then $U = \{Ui= 1,2,\ldots\}$ is a bounded locally finite open refinement of R. Therefore X is boundedly countably paracompact.

The converse of Theorem 1.14 above is not known.

1.15 Theorem. Every boundedly paracompact space is boundedly *Bd*-expandable.

Proof . Let X be a boundedly paracompact space, Let $F = \{F_\alpha : \alpha\varepsilon A\}$ be a bounded locally finite collection of closed subsets of X. Then there exists a positive integer n such that each point $x\varepsilon X$ belongs to atmost n members of F. Let $\Gamma=\{\gamma\subset A : |\gamma|\leq n\}$. For each $\gamma\varepsilon\Gamma$, define $V_\gamma = X - \cup_{\alpha\varepsilon\gamma} F_\alpha$. Then $V = \{V_\gamma : \gamma\varepsilon\Gamma\}$ is an open cover of X such that each member of V intersects atmost n members of F. Since X is boundedly paracompact, therefore, V has a bounded locally finite open refinement say $W = \{W_\delta : \delta\varepsilon\Delta\}$.

Now for each $\alpha\in A$, define $U_\alpha = St(F_\alpha, W) = \cup\{W_\delta \varepsilon\ W : W_\delta \cap F_\alpha \neq \phi\}$. Then surely $U=\{U_\alpha : \alpha\varepsilon A\}$ is bounded locally finite collection of open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha\varepsilon$A. Hence X is boundedly *Bd*-expandable.

The converse of Theorem 1.15 above is not true (cf Example 1.1).

Before proving a Theorem which is the crux of this paper. we give two examples which show that the notion of *Bd*-expandability is different from the notions of finite dimensionality coupled with Hausdorffness and hereditarily paracompactness, respectively.

1.16 Example. A *Bd*-expandable papracompact Hausdorff space which is not finite dimensional.

The Hilbert cube being compact Hausdorff is *Bd*-expandable paracompact Hausdorff space but it is not finite dimensional.

1.17 Example. A *Bd*-expandable paracompact Hausdorff space which is not hereditarily paracompact.

Let Ω be the first uncountable ordinal and ω be the first infinite ordinal. Let $X=[0,\Omega]\times[0,\omega]$ where both ordinal spaces $[0,\Omega]$ and $[0,\omega]$ are given the usual order topology. X is known as the Tychonoff plank. X being compact Hausdorff is *Bd*-expandable paracompact Hausdorff space but it is not hereditarily paracompact because the subspace $Y=X-\{(\Omega,\omega)\}$ is not paracompact.

Now we prove the Theorem which is the crux of this paper.

1.18 Theorem. Every *Bd*-expandable paracompact space is boundedly paracompact.

Proof. Let X be a *Bd*-expandable paracompact space. Let U be an open cover of X. Then U has a locally finite open refinement, say $V=\{V_\beta : \beta\varepsilon B\}$. Then by Theorem 1.5 above, there exists a bounded locally finite open cover $W=[W\delta : \delta\varepsilon\Delta]$ of X and a positive integer n such that each member of W intersects atmost n members of V. Let $G=\{W_\delta \cup V_\beta : \delta\varepsilon\Delta, \beta\varepsilon B\}$. Then surely G is a bounded locally finite open refinement of V and hence a bounded locally finite open refinement of U. Therefore X is boundedly paracompact.

1.19 Corollary. Every *Bd*-expandable θ-refinable (respectively, metacompact boundedly metacompact, subparacompact) space is boundedly paracompact.

Note that the converse of Theorem 1.18 is not known.

1.20 Theorem. The following are equivalent for a normal space X:

a) X is collectionwise normal.
b) X is boundedly *Bd*-expandable.
c) X is boundedly expandable.
d) X is discretely expandable.
e) X is discretely *H.C.* expandable.

Proof. (a) ⇒ (b) and (b) ⇒ (c) follows from Theorems 1.8 (iv), and 1.8 (iii), respectively. (c) ⇒ (d) is proved in Theorem 2.7 in [10]. (d) ⇒ (e) follows immediately by definitions and (e) ⇒ (a) is (d) ⇒ (a) in Theorem 2.6 in [10].

1.21 Corollary. Every *Bd*-expandable normal space is collectionwise normal.

1.22 Theorem. Every *Bd*-expandable screenable space is boundedly paracompact.

Proof. The proof follows from Theorems 1.8 (ii), 1.18 and Theorem 2.2 in [9].

1.23 Corollary. Every *Bd*-expandable strongly screenable space is boundedly paracompact.

1.24 Theorem. If X is a *Bd*-expandable space, then every A_σ - cover of X has a bounded locally finite open refinement.

Proof. Obvious.

1.25 Corollary. If X is a *Bd*-expandable space, then every A-cover of X has a bounded locally finite open refinement.

1.26 Corollary. Every Bd-expandable Lindelöf space is boundedly paracompact.

1.27 Theorem. A topological space X is boundedly *Bd*-expandable iff every B-cover of X has a bounded locally finite open refinement.

Proof. The proof follows from a simple modification of the proof of Theorem 3.7 in [10].

From the above results together with other known results we have the following implication diagram :

$$\begin{tikzcd}
 & \textit{Boundedly countably paracompact} \arrow[d, shift left, "/" marking] & \\
\textit{Compact} \arrow[r, shift right] \arrow[d, shift right] & \textit{boundedly paracompact} \arrow[l, shift right, "/" marking] \arrow[u, shift left] \arrow[r, shift right] \arrow[d, shift right] & \textit{paracompact} \arrow[l, shift right, "/" marking] \arrow[d, shift right] \arrow[dl, shift left, "/" marking] \\
\textit{Countably Compact} \arrow[u, shift right, "/" marking] \arrow[r] & \textit{Bd-expandable} \arrow[u, shift right, "/" marking] \arrow[u, shift left=3] \arrow[ur, shift left, "/" marking] \arrow[r, shift right] \arrow[d, shift right] & \textit{expandable} \arrow[l, shift right, "/" marking] \arrow[u, shift right, "/" marking] \arrow[d, shift right] \\
 & \textit{boundedly Bd-expandable} \arrow[u, shift right, "/" marking] \arrow[r] \arrow[d, shift left, "/" marking] & \textit{Boundedly expandable} \arrow[u, shift right, "/" marking] \\
 & \text{Collectionwise normal} \arrow[u, shift left] &
\end{tikzcd}$$

2. Open Problems. The following questions remain open :

2.1 Is every boundedly paracompact space *Bd*-expendable?

2.2 Is every *Bd*-expandable space countably compact'?

2.3 Is every boundedly countably paracompact space *Bd*-expandahle?

2.4 Is every boundedly expandable space boundedly *Bd*-expandable ?

If

not,whether these spaces are equivalent in llausdorff spaces.

Our feeling is such that the answers to problems 2.1 to 2.3 above are in negation, however, we do not know the required counter examples.

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(Dedicated to the memory of Professor J.N. Kapur)

STRONGLY EXPANDABLE SPACE

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ABSTRACT

The notion of strongly expandable spaces is introduced and studied to answer the following question : Is there any topological property other than normality coupled with Lindelöfness which strengthens paracompactness to strong paracompactness? Strong expandability is independent of normality coupled with Lindelöfness. The above question is answered in the affirmative by proving that strong expandability strengthens paracompactness (respectively, θ-refinability, metacompactness) to strong paracompactness. In fact strongly paracompact spaces are decomposed into strongly expandable and paracompact (respectively, θ-refinable, metacompact) spaces. Also, strongly χ_0-expandable spaces are introduced and it is shown that the notion of strong χ_0-expandability is equivalent to that of strong countable paracompactness. Various inclusion relations are obtained and several counter examples are given which demonstrate the invalidity of a number of plausible conjectures concerning strongly expandable spaces.

Introduction : In 1948 Morita [12] introduced the notion of star-finite property. A topological space X is said to have the star-finite property if every open covering of X has a star-finite open refinement . Begle[1] studied these spaces under the name S-spaces and Hu [3] called them hypocompact spaces. Since every hypocompact space is paracompact, hypocompact spaces were also called strongly paracompact spaces. Also, a topological space X is said to be strongly countably paracompact, if every countable open coverng of X has a star-finite open refinement. The class of strongly paracompact spaces contains the class of compact spaces and is contained in the class of paracompact spaces as well as in the class of strongly countably paracompact spaces. The class of strongly countably paracompact spaces is contained in the class of countably paracompact spaces. In normal spaces the notions of strong countable paracompactness, countable paracompactness and countable metacompactness are equivalent. There are examples of paracompact spaces which are not strongly paracompact [13. Example V-6]. In fact even metrizable spaces need not be strongly paracompact. However, every separable metric space is strongly paracompact. Every strongly countably paracompact Lindelöf space is strongly paracompact. Therefore, in a normal Lindelöf space the notion of countable paracompactness and hence that of paracompactness is equivalent to the notion of strong paracompactness. But it is not known, whether there is any other topoligcal property which strengthens paracompactness to strong paracompactness. Henec the following question arises :

(*) Is there any topological property other than normality coupled with Lindelöfness which strengthens paracompactness to strong paracompactness ?

In the present paper we answer the above posed question (*) in the affirmative by introducing the concept of strongly expandable spaces. The notion of strong expandability property is independent of normality coupled with Lindelöfness. The whole paper is divided into two sections. In section 1 preliminary definitions and results needed for this paper are given . In section 2 the notions of strongly expandable and -strongly χ_0-expandable spaces are introduced and the above posed question (*) is answered in the affirmative by proving that in strongly expandable spaces the notion of paracompactness is equivalent to that of strong paracompactness. Moreover, the strong expandability property strengthens θ-refinability and hence metacompactness to strong paracompactness. In fact strongly paracompact spaces are decomposed into strongly expandable and paracompact (respectively, metacompact, θ-refinable) spaces. Various inclusion relations are obtained and several counter examples are given which demonstrate the invalidity of a number of plausible conjectures concerning strongly expandable spaces. Lastly, the relations of strongly expandable spaces with other topological spaces are summarized in a implication diagram.

1. Preliminary Definitions and Results.

1.1 Definitions [13]. A collection A of subsets of a topological space X is said to be star-finite if each member of A intersects atmost finitely many members of A. A collection A of subsets of a topological space X is said to be σ-star-finite if it is a countable union of star-finite collections.

1.2 Theorem [5]. A normal space is strongly countably paracompact if X is countably paracompact.

1.3 Definition[14]. A topological space X is said to be strongly m-paracompact. where m is an infinite cardinal, if each open covering of X of order atmost m has a star-finite open refinement.

1.4 Theorem [14]. Normal space X is strongly m-paracompact iff X is strongly counably paracompact and every open cover of X of power atmost m has a σ-star-finite open refinement.

The above theorem remains true when "strongly m-paracompact" is replaced by srongly paracompact" and "every open cover of X of power atmost m" is replaced by "every open cover of X".

1.5 Theorem [13]. Every regular Lindelöf space is strongly paracompact.

1.6 Theorem[7]. A topological space X is expandable iff for each locally finite collection F of (Closed) subsets of X, there exists a locally finite open cover U of X such that each member of U intersects atmost finitely many members of F.

The above theorem remains true when "expandable" is replaced by χ_0-expandable" and "locally finite collection F" is replaced by "locally finite countable collection F".

1.7 Theorem [15]. Every expandable screenable space is paracompact.

1.8 Theorem [9]. A topological space X is χ_0-expandable iff X is countably paracompact.

1.9 Theorem [16]. A topological space X is almost χ_0-expandable iff X is countably metacompact.

2. Strongly Expandable Space

2.1 Definition. A topological space X is said to be strongly expandable (respectively, strongly χ_0-expandable) if for each locally finite collection $F = \{F\alpha : \alpha \varepsilon A\}$ of subsets of X (respectively, with $|A| \leq \chi_0$), there exists a star filint open cover U of X such that each member of U intersects atmost finitely many members of F.

2.2 Note. Obviously every strongly expandable space is strongly χ_0-expandable but the converse is not true. The Michael's modification of Bing's space [11] is strongly χ_0-expandable space which is not strongly expandable In the definitions 2.1 above the collection F of subsects of X may be taken as a collection of closed subsets of X.

2.3 Theorem. Every countably compact space is strongly expandable.

Proof . Obvious in view of the fact that in a countable compact space every locally finite collection of subsets is finite.

The converse of above theorem is not true. Following is an example of a strongly expandable space which is not countably compact :

2.4 Example. A strongly expandable space which is not countably compact.

Let IN be the set of all natural numbers with the discrete topology. Let $X\text{-}= II\{IN_i : i\varepsilon IN\}$ be the countable Cartesian product of copies of IN with the Tychonoff product topology τ. Then (X,τ) is strongly expandable but it is not countably compact.

2.5 Theorem. Every strongly paracompact space is strongly expandable.

Proof. *Let* X be a strongly paracompact space. Let $F = \{F_\alpha : \alpha\varepsilon A\}$ be a locally finite collection of closed subsets of X. Let $\Gamma = \{\gamma \subset A : \gamma \text{ is finite}\}$. For each $\gamma\varepsilon\Gamma$, define $V_\gamma = X \cup \{F_\alpha : \alpha \notin \gamma\}$ Then $V = \{V_\gamma : \gamma\varepsilon\Gamma\}$ is an open cover of X such that each member of V intersects atmost finitely many members of F. Since X is strongly paracompact,therefore, V has a star-finite open refinement, say W. Then W is a star-finite open cover of X such that each member of W intersects atmost finitely many members of F. Hence X is strongly expandable.

The converse of above theorem is not true. Even a strongly expandable space need not be paracompact. Following is an example of a strongly expandable space which is not paracompact :

2.6 Example. A strongly expandable space which is not paracompact.

Let Ω be the first uncountable ordinal and let $X = [0, \Omega]$ with the usual order topology. Then X is strongly expandable space which is not paracompact and hence also not strongly paracompact.

The example 2.6 above together with the following example shows that the notion of strong expandability is independent of that of paracompaetness.

2.7 Example . A paracompact space which is not strongly expandable.

Let A be an uncountable index set. Let I_α, $\alpha \varepsilon A$ be copies of the unit interval [0,1]. In their union $\cup_{\alpha\varepsilon A} I_\alpha$, we identify all zeros to get a star-shaped set $S(A)$. Then we define the distance between two points of $S(A)$ by $\rho(p,q) = p-q$ if $p, q \varepsilon I_\alpha$ and $\rho(p,q) = p+q$ if $p \varepsilon I_\alpha$, $q \varepsilon I_\beta$ where $\alpha \neq \beta$. Then $S(A)$ is a metric space with metric ρ. Consider the metric topology $\tau(\rho)$ on $S(A)$. Then $(S(A), \tau(\rho))$ is a paracompact space which is not strongly expandable. Since every paracompact space is expandable, therefore, this is also an example of an expandable space, which is not strongly expandable.

2.8 Theorem .

i) Every strongly expandable space is expandable.

ii) Every strongly χ_0 -expandable space is χ_0-expandable.

Proof . Obvious in view of theorem 1.6 and the fact that every star-finite open cover of a topological space X is also locally finite open cover of X.

The converse of theorem 2.8 (i) above is not true (cf. Example 2.7). Also, the converse of theorem 2.8 (ii) above is not true because of the facts that χ_0-expandability is equivalent to countable paracompactness (cf.Theorem 2.9 below) and that it is well known that a countably paracompact space need not be strongly countably paracompact.

2.9 Theorem. A topological space X is strongly χ_0-expandable iff X is strongly countably paracompuet.

Proof. Let X be a strongly χ_0-expandable space. Let $R = \{Ri : i = 1,2...\}$ be a countable open cover of X. Since X is strongly χ_0-expandable, therefore, by Theorem 2.8 (ii) above X is χ_0-expandable and hence by theorem 1.8, X is countably paracompact. Therefore, there exists a locally finite countable open cover $G = \{G_i : i=1,2...\}$ of X such that $G_i \subset R_i$ for each i. Then there exists a star-finite open cover $U = \{U_\alpha : \alpha \varepsilon A\}$ of X such that each member of U intersects atmost finitely many members of G. Then clearly $V = \{U_\alpha \cap G_i : \alpha \varepsilon A, i=1,2...\}$ is a star-finite open refinement of R. Hence X is strongly countably paracompact.

Conversely suppose that X is strongly countably paracompact. Then proceeding as in the proof of Theorem 2.5, by taking $|A| \leq \chi_0$ it is easy to prove that X is strongly χ_0-expandable.

2.10 Theorem. The following are equivalent for a normal space X

a) X is strongly χ_0-expandable.

b) X is χ_0-expandable.

c) X is almost χ_0-expandable.
d) X is countably metacompact.
e) X is countably paracompact.
l) X is strongly countably paracompact.

Proof. Obvious in view of Theorems 1.2, 1.9 and 2.9 above.

Now before giving the theorem which is the crux of this paper we give the following two examples which demonstrate that the notion of strong expandability is independent of that of normality coupled with Lindelöfness.

2.11 Example. A strongly expandable space which is neither normal nor Lindelöf.

The long line L is constructed from the ordinal space $[0, \Omega]$ (where Ω is the first uncountable ordinal) by placing between each ordinal α and its successor $\alpha + 1$ a copy of the unit integral$]0,1[$. L is then linearly ordered, and we give it the order topology. To the long line L we add a point p. Open sets of $L\cup\{p\}$ are the open sets of L together with those generated by the following neighbourhood of p: $U_\beta(p) = \{p\}\cup\{\cup^{\Omega}_{\alpha=\beta}]\alpha,\alpha+1[\}$ (where $1 \leq \beta \leq \Omega$). $U_\beta(p)$ is then a right hand ray less the ordinal points. We consider p to be the greatest element of $L\cup\{p\}$. Then $L\cup\{p\}$ is a strongly expandable space which is neither normal nor Lindelöf.

2.12 Example. A normal Lindelöf space which is not strongly expandable Let R be the set of all real numbers. For each $r\varepsilon R$, define $S_r = \{ x\varepsilon R : x > r)$. Then $B=\{S_r : r\in R\}$ is a base for some topology say τ, on R. The topology τ is known as right order topology on R. Then (R, τ) is normal Lindelöf space which is not countably paracompact and hence it is not strongly expandable.

2.13 Theorem. A topological space X is strongly paracompact iff X is strongly expandable and paracompact.

Proof. Only one implication needs proof. Let X be a strongly expandable paracompact space. Let U be an open cover of X. Since X is paracompact, therefore, U has a locally finite open refinement, say $V=\{V_\alpha : \alpha\varepsilon A\}$. Since X is strongly expandable and V is locally finite, therefore, there exists a star-finite open cover $W = \{W_\delta : \delta\varepsilon\Delta\}$ of X such that each member of W intersects atmost finitely members of V. Define G $\{W_\delta\cap V_\alpha : \delta\varepsilon\Delta, \alpha\varepsilon A\}$. Then clearly G is a star-finite open refinement of U. Hence X is strongly paracompact.

2.14 Corollary. A topological space X is strongly paracompact iff X is θ-refinable (respectively, metacompact) and strongly expandable.

2.15 Corollary. Every strongly expandable subparacompact space is strongly paracompact.

2.16 Theorem. Every strongly expandable screenable space is strongly paracompact.

Proof. Let X be a strongly expandable screenable space. Then by Theorem 2.8(i), X is an expandable screenable space and hence by Theorem 1.7, X is paracompact. Therefore, by Theorem 2.13, X is strongly paracompact.

2.17 Theorem. A normal almost χ_0-expandable screenable space is strongly paracompact.

Proof. Let X be a normal almost χ_0-expandable screenable space. Let $U=\{U_\alpha : \alpha\varepsilon A\}$ be an open cover of X. Since X is screenable, therefore, U has a σ-disjoint open refinement, say $V=\cup^{\infty}_{i=1} V_i$,where $V_i=\{V_{\alpha,i} : \alpha\varepsilon A_i\}$ for each i. Since each V_i is a pairwise disjoint open collection, therefore, each V_i is a σ-star-finite collection of open subsets of X. Therefore V is a σ-star-finite open refinement of U. Now since X is normal and almost χ_0-expandable, therefore, by theorem 2.10, X is normal strongly countably paracompact space and hence by Theorem 1.4 X is strongly paracompact.

2.18 Corollary. A normal almost expandable (respectively, expandable) screenable space is strongly paracompact.

Like the class of strongly expandable spaces the class of *Bd*-expandable spaces [6] also lies between the classes of countably compact and expandable spaces, but in the following example we show that these two concepts are different:

2.19 Example. A strongly expandable space which is not *Bd*-expandable.

For each positive integer n we define the compact subspace Q^n of E^n by $Q^n=\{(x_1, x_2,\dots,x_n): 0\leq x_i \leq 1/n \text{ for } i=1,2,\dots,n\}$ and we define X to be the discrete union of the topological spaces Q^n for $n=1,2\dots$ The space X clearly has a countable basis, Therefore, X is second countable and hence Lindelöf. Also a complete metric can be introduced into X which assigns the usual Euclidean distance between two points of the same Q^n and the distance 1 to any two points of two distinct subspaces Q^n. Since X is separable metric space, therefore, X is strongly paracompact and hence X is strongly expandable. Wenner [18, Example 3.1] has proved that X is not boundedly paracompact. Hence X cannot be *Bd*-expandable, because of the fact that a *Bd*-expandable paracompact space is boundedly paracompact.

Now from the above results together with other known results, we have the following implication diagram :

Compact ⇆ *strongly paracompact* ⇆ *paracompact*
⇅ ⇅ ⇅
Countably ⇆ *strongly expandable* ⇆ *expandable*
Compact
⇅ ⇅
Strongly χ_0*-expandable* χ_0⇆*expandable*
⇅ ⇅
Strongly countably ⇆ *countably paracompact*
Paracomact

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(Dedicated to the memory of Professor J.N. Kapur)

NEARLY METACOMPACT SPACES

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ABSTRACT

The notion of near metacompactness is introduced by generalizing the notion of metacompactness as well as that of near paracompactness. A topological space *X* is said to be nearly metacompact if every regular open cover of *X* has a point-finite open refinement. The class of nearly metacompact spaces contains the class of metacompact spaces as well as that of nearly paracompact spaces both. In collectionwise normal spaces the notion of near metacompactness is equivalent to that of near paracompactness. In semi-regular spaces the notion of near metacompactness coincide with that of metacompactness. The near metacompactness strengthens countably compactness to near compactness.

Introduction. In the present paper we introduce the notion of near metacompactness by generalizing the notion of metacompactness as well as that of near paracompactness. A topological space *X* is said to be nearly metacompact if every regular open cover of *X* has a point-finite open refinement. The class of nearly metacompact spaces contains the class of metacompact spaces as well as that of nearly paracompact spaces both. In collectionwise normal spaces the notion of near metacompactness is equivalent to that of near paracompactness. In semi-regular spaces the notion of near metacompactness is equivalent to that of metacompactness. The near metacompactness strengthens countably compactness to near compactness. The notion of near compactness is due to Singal and Mathur [5]. A topological space *X* is said to be nearly compact if every regular open cover of *X* has a finite subcover. The notion of near paracompactness is due to Singal and Arya [4]. A topological space X is said to be nearly paracompact if every regular open cover of X has a locally finite open refinement, the whole paper is divided into five sections. In section 1 the notion of near metacompactness is introduced and studied. In section 2 several counter examples are given which demonstrate the invalidity of several possible conjectures concerning nearly metacompact spaces. In section 3 nearly metacompact spaces and sum and product theorems are given. In section 4 nearly metacompact spaces and subset theorems are given. Lastly in section 3 nearly metacompact spaces and mapping theorems are given.

1. Nearly Metacompact Spaces

1.1 Definition . A topological space X is said to be nearly metacompact if every regular open cover of X has a point-finite open refinement.

1.2 Theorem. A topological space X is nearly metacompact iff every δ-open cover of X has a point-finite open refinement.

Proof. Obvious in view of the fact that regularly open sets form a base of δ-open sets.

1.3 Theorem . Every nearly paracompact space is nearly metacompact.

Proof. Obvious in view of the fact that every locally finite collection is point-finite.

1.4 Theorem. Every metacompact space is nearly metacompact.

Proof. Obvious in view of the fact that every regular open cover is also an open cover.

1.5 Theorem. A semi-regular space X is metacompact iff X is nearly metacompact.

Proof. If X is metacompact then by Theorem 1.4 above, X is nearly metacompact. So assume that X is a semi-regular nearly metacompact space. Let U be an open cover of X. Since in a semiregular space every open set is δ-open. therefore, U is also a δ-open cover of X. Since X is nearly metacompact, therefore, by Theorem 1.2, U has a point-finite open refinement. Hence X is metacompact.

Michael [2] has proved the following theorem:

1.6 Theorem. [2] Every point-finite open covering of a collectionwise normal space has a locally finite open refinement.

Using Theorem 1.6 above we prove the following theorem :

1.7 Theorem . A collectionwise normal space X is nearly paracompact iff

X is nearly metacompact.

Proof. Let X be a collectionwise normal space. If X is nearly paracompact then by Theorem 1.3. X is nearly metacompact. So assume that X is a nearly metacompact space. Let U be a regular open cover of X. Then by near metacompactness of X, U has a point-finite open refinement, say V. Now by Theorem 1.6 above, V has a locally finite open refinement, say W. Then W is a locally finite open refinement of U. Hence X is nearly paracompact.

The following theorem is due to Richard and Dugundji [3]:

1.8 Theorem. [3] Let $\{A_\alpha : \alpha \varepsilon \Delta\}$ be a point-finite covering of a topological space X. Then there exists an irreducible subcovering, that is, a subcovering that, when any single set is removed, is no longer a covering of X.

Using theorem 1.8 above we prove the following theorem:

1.9 Theorem. *Every countably compact nearly metacompact space is nearly compact.*

Proof. Let X be a countably compact nearly metacompact space. Let $U = \{U_\alpha : \alpha \varepsilon A\}$ be a regular open cover of X. By the near metacompactness of X, U has a point-finite open refinement, say $V : \{V_\beta : \beta \varepsilon B\}$. Then by Theorem 1.8 above, V has an irreducible subcovering, say $\{V_\gamma : \gamma \varepsilon \Gamma\}$. This minimal covering must be finite. For, we can find in each V_γ a point y_γ belonging to no set other than V_γ and if $V = \{V_\gamma : \gamma \varepsilon \Gamma\}$ is not finite then $\{y_\gamma : \gamma \varepsilon \Gamma\}$ is an infinite subset of X. Let Δ be a countably infinite subset of Γ. We may index the element of Δ as follows : $\Delta = \{\gamma_1, \gamma_2 \ldots\}$. Then clearly $\{y_{\gamma 1}, y_{\gamma 2}, \ldots\}$ is a sequence in X. Let $x \varepsilon X$. Then there exists a $\gamma \varepsilon \Gamma$ such that $x \varepsilon V\gamma$. Now if $\gamma \varepsilon \Delta$, then there is some positive integer i such that $y_{\gamma i} \varepsilon V_\gamma$. Now consider the positive integer $i + 1$, then clearly by the construction of the set $\{y_\gamma : \gamma \varepsilon \Gamma\}$ and the sequence $\{y_{\gamma 1}, y_{\gamma 2}, \ldots\}$ it follows that there is no positive interger $n \geq i+1$ such that $y_{\gamma n} \varepsilon V_\gamma$. and if $\gamma \notin \Delta$, then clearly V_γ contains no point of the sequence $\{y_{\gamma 1}, y_{\gamma 2}, \ldots\}$. Hence x is not an accumulation point of the sequence $\{y_{\gamma 1}, y_{\gamma 2}, \ldots\}$. From this it follows that the sequence $\{y_{\gamma 1}, y_{\gamma 2}, \ldots\}$ in X has no accumulation point which contradicts the fact that X is cuntably compact. Therefore, $\{V_\gamma . \gamma \varepsilon \Gamma\}$ is a finite open covering of X. Choose for each $\gamma \varepsilon \Gamma$ an $\alpha(\gamma) \varepsilon A$ such that $V_\gamma \subset U_{\alpha(\gamma)}$. Then $\{U_{\alpha(\gamma)} . \gamma \varepsilon \Gamma\}$ is a finite subcovering of U. Therefore, X is nearly compact.

2. Counter-Examples.

2.1 *Example.* Nearly metacompact space which is not metacompact. Let $X = \{a_{ij}, a_i, a : i, j = 1, 2, \ldots\}$. Let each a_{ij} be an isolated point. Let $\{U^n(a_i):$

$n=1,2,\ldots\}$ be the fundamental system of neighborhoods of a_i, where $\{U^n(a_i) = \{a_i, a_{1i}, a_{ij} : j \geq n\}$ for each $n=1,2,\ldots$. Let $\{V^n(a)\colon n=1,2,\ldots\}$ be the fundamental system of neighborhoods of a, where $V^n(a) = \{a, a_{11}\, a_{ij} : i \geq n, j \geq n\}$ for each $n=1,2,\ldots$. Then $\boldsymbol{B} = \{\{a_{ij}\}\ i,j = 1,2,\ldots\}\ U\ \{\{ U^n(a){:}n=1,2,\ldots\} : i = 1,2,\ldots\}\ U\ \{V^n(a) : n=1,2,\ldots\}$ is a base for a topology, say τ, on X. Then (X,τ) is a nearly metacompact space but it is not metacompact.

2.2 ***Example.*** Nearly metacompact normal Hausdorff space which is not *nearly paracompact.*

The Michael's modification [2] of the Bing's space [1] is an example of space which is nearly metacompact normal and Hausdorff but not nearly paracompact.

2.3 ***Example.*** A nearly metacompact space with a subset which is not *nearly metacompact.*

The Tychonoff plank $X = [0,\Omega] \times [0,\omega]$ is a nearly metacompact space but the deleted Tychonoff plank $Y = X - \{\Omega, \omega)\}$ is not a nearly metacompact space.

2.4 ***Example.*** A nearly metacompact space X such that $X \times X$ is not *nearly metacompact.*

The Sorgenfrey line S is nearly metacompact but the Sorgenfrey plank $S \times S$ is not nearly metacompact.

2.5 Note . Frolm the results of section 2.1 and 2.2 above together with other known results we have the following implication diagram:

$$\begin{tikzcd}
\textit{paracompact} \arrow[r, shift left] \arrow[d, shift left] & \textit{metacompact} \arrow[l, shift left, "/" marking] \arrow[d, shift right] \\
\textit{nearly paracompact} \arrow[u, shift left, "/" marking] \arrow[r, shift left] & \textit{nearly metacompact.} \arrow[l, shift left, "/" marking] \arrow[u, shift right, "/" marking]
\end{tikzcd}$$

3. Nearly Metacompact Spaces and Sum and Product Theorem

3.1 Theorem. The disjoint topological sum of nearly metacompact space is nearly metacompact.

Proof. Lef $\{X_\alpha\colon \alpha\varepsilon A\}$ be a disjoint family of nearly metacompact spaces. Let X denote the disjoint topological sum of this family. Let $\boldsymbol{U} = \{U_\beta :: \beta\varepsilon B\}$ be a regularly open cover of X. Then for each $\alpha\varepsilon A$, $\{U_\beta \cap X_\alpha : \beta\varepsilon B\}$ is an open cover of X_α and so $\{Int_{X\alpha}\, Cl_{X\alpha}(U_\beta \cap X_\alpha) : \beta\varepsilon B\}$ is a regularly open cover of X_α Since X_α is nearly metacompact, therefore $\{Int_{X\alpha}\, Cl_{X\alpha}(U_\beta \cap X_\alpha) : \beta\varepsilon B\}$ has a point-finite open *(in X_α)*, refinement, say $\boldsymbol{V}_\alpha$ .Then surely $\boldsymbol{U} = \cup_{\alpha\varepsilon A} V_\alpha$ is a point-finite open refinement of $\boldsymbol{U}$. Hence X is nearly metacompact.

3.2 Theorem. Let $\{G_\alpha : \alpha\varepsilon A\}$ be a family of subsets of topological space X

such that $\{Int_X G_\alpha : \alpha\varepsilon A\}$ forms a point-finite open covering of X. If each G_α is nearly metacompact, then X is nearly metacompact.

Proof. Let $\{U_\beta : \beta\varepsilon B\}$ be any regular open covering of X. Then for each $\alpha\varepsilon A$, $\{U_\beta : \beta\varepsilon B\}$ is a relatively open covering of G_α. Then $\{Int_{G\alpha}\, Cl_{G\alpha}(U_\beta \cap G_\alpha) : \beta\varepsilon B\}$. is a regularly open (in G_α) cover of G_α for each $\alpha\varepsilon A$. Since G_α is nearly metacompact, therefore $\{Int_{G\alpha}\, Cl_{G\alpha}(U_\beta \cap G_\alpha) : \beta\varepsilon B\}$ has a point-finite (in G_α) open (in G_α) refinement, say $\{V_\beta \cap G_\alpha : \gamma\varepsilon\Gamma_\alpha\}$ where each V_α is open in X. Then surely $\{V_\gamma \cap Int_x G_\alpha : \alpha\varepsilon A\}$ is a point-finite open refinement of $\{U_\beta : \beta\varepsilon B\}$. Hence X is nearly metacompact.

3.3 Corollary. Let $\{G_\alpha : \alpha\varepsilon A\}$ be a point - finite open covering of a topological space X. If each G_α is nearly metacompact, then X is nearly metacompact.

3.4 Theorem. The product of a nearly metacompact space with a nearly compact space is nearly metacompact.

Proof. Let X be a nearly metacompact and Y a nearly compact space. Let $\mathbf{U}$ be a regularly open cover of $X \times Y$. Let $(x,y)\ \varepsilon\ X \times Y$. Then there exists regularly open subset V_{xy} and W_{xy} of X and Y respectively, such that $(x,y)\ \varepsilon\ V_{xy}\, W_{xy} \subset U$ for some $U\varepsilon\ \mathbf{U}$. Let $I^x = \{x\}\times Y$ for each $x\ \varepsilon\ X$. Then $\{W_{xy} : (x,y)\ \varepsilon\ I^x\}$ is a regularly open covering of the nearly compact space Y and therefore there exists a finite subset J^x of I^x such that $\{W_{xy} : (x,y)\ \varepsilon\ J^x\}$ is a covering of Y. For each $x\varepsilon X$ let $V_x = \cap\ \{V_{xy}\ (x,y)\ \varepsilon\ J^x\}$. Then V_x is a regularly open subset of X containing x. Let $\mathbf{V}\ \{V_x : x\varepsilon\ X\}$. Then $\mathbf{V}$ is a regularly open covering of X. Since X is nearly metacompact, therefore, V has a point -finite open (in X) refinement, say $\mathbf{G}$. Now for each $G\varepsilon\ \mathbf{G}$ there exists $x_G\,\varepsilon\, X$ such that $G \subset V_{x_G}$. Now let $\mathbf{H} = \{G \cap W_{xy} : G\varepsilon \mathbf{G},\ (x,y)\varepsilon J^x,\ x\,\varepsilon\, X\}$. Then surely $\mathbf{H}$ is a point finite open refinement of $\mathbf{U}$. Hence $X \times Y$ is nearly metacompact.

3.5 Corollary. The product of a nearly metacompact space with a compact space is nearly metacompact.

4. Nearly Metacompact Spaces and Subset Theorem.

4.1 Theorem. Every clo-open subset of nearly metacompact space is nearly metacompact.

Proof. Let X be a nearly metacompact space and Y be a clo-open subset of X. Let $\mathbf{U} = \{U_\alpha : \alpha\varepsilon A\}$ be a cover of Y by regularly open subsets of Y. Since regularly open subsets of a clo-open subset Y of X is a regularly open subset of X, therefore, each U_α is a regularly open subset of X. Also clearly $X-Y$ is a regularly open subset of X. Therefore, $\mathbf{U} \cup \{X-Y\}$ is a regularly open cover of X. Since X is nearly metacompact, therefore, there exists a point-finite open (in X) refinement, say $V=\{V_\beta : \beta \varepsilon B\}$ of $U \cup \{X-Y\}$. Then surely $V \cap Y = \{V_\beta \cap Y : \beta \varepsilon B\}$ is a point-finite (in Y) open (in Y) refinement of $\boldsymbol{U}$. Therefore, Y is nearly metacompact.

5. Nearly Metacompact Space and Mapping Theorem.

5.1 Theorem . Let $f : X \rightarrow Y$ be an almost conitinuous open surjection such that $f^{-1}(y)$ is a finite subset of X for each $y \varepsilon Y$. Then Y is nearly metacompact, if X is nearly metacompact.

Proof. Let X be a nearly metacompact space. Let $\boldsymbol{U} = \{U_\alpha : \alpha \varepsilon A\}$ be a regularly open cover of Y. Since f is almost continuous open surjection and hence is almost continuous almost open surjection, therefore, $f^{-1}(U) = \{f^{-1}(U_\alpha) : \alpha \varepsilon A\}$ is a regularly open cover of X. Therefore, there exists a point-finite open refinement, say $\boldsymbol{V} = \{V_\beta : \beta \varepsilon B\}$ of $f^{-1}(U)$. Then surely $f(V) = \{f(V_\beta) : \beta \varepsilon B\}$ is a point - finite open refinement of $\boldsymbol{U}$. Hence Y is nearly metacompact.

5.2 Theorem. Let $f : X \rightarrow Y$ be a continuous δ - closed surjection such that $f^{-1}(y)$ is N-closed for each $y \varepsilon Y$. Then X is nearly metacompact, if Y is nearly metacompact.

Proof. Let Y be a nearly metacompact space. Let $\boldsymbol{U} = \{U_\alpha : \alpha \varepsilon A\}$ be a regularly open cover of X. Let Γ be that collection of all finite subsets of A. Let $V_\gamma = Y - f(X - \cup_{\alpha \varepsilon \gamma} U_\alpha)$ for each $\gamma \varepsilon \Gamma$. Then surely $\boldsymbol{V} = \{V_\gamma : \gamma \varepsilon \Gamma\}$ is a δ-open cover of Y such that $f^{-1}(V_\gamma) \subset \cup_{\alpha \varepsilon \gamma} U_\alpha)$ for each $\gamma \varepsilon \Gamma$. Since $\boldsymbol{Y}$ is nearly metacompact. therefore, $\boldsymbol{V}$ has a point-finite open refinement, say $W = \{W\delta : \delta \varepsilon A\}$. Since for each $\delta \varepsilon A$ there is $\gamma_\delta \varepsilon \Gamma$ such that $W_\delta \subset V_{\gamma\delta}$ therefore, for each $\delta \varepsilon \Delta$, there exists a $y_\delta \varepsilon \Gamma$ such that $f^{-1}(W_\delta) \subset f^{1}(V_{\gamma\delta}) \subset \cup_{\alpha \varepsilon \gamma\delta} U_\alpha$ Then surely $\{f^{-1}(W_\delta) \cap U_\alpha : \alpha \varepsilon \gamma_\delta, \delta \varepsilon \Delta$ is a point- finite open refinement of U. Hence X is nearly metacompat.

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COMMON FIXED POINTS OF COMPATIBLE MAPPINGS OF TYPE (α)

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ABSTRACT

In this paper we shall prove a common fixed point theorem for six mappings in complete metric space. Our theorem generalizes results of Banach [8], Kannan [7], Fisher [1], Chatterjee [9] and Cho [14].

1. Introduction. Sessa [12] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [2] introduced more generalized commutativity, so called compatibility, which is more general then that of weak commutativity. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae [11]. Also, Jungck [3] extended the results of Khan and Imdad [5] and proved common fixed point theorem for four mappings by using one of the mappings continuous and employing condition of compatible mappings. Kang, Cho and Jungck [10] extended the results of Ding [13] and Diviccaro and Sessa [6].

In 1993, Jungck, Murthy and Cho [4] introduced the concept of compatible mappings of type (α) in metric spaces. In this paper, we prove a common fixed point theorem for six mappings under the condition of compatible mappings of type (α), in complete metric space.

2.Definitions. In this section, we give some definitions.

Definition 2.1. A sequence $\{x_n\}$ in a metric space (X,d) is said to be convergent to a point x in X if $\lim_{n\to\infty} d(x_n,x)=0$.

Definition 2.2. A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Definition 2.3. A metric space (X, d) is said to be complete if every Cauchy sequency in X is convergent.

Definition 2.4 [2]. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be compatible. if

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0,$$

where $\{x_n\}$ is a sequence in x such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Definition 2.5[4]. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be compatible of type (α) if

$$\lim_{n\to\infty} d(ABx_n, BBx_n) = 0$$

and

$$\lim_{n\to\infty} d(ABx_n, AAx_n) = 0,$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Proposition 2.1[4]. Let (X, d) be a metric space. Let A and B be continuous mappings from X into itself. Then A and B are compatible if and only if they are compatible of type (α).

Proposition 2.2 [4]. Let (X,d) be a nietric space and A and B are mappings from X into itself. If A and B are compatible of type (α) and $Az = Bz$ for some $z \in X$, then

$$ABz = BBz = BAz = AAz.$$

Proposition 2.3 [4]. Let (X, d) be a metric space and A and B are mappings from X into itself. If A and B are compatible of type (α) and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X. \text{ then}$$

(i) $\lim_{n\to\infty} BAx_n = Az$ *if* A is continuous at z,

(ii) $ABz = BAz$ and $Az = Bz$, if A and B are continuous at z.

3. Common fixed point theorem.

Theorem 3.1. Let (X,d) be a complete metric space. Let $A, B, S, T, P,$ and Q, be mappings from X into itself such that

(3.1) $P(X) \subset AB(X)$ and $Q(X) \subset ST(X)$,

(3.2) $AB = BA$, $ST = TS$, $PA = AP$, $BP = PB$, $QS = SQ$, $QT = TQ$,

(3.3) A, B, S, and T are continuous

(3.4) the pair $\{P, AB\}$ and $\{Q, ST\}$ are compatible of type (α),

(3.5) $$d(Px, Qy) \leq \alpha_1 \left[\frac{d(STy, Qy)\, d(ABx, ABy) + d(ABy, Qy)\, d(ABx, Px)}{d(ABx, Qy) + d(ABx, STy)}\right]$$
$$+ \alpha_2 [d(STy, Px) + d(STy, Qy) + d(ABx, STy)]$$
$$+ \alpha_3 [d(ABx, Qy) + d(ABy, Px) + d(STy, ABy)]$$
$$+ \alpha_4 [d(ABx, Px) + d(ABy, Qy)],$$

for all $x, y \in X$, where $\alpha_i \geq 0$ and $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Then A,B,S,T,P and Q have a unique common fxied point in X.

Proof. By (3.1), since $P(X) \subset AB(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$. Since $Q(X) \subset ST(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Qx_1 = STx_2$. Inductively, we can define a sequence $\{y_n\}$ in X as follows

$$y_{2n} = Px_{2n} = ABx_{2n+1},$$
$$y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$$

for $n = 0, 1, 2, \ldots$.

By putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (3.5) we write

$$d(Px_{2n+1}, Qx_{2n+2})$$
$$\leq \alpha_1 \left[\frac{d(STx_{2n+2}, Qx_{2n+2})\, d(ABx_{2n+1}, ABx_{2n+2}) + d(ABx_{2n+2}, Qx_{2n+2})\, d(ABx_{2n+1}, Px_{2n+1})}{d(ABx_{2n+1}, Qx_{2n+2}) + d(ABx_{2n+1}, STx_{2n+2})}\right]$$
$$+ \alpha_2 [d(STx_{2n+2}, Px_{2n+1}) + d(STx_{2n+2}, Qx_{2n+2}) + d(ABx_{2n+1}, STx_{2n+2})]$$
$$+ \alpha_3 [d(ABx_{2n+1}, Qx_{2n+2}) + d(ABx_{2n+2}, Px_{2n+1}) + d(STx_{2n+2}, ABx_{2n+2})]$$
$$+ \alpha_4 [d(ABx_{2n+1}, Px_{2n+1}) + d(ABx_{2n+2}, Qx_{2n+2})].$$

$$d(y_{2n+1}, y_{2n+2})$$
$$\leq \alpha_1 \left[\frac{d(y_{2n+1}, y_{2n+2})\, d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})\, d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n+2}) + d(y_{2n}, y_{2n+1})}\right]$$
$$+ \alpha_2 [d(y_{2n+1}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1})]$$
$$+ \alpha_3 [d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1}) + d(y_{2n+1}, y_{2n+1})]$$
$$+ \alpha_4 [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})].$$

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{(2\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_2-\alpha_3-\alpha_4)} d(y_{2n}, y_{2n+1}).$$

Putting $h = \dfrac{(2\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_2+\alpha_3+\alpha_4)}$,

we find $h < 1$, *since* $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Hence

$$d(y_{2n+1}, y_{2n+2}) \leq hd(y_{2n}, y_{2n+1}).$$

Similarly we can see that

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

Proceeding in this way, we have

$$d(y_{2n}, y_{2n+1}) \leq h^{2n-1}(y_0, y_1).$$

By routine caculations the following inequalities hold for k > n

$$\begin{aligned} d(y_n, y_{n+k}) &\leq \sum_{i=1}^{k} d(y_{n+i-1}, y_{n+i}) \\ &\leq \sum_{i=1}^{k} h^{n+i-1} d(y_0, y_1) \\ &\leq \frac{h^n}{1-h}\ d(y_0, y_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Here $h < 1$. Hence $\{y_n\}$ is a Cauchy sequence and by completeness of X we see that $\{y_n\}$ is convergent to a point z in X.

Since Px_{2n}, Qx_{2n+1}, ABx_{2n+1} and STx_{2n+2} are subsequences of $\{y_n\}$, they also converge to a point z, that is as $n \to \infty$,

$$Px_{2n},\ Qx_{2n+1}, STx_{2n+1}, \to z.$$

Since A and B are continuous and the pair $\{P, AB\}$ is compatible of type (α), by Proposition (2.3) we have, as $n \to \infty$

$$P(AB)x_{2n+1} \to ABz,\ (AB)^2x_{2n+1} \to ABz.$$

Now Putting $x = ABx_{2n+1}$ and $y = x_{2n+2}$ in (3.5), we write

$$\begin{aligned} &d(P(AB)x_{2n+1}, Qx_{2n+2}) \\ &\leq \alpha_1 \left[\frac{d(STx_{2n+2}, Qx_{2n+1})\, d((AB)^2x_{2n+1}, ABx_{2n+2}) + d(ABx_{2n+2}, Qx_{2n+2})\, d((AB)^2x_{2n+1}, P(AB)x_{2n+1})}{d((AB)^2x_{2n+1}, Qx_{2n+2}) + d((AB)^2x_{2n+1}, STx_{2n+2})} \right] \\ &\quad + \alpha_2 [d(STx_{2n+2}, P(AB)x_{2n+2}) + d(STx_{2n+1}, Qx_{2n+2}) + d((AB)^2x_{2n+1}, STx_{2n+2})] \\ &\quad + \alpha_3 [d((AB)^2x_{2n+1}, Qx_{2n+2}) + d(ABx_{2n+2}, P(AB)x_{2n+1}) + d(STx_{2n+2}, ABx_{2n+2})] \\ &\quad + \alpha_4 [d((AB)^2x_{2n+1}, P(AB)x_{2n+1}) + d(ABx_{2n+2}, Qx_{2n+2})] \end{aligned}$$

which implies that, as $n \to \infty$

$$d(ABz, z) \leq (2\alpha_2 + 2\alpha_3)\, d\,(ABz, z),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore, we have $ABz = z$.

By putting $x = Px_{2n}$ and $y = x_{2n+1}$ in (3.5), we write

$$d\,(P(Px_{2n}), Qx_{2n+1})$$

$$\leq \alpha_1 \left[\frac{d(STx_{2n+1}, Qx_{2n+1})\, d((AB(Px_{2n}), ABx_{2n+1}) + d(ABx_{2n+1}, Qx_{2n+1})\, d((AB(Px_{2n}), P(Px_{2n}))}{d((AB(Px_{2n}), Qx_{2n+1}) + d(AB(Px_{2n}), STx_{2n+1})} \right]$$

$$+ \alpha_2\, [d(STx_{2n+1}, P(Px_{2n})) + d(STx_{2n+1}, Qx_{2n+1}) + d((AB(Px_{2n}), STx_{2n+1})]$$
$$+ \alpha_3\, [d(AB(Px_{2n}), Qx_{2n+1}) + d(ABx_{2n+1}, P(Px_{2n})) + d(STx_{2n+1}, ABx_{2n+1})]$$
$$+ \alpha_4\, [d(AB)(Px_{2n}), P(Px_{2n}) + d(ABx_{2n+1}, Qx_{2n+1})\,].$$

Talking the limit, $n \to \infty$, we have

$$d(Pz, z) \leq (2\alpha_2 + 2\alpha_3 + \alpha_4)\, d(Pz, z),$$

which is a contradicition $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore, we have $Pz = z. = ABz$.

Now we show that $Bz = z$. By putting $x = Bz$ and $y = x_{2n+1}$ in (3.5), we write

$$d(P(Bz), Qx_{2n+1})$$

$$\leq \alpha_1 \left[\frac{d(STx_{2n+1}, Qx_{2n+1})\, d((AB(Bz), ABx_{2n+1}) + d(ABx_{2n+1}, Qx_{2n+1})\, d(AB(Bz), P(Bz))}{d((AB(Bz), Qx_{2n+1}) + d((AB(Bz), STx_{2n+1})} \right]$$

$$+ \alpha_2\, [d(STx_{2n+1}, P(Bz)) + d(STx_{2n+1}, Qx_{2n+1}) + d(AB(Bz), STx_{2n+1})]$$
$$+ \alpha_3\, [d(AB(Bz), Qx_{2n+1}) + d(ABx_{2n+1}, P(Bz)) + d(STx_{2n+1}, ABx_{2n+1})]$$
$$+ \alpha_4 [d(AB)(Bz), P(Bz) + d(ABx_{2n+1}, Qx_{2n+1})].$$

Taking the limit $n \to \infty$ we have

$$d(Bz, z) \leq (2\alpha_2 + 2\alpha_3)\, d(Bz, z),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$,

which implies that $Bz = z$, since $ABz = z$, $Az = z$.

Now by putting $x = z$ and $y = STx_{2n+2}$ in (3.5) we write

$$d(Pz, Q(ST)x_{2n+2})$$

$$\leq \alpha_1 \left[\frac{d((ST)^2x_{2n+2}, Q(ST)x_{2n+2})\, d((AB), AB(ST)x_{2n+2}) + d(AB(ST)x_{2n+2}, Q(ST)x_{2n+2})\, d(ABz, Pz)}{d(ABz), Q(ST)x_{2n+2}) + d((ABz, (ST)^2x_{2n+2})} \right]$$

$$+\alpha_2 [d((ST)^2 x_{2n+2}, Pz) + d((ST)^2 x_{2n+2}, Q(ST)x_{2n+2}) + d((ABz,(ST)^2 x_{2n+2})]$$
$$+\alpha_3 [d(ABz,Q(ST)x_{2n+2}) + d(AB(ST)x_{2n+2}, Pz) + d((ST)^2 x_{2n+2}, AB(ST)x_{2n+2})]$$
$$+\alpha_4 [d(ABz, Pz) + d(AB(ST)x_{2n+2}, Q(ST)x_{2n+2})],$$

which implies that , as $n \to \infty$

$$d(z, STz) \leq (2\alpha_2 + 2\alpha_3 + \alpha_4)\, d(z, STz),$$

which is a contradiction, since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore we have $STz = z$.

Now putting $x = z$ and $y = Qx_{2n+1}$ in (3.5), we write

$$d(Pz, Qx_{2n+1})$$
$$\leq \alpha_1 \left[\frac{d(ST(Qx_{2n+1}), Q(Qx_{2n+1}))\, d(ABz, AB(Qx_{2n+1})) + d(AB(Qx_{2n+1}), Q(Qx_{2n+1}))\, d(ABz), Pz)}{d(ABz, Q(Qx_{2n+1}) + d(ABz, ST(Qx_{2n+1}))} \right]$$
$$+\alpha_2 [d(ST(Qx_{2n+1}), Pz) + d(ST(Qx_{2n+1}, Q(Qx_{2n+1})) + d(ABz, ST(Qx_{2n+1}))]$$
$$+\alpha_3 [d(ABz, Q(Qx_{2n+1})) + d(AB(Qx_{2n+1}), Pz)) + d(ST(Qx_{2n+1}), AB(Qx_{2n+1}))]$$
$$+\alpha_4 [d(ABz, Pz) + d(AB(Qx_{2n+1}), Q(Qx_{2n+1}))].$$

Taking the limit $n \to \infty$ we have

$$d(z, Qz) \leq (\alpha_2 + \alpha_3 + \alpha_4)\, d(z, Qz),$$

which is a contradiction, since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore, we have $Qz = z$ and hence $STz = z = Qz$.

Finally we show that $Tz = z$. By puting $x = z$ and $y = Tz$ in, (3.5), we write

$$d(Pz, Q(Tz))$$
$$\leq \alpha_1 \left[\frac{d(ST(Tz), Q(Tz)), d((ABz, AB(Tz)) + d(AB(Tz), Q(Tz))\, d(ABz, Pz)}{d(ABz, Q(Tz)) + d(ABz, ST(Tz))} \right]$$
$$+\alpha_2 [d(ST(Tz), Pz) + d((ST(Tz), Q(Tz)) + d(ABz, ST(Tz))]$$
$$+\alpha_3 [d(ABz, Q(Tz)) + d(AB(Tz), Pz)) + d(ST(Tz), AB(Tz))]$$
$$+\alpha_4 [d(ABz, Pz) + d(AB(Tz), Q(Tz))]$$

which implies that

$$d(z, Tz) < (2\alpha_2 + 2\alpha_3)\, d(z, Tz).$$

Therefore we have $Tz = z$. Since $STz = z$, we have $Sz = z$.

Therefore by combining the above results, we have

$$Az = Bz = Sz = Tz = Pz = Qz = z,$$

that is z is common fixed point of A, B, S, T, P and Q.

For uniqueness let $w (w \neq z)$ be another common fixed point A, B, S, T, P

and Q, then by (3.5), we write

$$d(Pz, Qw)$$

$$\leq \alpha_1 \left[\frac{d(STw,Qw), d((ABz,ABw) + d(ABw, Qw)\, d(ABz,Pz)}{d(ABz, Qw) + d(ABz, STw)} \right]$$

$$+ \alpha_2 \; [d(STw, Pz) + d((STw, Qw) + d(ABz,STw)]$$

$$+ \alpha_3 \; [d(ABz, Qw) + d(ABw,Pz)) + d(STw, ABw),]$$

$$+ \alpha_4 \; [d(ABz, Pz) + d(ABw, Qw)]$$

which implies that

$$d(z, w) < (2\alpha_2 + 2\alpha_3)\, d(z, w)$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$

Therefore, we have $z = w$.

This completes the proof of the theorem.

Remark 1. Taking $AB = ST = I$ (I is the identity map on X) and $P = Q$ and $\alpha_1 = \alpha_2 = \alpha_4 = 0$ in Theorem 3. 1, we obtain result due to Fisher [1].

Remark 2. Taking $AB = I$, $P = Q$ and in $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in Theorem 3 .1 we obtain result due to Kannan [7].

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(Dedicated to the memory of Professor J.N. Kapur)

COMMON FIXED POINT THEOREMS OF DENSIFYING MAPPING IN HILBERT SPACE

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ABSTRACT

In this paper, we have obtained some fixed point theorems of densifying mapping in Hilbert space, which are generalizations of known results of [1], [2], [4] and others.

Introduction. An inner product space is called a Hilbert space if the normed space induced by the inner product space is a Banach space.

Most of the author's studied the concept of densifying single mapping pair of mappings in metric spaces, and have obtained their fixed points.

In this paper, we have studied and have extended these results for densifying mappings in Hilbert spaces and to obtain their fixed points by using more generalized contractive mappings.

This paper is inspired by the results of Furi and Vignoli [4], Khan [6] Ahmed and Sakil [1], Sharma and Shrivastava [6], Ansari and Namdeo [2], Iseki [5].

Our Main Results

Theorem 1. Let S and T continuous densifying selfmaps of a Hilbert Space H satisfying

$$\|Sx-Ty\|^2 \le \infty\{\|x-Sx\| \, y-Ty\|+\|x-Ty\| \, \|y-Sx\|\}+\alpha_2\{\|x-Sx\| \, x-y\|+\|x-Ty\| \, \|y-Ty\|\}+\alpha_3\{\|x-Sx\| \, \|Sx-Ty\| \, \|y-Ty\|+\|Sx-Ty\|\}+\alpha_4\|x-y\| \, \|Sx-Ty\| \qquad ...(1)$$

For all distinct $x, y \in H$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ge 0$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4 < 1$, $\alpha_1+\alpha_4 < 1$, if for some x_0 in H.

The sequence $\{x_n\}$ defined as $Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$, for $n=0,1,2,...$ is bounded, then either S or T has a fixed point in H. Further if z is a common fixed point S and T then it is unique fixed point of S and T.

Proof. Consider the set $B = \overset{\infty}{\underset{n=0}{U}} \{x_{2n}\}$ which is bounded and

$TS(B) = \overset{\infty}{\underset{n=0}{U}} \{x_{2n+2}\}$

and $TS(B) \subset B$ the continuity of S and T implies that

$TS(\bar{B}) \subset \overline{TS(B)} \subset \bar{B}$

Therefore $\bar{B}$ is invariant under TS and is bounded.

To prove $\bar{B}$ compact it is sufficient to prove $\alpha(B) = 0$.

As B is bounded $\alpha(B)$ is finite.

Let $\alpha(B) > 0$.

$$\begin{aligned} \text{Then } \alpha(B) &= \max\{\alpha(x_0), \alpha(TS(B)\} \\ &= \alpha(TS(B)) \\ &< \alpha(B). \end{aligned}$$

A contradiction. This shows that $\alpha(B) = 0$ and hence $\bar{B}$ is compact.

Now we define a non-negative real valued function f on $\bar{B}$ by

$$f(x) = \|Sx - x\|^2 \qquad ...(2)$$

for all x in $\bar{B}$.

Here f being the composite of continuous functions $\|.\|$ and S is continuous on compact subset $\bar{B}$ of H.

Hence $f(x)$ has a infimum in $\bar{B}$.

Therefore there exists $z \in \bar{B}$ such that

$f(z) = inf\{fx) : x \in \bar{B}\}$.

Suppose that $Sz \neq z$, $TSz \neq Sz$ then using (1) and (2), we have

$$\begin{aligned} f(TSz) &= \|STs_z - Ts_z\|^2 \\ &\le \alpha_1\{\|TSz - STsz\|\,\|Sz - TSz\| + \|TSz - Tsz\|\,\|TSz - STsz\|\} \\ &+ \alpha_2\{\|TSz - STsz\|\,\|Sz - TSz\| + \|TSz - Tsz\|\,\|Sz - Tsz\|\} \\ &+ \alpha_3\{\|TSz - STsz\|\,\|STsz - TSz\| + \|Sz - Tsz\|\,\|STsz - Tsz\|\} \\ &+ \alpha_4\{\|TSz - Sz\|\,\|STsz - TSz\|\}. \end{aligned}$$

$$\begin{aligned} \|STs_z - Ts_z\|^2 &\le \alpha_1\{\|TSz - STsz\|\,\|Sz - TSz\| + \alpha_2\{\|TSz - STsz\| \\ &\|Sz - TSz\| + \alpha_3\{\|TSz - STsz\|\,\|STsz - TSz\| + \|Sz - Tsz\| \\ &\|STsz - Tsz\|\} + \alpha_4\{\|TSz - Sz\|\,\|STsz - TSz\|\} \\ \le \alpha_1\{\|TSz - STsz\|\,\|Sz - TSz\| &+ \alpha_2\|TSz - STsz\|\,\|Sz - TSz\| \\ &+ \alpha_3\|TSz - STsz\|^2 + \alpha_3\{\|Sz - Tsz\|\,\|STsz - TSz\| \\ &+ \alpha_4\|TSz - Sz\|\,\|\upsilon STsz - TSz\| \end{aligned}$$

$$\leq (\alpha_1+\alpha_2+\alpha_3+\alpha_4) + \|Sz-Tsz\| \; \|STsz-TSz\| + \alpha_3\|Sz-Tsz\|^2 .$$

$$(1-\alpha_3)\|STSz-Tsz\|^2 \leq (\alpha_1+\alpha_2+\alpha_3+\alpha_4) \, \|Sz-Tsz\| \, \|STSz-Tsz\|.$$

$$\|STsz-Tsz\| \leq \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_3)} \|Sz-Tsz\|$$

$$< \|Sz-Tsz\| \qquad \text{as } \alpha_1+\alpha_2+2\alpha_3+\alpha_4 < 1$$

$$< \| z-Sz\|$$

i.e. $f(TSz) < f(z)$.

Hence $z = Sz$.

$z = Sz$ indicating that z is a fixed point of S

or $TSz = Sz$ in this case Sz is a fixed point of T.

Hence either S or T must have a fixed point.

Uniqueness. Let z be a common fixed point of S and T suppose w be another fixed point of S and T.

We have

$$\|w-z\|^2 = \|Sw-Tz\|^2$$

$$\leq \alpha_1 \|w-Sw\| \, \|z-Tz\| + \|w-Tz\| \, \|z-Sw\|\} + \alpha_2 \{\|w-Sw\| \, \|w-z\| + \|w-Tz\| \, \|z-Tz\|\}$$

$$+\alpha_3 \{\|w-Sw\| \, \|Sw-Tz\| + \|z-Tz\| \, \|Sw-Tz\|\} + \alpha_4 \{\|w-z\| \, \|Sw-Tz\|,$$

$$\|w-z\|^2 \leq \alpha_1 \|w-z\| \, \|z-w\| + \alpha_4 \|w-z\| \, \|w-z\|,$$

$$\|w-z\|^2 \leq (\alpha_1+\alpha_4) \|w-z\|^2 ,$$

$$\|w-z\|^2 < \|w-z\|^2 , \qquad \text{as } (\alpha_1+\alpha_4) < 1$$

which is contradiction and which shows that $w = z$.

Hence z is the unique fixed point of S.

Similarly z is the unique fixed point of T.

This completes the proof.

Corollary-1. Let S and T be continuous self mappings of a compact Hilbert space H satisfying (1). Then S and T have a unique commmon fixed point in H.

Theorem 2. Let S and T be two commuting continuous densifying self mappings of a bounded Hilbert space H satisfying

$$\|Sx-STy\|^2 \leq a \{\|x-Sx\| \, \|Ty-STy\| + b\|x-Sx\| \, \|Sx-Ty\| + c\|x-Ty\| \, \|Sx-Ty\| + d \, \|x-Ty\| \, \|Ty-STy\| + e\|Sx-Ty\| \, \|Ty-STy\| \qquad \text{...(1)}$$

For all $x, y \in H$, where a,b,c,d,e are non negative real constants with $a+d < 1$, $c < 1$.

Then S and T have a unique common fixed point in H.

Proof. Let x_0 be an arbitrary point in H and the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$, $n = 0,1,2\ldots$.

Let $B = \bigcup_{n=0}^{\infty}\{x_{2n+1}\}$ and $ST\,(B) \bigcup_{n=1}^{\infty}\{x_{2n+1}\}$,

so that $ST\,(B) \subset B$.

As S and T are cotinuous $ST\,(\bar{B}) \subset \overline{TS\,(B)} \subset \bar{B}$,

and hence $\bar{B}$ is invariant under ST and is bounded.

To prove $\bar{B}$ is compact, it is sufficient to prove $\alpha\,(B) = 0$.

As B is bounded, $\alpha\,(B)$ is finite.

Let $\alpha\,(B) > 0$.

Then $B = \{x_i\} < \cup\, ST\,(B)$

$(B)\ max\ \{\alpha\,(x_i),\ \alpha\,(\,TS\,(B)\,)\,\}$

$= < \alpha\,(ST(B))$

$< \alpha\,(B)$,

which is contradiction as S and T are densifying mappings.

Hence $\alpha\,(B) = 0$ and thus $\bar{B}$ is compact,

define a non-negative real valued fuinction f on H by

$$f(x) = \|Tx - STx\|^2, \text{ for all } x \in \bar{B}$$

then f being the composite of continuous functions $\|\,.\,\|$ S and T, is continuous in compact space $\bar{B}$ and hence there exists $w \in \bar{B}$ such that

$$f(w\,) = inf\ \{\{fx\}: x \in \bar{B}\}.$$

Suppose that $Sw \neq w$, then using (1), (2) and parallelogram law, we have

$$f\,(SW) = \|TSw - STSw\|^2$$

$$= \|STw - STSw\|^2$$

$\leq a\,\|Tw - STw\|\,\|TSw - STSw\| + b\|Tw - STw\|\,\|STw - TSw\|$

$+ c\,\|Tw - TSw\|\,\|STw - TSw\| + d\,\|Tw - TSw\|\,\|TSw - STSw\|$

$+ e\,\|STw - TSw\|\,\|TSw - STSw\|$

$\leq a\,\|Tw - STw\|\,\|TSw - STSw\| + d\|Tw - TSw\|\,\|TSw - STSw\|$

$\|STw - STSw\|^2 \leq (a+d)\|Tw - STw\|\ \|TSw - STSw\|$.

Therefore $\|STw - STSw\| \leq (a+d)\,\|Tw - STw\|$

$< \|Tw - STw\|$ as $a+d < 1$.

A contradiction, therefore $Sw = w$ and hence w is a fixed point of S.

We have $STw - TSw = Tw$.

Now we shall prove that $Tw = w$.

For this suppose that $Tw \neq w$ then by(1), we have

$$\|w - Tw\|^2 = \|Sw - STw\|^2$$

$\leq a\,\|w - Sw\|\,\|Tw - STw\| + b\,\|w - Sw\|\ \|Sw - Tw\|$

$$+\,c\|w-Tw\|\,\|Sw-Tw\|+d\,\|w-Tw\|\,\|Tw-STw\|$$

$$+\,e\,\|Sw-Tw\|\,\|Tw-STw\|,$$

$$\|Sw-STw\|^2\leq c\|w-Tw\|\,\|Sw-Tw\|.$$

$$\leq c\,\|w-Tw\|\,\|w-Tw\|$$

$$\leq c\,\|w-Tw\|^2,$$

$$<\|w-Tw\|^2 \text{ as } c<1\,.$$

This contradiction proves that $Tw=w$.

Uniqueness . To prove the uniqueness if possible let z be another fixed point of $T\,(w\neq z)$ then by (1), we have

$$\|w-z\|^2=\|Sw-STz\|^2$$

$$\leq a\,\|w-Sw\|\,\|Tz-STz\|+b\|\,w-Sw\|\,\|Sw-Tz\|+c\,\|w-Tz\|$$

$$\|Sw-Tz\|+d\,\|w-Tz\|\,\|Tz-STz\|+e\,\|Sw-Tz\|\,\|Tz-Tz\|$$

$$\leq a\,\|w-w\|\,\|z-z\|+b\,\|w-w\|\,\|w-z\|+c\|\,w-z\|\,\|w-z\|$$

$$+d\,\|w-z\|\,\|z-z\|+e\,\|w-z\|\,\|z-z\|$$

$$+\,e\,\|w-z\|\,\|z-z\|$$

$$\leq c\,\|w-z\|^2$$

$$<\|w-z\|^2 \qquad \text{as } c<1\,.$$

A contradiction. Therefore w is a unique common fixed point of S and T.

Corollary. Let S and T be two commuting self mappings of a compact Hilbert space H satisifying the condition (1) of above theorem. Then S and T have a unique common fixed point in H.

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(Dedicated to the memory of Professor J.N. Kapur)

COMMON FIXED POINT THEOREM IN REFLEXIVE BANACH SPACES

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ABSTRACT

In this paper few common fixed point theorems of mappings which map a reflexive Banach Space into itself are proved. The results of this paper extend the results which have been proved in {[1-3],[5],[10]}.

Introduction. A large number of literatures are available which deal with common fixed points of two mappings or a family of mappings in a complete metric space or in a Hilbert Space etc. The conditions considered in the theorems of these literatures are taken in such a way that the method of iterations provides the way to get fixed points easily. But if the condition is extended or multiplier is extended it becomes very difficult to obtain fixed points of mappings by iteration method. In this case either some conditions require to impose on the space or on the mappings or both. Also the method is different. One can see { [1- 3],[5],[10]}.

In this paper we have proved a common fixed point for two mappings in the *Ist* theorem and in the *2nd* theorem for a sequence of mappings in a reflexive Banach space with some additional condition on the space as well as on the operators. The following is quoted from [5].

Definition [5]. Let A be a bounded subset of a Banach Space X. A point $a \in X$ is said to be a non-diametrial point of A if sup $\{\| x-a \|, x \in A\} < \delta)(A)$.

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A bounded convex subset K of X is said to have normal structure if for each convex subset H of K which contains more than one point there exists an $x \in H$ which is a non-diametral point of H.

We now proved the following theorems.

Theorem 1. Let X be a reflexive Banach Space and K be a non-empty bounded closed convex subset of X. Let $T_1, T_2 : K \to K$ be such that

(A) $\|T_1x\, T_2y\| \le max\, \{\|x-y\|, \|x-T_1x\|, \|y-T_2y\|$ for all $x, y \in K$

(B) K has a Normal Structure,

(C) $T_1C \subset C$ if and only if $T_2(C) \subset C$ for each closed subset C of K, and

(D) Either $\underset{x \in F}{Sup}\, \|\| x-T_1x\| < \delta\, (F$ or $\underset{y \in F}{Sup}\, \|x-T_2x\| < \delta\, (F)$,

for every non-empty bounded closed convex subset of K which is mapped into itself by T_1 or T_2. Then T_1 and T_2 have a common fixed point in X.

Proof. Let Y denotes the family of all non-empty bounded closed convex subsets of K ordered by set inclusion which are mapped into itself by T. By Smulian's result [9] i.e. X is reflexive if and only if every decreasing sequence of non-empty bounded closed convex subsets of X has a non-empty intersection and by Zorn's lemma, it follows that X possesses a minimal element F, say. If F contains only one element then that element becomes a fixed point of T. We shall show that F contains only one element.

We suppose, on the contrary, that F contains more than one point, which we shall show implies a contradiction.

Let $A = \underset{y \in F}{Sup}\, \|T_2y - y\|$. By the condition, $A < \delta\, (F)$ we now define the following terms.

For $x \in F$, let,
$$\upsilon_x(F) = max\{\, \underset{y \in F}{Sup}\|x - y\|,\ A\},$$
$$\upsilon_x\, (F) = inf\, \{V_x(F),\ x \in F\},$$
$$F_c = \{x \in F\ ;\ \upsilon_x(F) = \upsilon(F)\}.$$

We now show that F_c is nonempty, closed and convex.

For a positive integer n and for $x \in F$, let
$$F(x,\, n) = \{y \in F\ ;\ \|\, x - y\| \le \upsilon\, (F) + 1/n\} \text{ and}$$
$$C_n = \bigcap_{y \in F} F\, (x,n).$$

We show first that C_n is non-empty. If possible let $C_n = \phi$, then there exist $x_1\, x_2 \in F$ such that $F(x_i,n) \cap F(x_2,n) = \phi$. By construction

$F(x_1,n) = \{\, y \in F : \|x_1 - y\| < \upsilon\, (F) + 1/n\}$ and similarly for $F(x_2,n)$. We obtain from the disjointness of $F(x_1,n)$ and $F(x_2,n)$

$$\|x_2 - x_2\| \leq 2\upsilon(F) + 2/n$$

Now for $x \in F$, $\sup_{y \in F} \|x-y\|$ $\frac{\delta(F)}{2}$ (F) and so $\upsilon_x(F) \geq \frac{\delta(F)}{2}$ and this implies.

$$\frac{\delta(F)}{2} \leq \upsilon(F)$$

Therefore $\delta(F) < 2\upsilon(F) + 2/n$. So from(1)

$\|x_2 - x_2\| \delta(F)$ which is a contradiction because $x_1, x_2 \in F$.

C_n is non-empty.

It may further be verified that C_n is closed bounded convex and that $C_{n+1} \subset C_n$.

We wish to show that $F_c = \bigcap_{n=1}^{\infty} C_n$.

For this, let $y \in F_c$ then $\upsilon \in (F) = \upsilon(F)$.

So $\max\{\sup_{x \in F} \|y - x\|, A\} = \upsilon(F)$ and so

$$\sup_{x \in F} \|y - x\| \leq \upsilon(F). \qquad ...(2)$$

We verify that $y \in F(x,n)$ for all $x \in F$ and for all n.

If possible, let $y \notin F(x,n)$ for some x and for some n. Then

$$\|x - y\| > \upsilon(F) + 1/n \qquad ...(3)$$

From (2) we see that $\|x - y\| \leq \upsilon(F)$ which is a contradiction to (3).

Therefore $y \in \bigcap_{n=1}^{\infty} C_n$ *and so* $F_C \subset \bigcap_{n=1}^{\infty} C_n$.

Next let $y \in \bigcap_{n=1}^{\infty} C_n$. Then $y \in F(x,n)$ for all x and for all n and this implies that $\sup_{x \in F} \|y - x\| \leq \upsilon(F)$.

Also $A \leq v(F)$. These two together give $\upsilon_y(F) \leq \upsilon(F)$. But $\upsilon(F) \leq \upsilon_y(F)$ always, and therefore $\upsilon_y(F) = \upsilon(F)$ and this gives $y \in F_c$. Thus $\bigcap_{n=1}^{\infty} C_n \subset F_c$ $\therefore F_c = \bigcap_{n=1}^{\infty} C_n$. This equality further gives that F_c is closed and convex by Smulians result [9] non empty.

Next we show that $\delta(Fc) < \delta(F)$.

Since K has normal structure and $A < \delta(F)$. There exists a point $x \in F$ such that $\upsilon_x(F) < \delta(F)$. If $x_1, x_2 \in Fc$ then $\|x_1 - x_2\| < \upsilon\, x_1(F) = \upsilon(F)$.

So
$$\delta(F_c) = \sup\{\|x_1 - x_2\| : x_1, x_2 \in F_c\}$$
$$< \upsilon(F) \leq \upsilon_x(F) < \delta(F) \qquad ...(4)$$

If $x \in F_c$ and y is an arbitrary element of F we obtain

$$\|T_1 x - T_2 y\| < \max\{\|x - y\|, \|x - T_1 x\|\}$$
$$\leq \max\{\sup_{y \in F} \|x-y\|, \sup_{y \in F} \|y - T_2 y\|, \sup_{y \in F} \|x - T_1 x\|\}$$
$$\leq \max\{\sup_{y \in F} \|x - y\|, A\} = \upsilon_x(F) = \upsilon(F).$$

So the set $T_2(F)$ is contained in a closed sphere with centre at T_1x and radius $\upsilon(F)$. We denote this sphere by $\bar{U}$.

Clearly $T_2(F \cap \bar{U}) \subset F \cap \bar{U}$ and because F is minimal, $F \subset \bar{U}$ and so,

$$\sup_{y \in F} \|T_1x - y\| \le \upsilon(F) \qquad \text{...(5)}$$

Now

$$\begin{aligned} \upsilon_{T_1x}(F) &= max\ \{\sup_{y\in F} \|T_1x - y\|, A\} \\ &\le max\ \{\upsilon(F), A\}, \text{ from (5)}, \\ &= \upsilon(F), \text{ because } \upsilon(F) \ge A. \end{aligned}$$

Hence $\upsilon_{T_1x}(F) \le \upsilon(F)$. But we always have $\upsilon(F) < \upsilon_{T_1x}(F)$. So $\upsilon_{T_1x}(F) = \upsilon(F)$. This implies that $T_1(x) \in F_c$. Similarly $T_2(x) \in F_c$. Therefore F_c is a non empty closed convex subset of F which is mapped into itself by T_1 and T_2 and because of (4), $\delta(F_c) < \delta(F)$. Therefore F_c is a proper subset of F. This contradicts the fact that F is minimal. Thus F can not contain more than one element. But F is non-empty. Hence F contains only one element which is clearly fixed point of T_1 and T_2.

Note 1. If $T_1 = T_2$ the theorem proved in Achari and Lahiri [1] and Tiwary and Lahiri [10] follows.

Note 2. $T_1 = T_2$ and if the condition (C) is withdrawn the theorem proved in Kirk[5] follows.

Theorem 2. Let X be a reflexive Banach space and K be a non-empty bounded closed convex subset of X. Let $\{T_n\}$ be a sequence of mappings which map K into itself by $\{T_n\}$ and satisfy

(A) $\|T_ix - T_jy\| \le max\{\|x-y\|, \|x - T_ix\|, \|y - T_jy\|\}$ *for all* $x, y \in K$

(B) K has a normal structure,

(C) $T_i(C) \subset C$ if and only if $T_j(C) \subset C$ for each closed subset C of K and $T_i, T_j \in \{T_n\}$ and

(D) Either $\sup_{x\in F} \|x - T_ix\| < \delta(F)$

or

$\sup_{x\in F} \|x - T\| < \delta(F)$,

for every non empty bounded closed convex subset of K which is mapped into itself by either T_i or T_j.

Then $\{T_n\}$ has a common fixed point in X.

Proof. Let F denote the family of all non empty bounded closed convex subsets of K which are mapped into itself by $\{T_n\}$ and ordered by set inclusion. The family F is non empty and because X is reflexive by Smulian's result [9] every decreasing sequence of non empty bounded closed convex subsets of X has a non empty intersection and by Zorn's lemma, it follows that X possesses a minimal element F, say.

Picking any two mapping T_i and T_j from $\{T_n\}$ and following the proof of the Theorem 1 it follows that, T_i and T_j have a common fixed point in X.

Since T_i and T_j are any two mappings, it follows that $\{T_n\}$ has a common fixed point in X.

This completes the proof of the theorem.

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(*Dedicated to the memory of Professor J.N. Kapur*)

A PROBLEM ON HEAT CONDUCTION IN A ROD UNDER THE ROBIN CONDITION

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ABSTRACT

In the present paper, we discuss the problem on heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends in side the medium). Its special cases are also discussed.

It is remarked that all the results due to Chaurasia-Gupta ([4], (2.1), (2.2), (3.1), (3.3), (4.1) to (4.12)) are wrongly expressed and their correct forms are special cases of our results. Our results are also generalization of the results due to Chaurasia and Patni[3]. Their all results ([3],(8), (11) to (19)) are also wrongly expressed.

1. Introduction. Chandel and Tiwari [1] discussed two boundary value problems with applications of multiple hypergeometric function of Srivastava and Daoust ([8], [9],[10]). Chaurasia and Patni[3] discussed a heat conduction problem and certain product of the multivariable H-function due to Srivastava and Panda ([11],[12] also see [13]) and two general classes of polynomials due to Srivastava [7]. Chaurasia and Gupta [4] also discussed a solution of the partial differential equation of heat conduction in a rod under the Robin condition. Srivastava and Srivastava [17] dicussed application of Meiger's G-function of one variable in two boundary value problems. Recently, Chandel and Sengar [2] discussed two boundary value problems employing multivariable H-function of Srivastava and Panda ([11],[12] also see [13]) and product of several general classes of poynomials of Srivastava [7].

In the present paper, we shall discuss the problem on heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends in side the medium). If the thermal coefficient is constant and there is no source of thermal enegry, then we shall find the function $\theta(x, t)$ satiafying partial differatial equation

$$(1.1) \qquad \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \;, \quad t \geq 0$$

with initial condition

$$(1.2) \qquad u(x,0) = f(x),$$

and boundary conditions

$$(1.3) \qquad \frac{\partial u(0, t)}{\partial t} - h\; u(0, t) = 0,$$

$$(1.4) \qquad \frac{\partial}{\partial t}\, u(L, t) + h\; u(L, t) = 0,\; h > 0.$$

It is clear that the expression

$$(1.5) \qquad u(x,t) = e^{-\mu E_p^2 t} (m \cos E_p x + n \sin E_p x)$$

satisfies (1.1).

Then boundary conditions (1.3)and (1.4) give respectively

$$(1.6) \qquad m/n \quad = \quad E_p/h$$

and

$$(1.7) \qquad \tan (E_p L) \;=\; 2E_p h/(E_p^2 - h^2),$$

where E_p is p^{th} positive root of (1.7).

Then the elegant form of solution of the problem is given by

$$(1.8) \qquad u(x,t) = \sum_{p=1}^{\infty} R_p (\cos E_p x + h/E_p \sin E_p x) e^{-\mu E_p^2 t}\,.$$

Here we consider the problem of determining $u(x, t)$, where

$$(1.9) \qquad u(x, 0) = f(x) = (\sin(\pi x/L))^{W-1}\, S_{n_1}^{m_1}[y_1 (\sin \pi x/L)^{2\rho_1}] \ldots S_{n_r}^{m_r}[y_r (\sin \pi x/L)^{2\rho_r}]\,.\,H,$$

$$(1.10) \qquad H = H^{0,\,\lambda:\,(\mu',\,\upsilon');\ldots;(\mu^{(n)},\,\upsilon^{(n)})}_{A,C:\,(B',D');\ldots;(B^{(n)},D^{(n)})} \left(\begin{matrix} [(a):\theta';\ldots,\theta^{(n)}]: \\ [(c):\psi';\ldots;\psi^{(n)}]: \end{matrix} \begin{matrix} [(b'),:\phi');\ldots;[(b^{(n)}),\phi^{(n)}]; \\ [(d'),:\delta');\ldots;[(d^{(n)}),\delta^{(n)}]; \end{matrix}\; z_1 (\sin \pi x/L)^{2\xi_1};\ldots; z_n (\sin \pi x/L)^{2\xi_n} \right)$$

is multivariable H-function of Srivastava and Panda ([11], [12] and also see [13]) and $S_n^m[x]$ are generalized polynomids of Srivastava ([7], p.1, eqn. (1.1)) difined by

$$(1.11) \qquad S_n^m\, [x] = \sum_{s=0}^{[n/m]} \cdot \frac{(-n)_{m.s}}{s!} \;A_{n,s} x^s \quad , \; n=0,1,2,\ldots$$

where m is arbitrary positive integer and the coefficients $A_{n,s}$ $(n, s, \geq 0)$ are arbitrary constants, real or complex.

2. Formulae Required. The following well known results will be frequently used in our investigations

$$(2.1)\qquad \int_0^L (\sin \pi x/L)^{\sigma-1} \sin(E_p \pi x/L)\,dx = L\,2^{1-\sigma} \sin(E_p \pi/2)\frac{\Gamma(\sigma)}{\Gamma((\sigma \pm E_p+1)/2)}$$

$$(2.2)\qquad \int_0^L (\sin \pi x/L)^{\sigma-1} \cos(E_p \pi x/L)\,dx = L2^{1-\sigma} \cos(E_p\,\pi/2)\frac{\Gamma(\sigma)}{\Gamma((\sigma \pm E_p+1)/2)},$$

$$\mathrm{Re}\,(\sigma) > 0,$$

$$(2.3)\qquad \int_0^L (\cos E_q x + h/E_q \sin E_q x)\,(\cos E_p x + h/E_p \sin E_p x)\,d_x$$

$$= [2E_q^{\,2}\Gamma^{1}(E_q^{\,2}+h^2)\,L + 2h],\quad p=q$$

$$= 0,\quad p \neq q,$$

where E_q is qth positive root of non-algebraic equation

$$(2.4)\qquad \tan EL = 2hE/(E^2 - h^2).$$

3. Main integrals. In this section, we evaluate the following integrals, which will be used in our investigations :

$$(3.1)\qquad \int_0^L (\sin \pi x/L)^{W-1} \cos(\pi x \lambda_m/L) \prod_{i=1}^{r} S_{n_i}^{m_i}[y_i\,(\sin \pi x/L)^{2\rho_i}]\,H\,dx$$

$$\frac{L\cos(\pi\lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i, s_i}\, y_i^{s_i}}{4^{\rho_i s_i}\, s_i!} \quad H(s_1,\ldots,s_r;\lambda_m)$$

where

$$H(s_1,\ldots,s_r;\lambda_m) = H^{0,\,\lambda+1:\,(\mu',\upsilon');\ldots;(\mu^{(n)},\,\upsilon^{(n)})}_{A+1,C+2:\,(B',D');\ldots;(B^{(n)},D^{(n)})} \begin{bmatrix} [(a):\theta';\ldots,\theta^{(n)}], \\ [(e):\psi';\ldots;\psi^{(n)}], \\ [1-W-2(\rho_1 s_1+\ldots+\rho_r s_r):\,2\xi_1,\ldots,2\xi_n] \quad :[(b'):(\phi')];\ldots;[(b^{(n)}):(\phi^{(n)})], \\ [\{1-W-2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m\}/2:\,\xi_1,\ldots,\xi_n]:[(d'):(\delta')];\ldots;[(d^{(n)}):(\delta^{(n)})]. \end{bmatrix}$$

$$z_1/4^{\xi_1},\ldots,z_1/4^{\xi_n}),$$

$Re\,(W) > 0$, $Re(W + \sum_{i=1}^{n} \xi_i\, d_j^{(i)}/\delta_j^{(i)}) > 0$, all ξ_i are real positive numbers ,λ ,A,C, $\mu^{(i)}$, $\upsilon^{(i)}$,$B^{(i)}$,$D^{(i)}$ are such that $A \geq \lambda \geq 0$, $C > 0$, $D^{(i)} \geq \mu^{(i)} \geq 0$, $B^{(i)} \geq \upsilon^{(i)} \geq 0$ and $\theta_j^{(i)}$, $j=i,\ldots,A$; $\phi_j^{(i)}$, $j=1,\ldots, B^{(i)}$; $\psi_j^{(i)}$, $j = 1,\ldots, C$; $\delta_j^{(i)}$, $j=1,..,D^{(i)}$ are positive real numbers,

$$|\arg [z_i(\sin\pi x/L)^{2\xi_i}]| < \Delta_i \pi/2,$$

$$\Delta_i = -\sum_{j=\gamma+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\upsilon^{(i)}} \phi_j^{(i)} - \sum_{j=\upsilon^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$-\sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0;\ i=1,\ldots,n.$$

Also ρ_k are all positive numbers, n_k,m_k , are arbitrary positive integers, A_{n_k,s_k} are arbitrary functions of n_k and s_k real or complex independent of x, y_k, ρ_k ,$k = 1,\ldots,r$.

For $r = 2$, (3.1) reduces to the correct form of the result due to Chaurasia and Gupta ([4],(2.2)).

We shall also frequently use the integral due to authors ([2],(2.1))

4. Solution of the problem.

For t = 0, from (1.8) and (1.9) we derive

$$(4.1)\qquad (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} S_{n_i}^{m_i} [y_i (\sin \pi x/L)^{2\rho_i}]\ H$$

$$= \sum_{q=1}^{\infty} R_q (\cos E_q x + h/E_q \ \sin E_q x).$$

Then

$$(4.1)\qquad \int_0^L \sum_{q=1}^{r} R_q(\cos E_q x + h/E_q \sin E_q x)\, (\cos E_p + h/E_p \sin E_p x)\, dx$$

$$= \int_0^L (\sin \pi x/L)^{W-1} (\cos E_p x + h/E_p \sin E_p x) \prod_{i=1}^{r} S_{n_i}^{m_i} [y_i (\sin \pi x/L)^{2\rho_i}] .$$

Thus making an appeal to (2.1),(2.2) and (2.3), we derive

$$(4.2)\qquad R_p = \frac{L}{2^{W-2}} \frac{[E_p \cos (L E_p/2)] + h \ \sin (L E_p/2)]E_p}{(E_p^2 + h^2)L + 2h}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i, s_i} y_i^{s_i}}{4^{\rho_i s_i}\, s_i!} H(s_1,\ldots,s_r; L E_p/\pi).$$

Therefore, substituting the value of R_p in (1.8), the solution of the problem in given by

$$(4.3)\ u(x,t) = \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p \cos (L E_p/2)] + h \ \sin (L E_p/2)]}{(E_p^2 + h^2)L + 2h}$$

$$E_p(\cos E_p x + h \sin E_p x)\, e^{-\mu e_p^2 t} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i, s_i} y_i^{s_i}}{4^{\rho_i s_i}\, s_i!}$$

$$H(s_1,\ldots,s_r; L E_p/\pi),$$

where all the conditions of (3.1) are satisfied.

5. Expansion formula. For $t = 0$, (4.3) with (1.9), gives expansion formula

$$(5.1)\qquad (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} S_{n_i}^{m_i} [y_i (\sin \pi x/L)]^{2\rho_i}\ H$$

$$= \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p \cos (L E_p/2)] + h \ \sin (L E_p/2)][E_p \cos E_p x + h \sin E_p x]}{[E_p^2 + h^2)]L + 2h}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i, s_i} y_i^{s_i}}{4^{\rho_i s_i}\, s_i!} H(s_1,\ldots,s_r; L E_p/\pi),$$

valid if all the conditions of (3.1) are satisfied.

6. Special Cases.

Case I. Choosing each $m_i = 2$, $A_{n_i,s_i} = (-1)^{s_i}$, we have

$$S^2_{n_i}[y_i] \to y_i^{n_i/2} H_{n_i}(1/2\sqrt{y_i}),\ i = 1,\dots,r.$$

Therefore, for Hermite polynomials ([16], p.106, eq. (5.5.4)and [14], p. 158), our main integral (3.1) reduces to

$$(6.1)\qquad \int_0^L (\sin \pi x/L)^{W-1} (\cos \pi x\lambda_m/L) \prod_{i=1}^{r} y_i^{n_i/2} H_{n_i}(1/2\sqrt{y_i})\,.H\,dx$$

$$= \frac{L\cos(\pi\lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \prod_{i=1}^{r} \frac{(-n_i)_{2s_i}(-1)^{s_i}\, y_i^{s_i}}{4^{\rho_i s_i}\, s_i!} H(s_1,\dots,s_r;\lambda_m)$$

where all conditions of (3.1) are satisfied.

Then solution (4.3) of the problem reduces to

$$(6.2)\ u(x,t) = \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \cdot \frac{[E_p\cos(L E_p/2)] + h\ \sin(L E_p/2)]}{(E_p^2 + h^2)L + 2h}$$

$$[E_p \cos E_p x + h\ \sin E_p x] e^{-\mu E_p^2 t}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i}(-1)^{s_i}\, y_i^{s_i}}{4^{\rho_i s_i}\, s_i!} H(s_1,\dots,s_r; L E_p/\pi),$$

which is valid if all conditions of (3.1) are satisfied.

The expansion formula (5.1) reduces to

$$(6.3)\qquad (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} (\sin \pi x/L)^{n_i\rho_i}\, y_i^{n_i/2} H_{n_i}\left[\frac{1}{2\sqrt{y_i}(\sin \pi x/L)^{\rho_i}}\right].H$$

$$= \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p\cos(L E_p/2)] + h\sin(L E_p/2)][E_p\cos\ E_p x + h\sin E_p x]}{[E_p^2 + h^2)]L + 2h}$$

$$\sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \prod_{i=1}^{r} \frac{(-n_i)_{2s_i}\, y_i^{s_i}(-1)^{s_i}}{4^{\rho_i s_i}\, s_i!} H(s_1,\dots,s_r; L E_p/\pi).$$

Case II. Choosing each $m_i = 1$, $A_{n_i,s_i} = \binom{n_i + \alpha_i}{n_i} \frac{1}{(1+\alpha_i)_{s_i}}$,

we have $S^{(1)}_{n_i}[y_i] \to L^{(\alpha)}_{n_i}(y_i)$, $i = 1,\dots,r$,

therefore for Laguerre polynomials([16], p. 10, eq.(5.1.6) and [14], p. 158), our result (3.1) reduces to

$$(6.4)\qquad \int_0^L (\sin \pi x/L)^{W-1} (\cos \pi x\lambda_m/L) \prod_{i=1}^{r} L^{(\alpha_i)}_{n_i}\ [y_i(\sin \pi x/L)^{2\rho_i}]\ H\,dx$$

$$= \frac{L\cos(\pi\lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{n_1} \cdots \sum_{s_r=0}^{n_r} \prod_{i=1}^{r} \frac{(-n_i)_{s_i}\, y_i^{s_i}}{4^{\rho_i s_i}\, s_i!} \binom{n_i+\alpha_i}{n_i}$$

$$\frac{1}{(1+\alpha_i)_{s_i}}\; H(s_1,\ldots,s_r;\lambda_m)$$

and the solution (4.3) of the problem reduces to

$$(6.5)\quad u(x,t)=\frac{L}{2^{W-2}}\sum_{p=1}^{\infty}\frac{[E_p\cos(L\,E_p/2)]+h\sin(L\,E_p/2)]}{(E_p^2+h^2)L+2h}$$

$$[E_p\cos E_p x+h\;\sin E_p x]e^{-\mu E_p^2 t}\;\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\;\prod_{i=1}^{r}\binom{n_i+\alpha_i}{n_i}\;\frac{1}{(1+\alpha_i)_{s_i}}$$

$$\frac{(-n_i)_{s_i}\,y_i^{s_i}}{4^{\rho_i s_i}\,s_i!}\;H(s_1,\ldots,s_r;L\,E_p/\pi),$$

provided that all the conditions of (3.1) are satisfied.

Then expansion formula (5.1) reduces to

$$(6.6)\qquad(\sin\pi x/L)^{W-1}\prod_{i=1}^{r}L_{n_i}^{(\alpha_i)}[y_i(\sin\pi x/L)^{2\rho_i}]\;.\;H$$

$$=\frac{L}{2^{W-2}}\sum_{p=1}^{\infty}\frac{[E_p\cos(L\,E_p/2)]+h\sin(L\,E_p/2)][E_p\cos\;E_p x+h\sin E_p x]}{[E_p^2+h^2)]L+2h]}$$

$$\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\;\prod_{i=1}^{r}\;\frac{(-n_i)_{s_i}}{4^{\rho_i s_i}\,s_i!}\;\binom{n_i+\alpha_i}{n_i}\;\frac{y_i^{s_i}}{(1+\alpha_i)_{s_i}}\quad H(s_1,\ldots,s_r;L\,E_p/\pi).$$

Case III. Choosing each $m_i=1$, $A_{n_i,s_i}=\binom{n_i+\alpha_i}{n_i}\dfrac{(1+\alpha_i+\beta_i+n_i)}{(1+\alpha_i)_{s_i}}$

we have $S_{n_i}^{(1)}[y_i]\to P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i)$, $i=1,\ldots,r$,

therefore, for Jacobi polynomials ([16], p.68, eq. (4.3.2) and [14]p.159), (3.1)) reduces to

$$(6.7)\qquad\int_0^L(\sin\pi x/L)^{W-1}(\cos\pi x\lambda_m/L)\prod_{i=1}^{r}P_{n_i}^{(\alpha_i,\beta_i)}[1-2y_i(\sin\pi x/L)^{2\rho_i}]$$

$$=\frac{L\cos(\pi\lambda_m/2)}{2^{-1}}\;\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\;\prod_{i=1}^{r}\frac{(-n_i)_{s_i}\,y_i^{s_i}}{4^{\rho_i s_i}\,s_i!}\quad\binom{n_i+\alpha_i}{n_i}$$

$$\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}\,H(s_1,\ldots,s_r;\lambda_m)$$

and the solution (4.3) of the problem reduces to

$$(6.8)\quad u(x,t)=\frac{L}{2^{W-2}}\quad\sum_{p=1}^{\infty}\frac{[E_p\cos(L\,E_p/2)]+h\sin(L\,E_p/2)]}{(E_p^2+h^2)L+2h}$$

$$[E_p\cos E_p x+h\;\sin E_p x]e^{-\mu E_p^2 t}\;\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\;\prod_{i=1}^{r}\frac{(-n_i)_{s_i}\;y_i^{s_i}}{4^{\rho_i s_i}\,s_i!}$$

$$\binom{n_i+\alpha_i}{n_i} \frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}} \quad H(s_1,\ldots,s_r; L\,E_p/\pi),$$

where all conditions of (3.1) are satisfied.

Then expansion formula (5.1) reduces to

$$(6.9) \qquad (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} P_{n_i}^{(\alpha_i,\beta_i)}[1-2y_i(\sin \pi x/L)^{2\rho_i}]\,.\,H$$

$$= \frac{L}{2^{W-2}} \cdot \sum_{p=1}^{\infty} \frac{[E_p \cos (L\,E_p/2)] + h\sin (L\,E_p/2)][E_p\cos\ E_p x + h\sin E_p x]}{(E_p^2+h^2)L+2h} .$$

$$\sum_{s_1=0}^{n_1} \cdots \sum_{s_r=0}^{n_r} \frac{(-n_i)_{s_i}}{4^{\rho_i s_i}\, s_i!} \binom{n_i+\alpha_i}{n_i} \frac{(1+\alpha_i+\beta_i+n_i)_{s_i}\ y_i^{s_i}}{(1+\alpha_i)_{s_i}}$$

$$H(s_1,\ldots,s_r; L\,E_p/\pi),$$

valid if all conditions of (3.1) are satisfied.

7. Remarks.

Remark 1. Specially for $r=2$, our results (3.1), (4.1), (4.2), (4.3), (6.1), (6.2), (6.4) and (6.5) reduce to wrongly expressed results due to Chaurasia and Gupta ([4], (2.2), (3.2), (3.3), (3.1), (4.5),(4.6), (4.11) and (4.12)) respectively.

Remark 2. For $r = 2$, $\lambda = A$, $u^{(i)}=1$, $v^{(i)}= B^{(i)}$ *and replacing* $D^{(i)}$ *by* $D^{(i)}+1$, $i=1,\ldots,n$; our results (3.1) and (4.3) reduce to correct forms of the results due to Chaurasia and Gupta ([4], (4.8) and (4.9)) respectively.

Remark 3. For $r = 2$, $n=1$, our results (3.1) and (4.3) reduce to correct forms of ([4], (4.2) and (4.3)) respectively.

Remark 4. Here it is also remarked that all the remaining results due to Chaurasia and Gupta([4], (2.1), (4.1),(4.4),(4.7) and (4.10)) are wrongly expressed.

Remarks 5. The above remarks also suggest that all the results due to Chaurasia and Patni ([3], (8), (11) to (19)) are also worngly expressed. Actually every where $2^{-hs-h's}$ should be written within $\sum_{s=0}^{[n/m]} \sum_{s'=0}^{[n'/m']}$.

Specializing the parameters of multivariable H-function of Srivastava and Panda ([10], [11] also see [12]) and parameters and numbers of the polynomials [7], we can derive serveral interesting results very useful in Analysis, Applied Mathematics and Mathematical Physics.

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(Dedicated to the memory of Professor J.N. Kapur)

Dr. B.B. Lal, D. Litt
[April 01, 1918 - March 29, 2001]

**This page of Jñānābha, is being dedicated
to the memory of memory
of**

Dr. B.B. Lal, D. Litt
Ex - Principal, D.V. Postgraduate College, Orai
and
Former Vice - Chancellor, Bundelkhand University, Jhansi
[April 01, 1918 - March 29, 2001]

who sent his message to Professor J.N. Kapur on his
70th Birthday *(September 7, 1993)* Celebbrations
organised by VIJÑĀNA PARISHAD OF INDIA at HBTI, Kanpur

मनीषी, गणित गणपति, उदारमना
देवपुरुष डा० कपूर का अभिनन्दन

ज्ञानाभ परिवार के संरक्षक एवं 'ज्ञानाभ' का श्रीगणेश करने के लिये प्रेरक भूमिका का निर्वाह करते हुए, मुझे प्रस्तोता के रुप में सेवा का एवं इस शोध पत्रिका से संबद्ध होने का सौभाग्य मिला। 'ज्ञानाभ' की विश्वयात्रा की उपलब्धि की पृष्ठभूमि में श्रद्धेय महामना डा० कपूर का प्रशंस्य योग रहा जिसके लिए उनके प्रति 'ज्ञानाभ ' परिवार के साथ मैं भी नतमस्तक एवं कृतार्थ होकर विनम्र कृतज्ञता सादर ज्ञापित करता हूँ।

विश्व धर्म सम्मेलन में शिकागो में महर्षि विवेकानन्द को बोलने के लिए 'शून्य' प्रस्तावित किया गया। 'शून्य', गणित का शून्य जो ब्रह्म विद्या, राजयोग और आत्म दर्शन की भी व्याख्या भारत के प्रतिनिधि ने ही दी। इसी संदर्भ में मैंने डा० कपूर को 'मनीषी' देव पुरुष कहने का साहस किया है।

गणित एकाग्रता का ही पर्याय है तथा एकाग्रता समस्त योग और ब्रह्मविद्या की आधार शिला है। मेरी ऐसी मान्यता है कि गणितज्ञ अपने अध्ययन अनुशीलन विषय 'गणित' के ही कारण योगी की कोटि में आ जाता है। सारस्वत डा० कपूर एक गणितज्ञ हैं, गणित पुरुष हैं तथा राजयोगी हैं, ब्रह्मविद् हैं। उनका अभिनन्दन भारतीय मनीषी का अभिनन्दन है, भारतीय दर्षन का अभिवादन है, भारतीय आत्मा के प्रति विनम्र नमन है।

हमारी कामना और परमपिता परमात्मा से हार्दिक प्रार्थना है कि यशस्वी डा० कपूर शतायु हों स्वस्थ एवं प्रसन्न रहें। नयी पीढ़ी अपनी विकासोन्मुख दिशा में उनसे प्रेरणा प्राप्त करे, उनके, आशीर्वाद से अग्रसर हों तथा उनकी भावस्वरुपा छवि में उनकी उल्लेख्य ऊँचाइयों को प्राप्त करें।

एवमस्तु ! एवमस्तु !! एवमस्तु !!!

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